Weihrauch degrees of problems related to topological circles

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We classify the Weihrauch degree of translating negative to positive information about topological circles in $[0,1]^2$.

There is a significant body of results in the literature showing that every computably compact (negative information) set of particular homeomorphism types is also computably overt (positive information). This started with Joseph Miller proving this for topological circles in his PhD thesis [10]. The programme was taken up by Zvonko Iljazović and coauthors (eg. [6, 7, 8, 4, 5]. There also is some recent work by Amir and Hoyrup [1].

It may seem natural at first glance to expect the proofs here to be uniform, i.e. to involve an algorithm to compute the overt information from the compact information given the promise that the set has a particular homeomorphism type. However, already for sets as simple as *two distinct points* such a uniform translation is impossible (essentially in [9]). The proofs instead proceed by incorporating additional point-wise computable information, such as knowing a (rational) point in the inside of the circle.

Here, we explore the question of how non-uniform the argument needs to be, and whether the construction in the proof is any more non-uniform than the result itself for the original result in the programme, pertaining to topological circles. The framework for this is Weihrauch reducibility [3]. Recall that $\mathcal{O}(\mathbf{X})$, $\mathcal{A}(\mathbf{X})$, $\mathcal{V}(\mathbf{X})$ refers to the spaces of open, closed, overt subsets of $\mathbf{X}$. Our ambient space is the compact and Hausdorff space $[0,1]^2$, which means we do need to distinguish $\mathcal{A}(\mathbf{X})$ from the space of compact sets $\mathcal{K}(\mathbf{X})$. See [12] for more on these spaces.

Let TC be the collection of all subsets of $[0,1]^2$ homeomorphic to $\mathcal{S}_1$. For some $S \in \mathrm{TC}$, let $\mathfrak{D}S$ denote the disk whose boundary S is.

We will now show that the translation from negative information of a copy of circle shares a Weihrauch degree with finding a point on the inside of it (hence Miller's proof is not "wasteful" when it comes to non-uniformity), and this Weihrauch degree is the very familiar degree $\mathrm{C}_\mathbb{N}$ of closed choice on the natural numbers. This degree already took prominent roles in [2, 13, 11].

Theorem 1. The following operations are Weihrauch equivalent:

1. $\mathrm{C}_\mathbb{N}$
2. $\mathrm{id} : \mathcal{A}(\mathbb{R}^2)|_{\mathrm{TC}} \to \mathcal{V}(\mathbb{R}^2)$, i.e. finding the positive information from the negative information
3. $\mathrm{MiddlePoint} : \mathcal{A}(\mathbb{R}^2)|_{\mathrm{TC}} \rightrightarrows [0,1]^2$, mapping S to some point in $(\mathfrak{D}S)^\circ$
4. $\mathrm{Split} : \mathcal{A}(\mathbb{R}^2)|_{\mathrm{TC}} \to \mathcal{O}(\mathbb{R}^2) \times \mathcal{V}(\mathbb{R}^2) \times \mathcal{O}(\mathbb{R}^2)$ mapping S to $((\mathfrak{D}S)^C, S, (\mathfrak{D}S)^\circ)$.
5. $\mathrm{InnerRadius} : \mathcal{A}(\mathbb{R}^2)|_{\mathrm{TC}} \to \mathbb{R}$, mapping S to $\sup\{\varepsilon > 0 \mid \exists x \in [0,1]^2 \ B(x,\varepsilon) \subseteq \mathfrak{D}S\}$
6. $\mathrm{BoundInnerRadius} : \mathcal{A}(\mathbb{R}^2)|_{\mathrm{TC}} \rightrightarrows \mathbb{N}$, mapping S to some $n \in \mathbb{N}$ such that there exists some $x \in [0,1]^2$ with $B(x, 2^{-n}) \subseteq \mathfrak{D}S$

7. $\text{OuterRadius} : \mathcal{A}(\mathbb{R}^2)|_{\text{TC}} \to \mathbb{R}$, mapping S to $\inf\{\varepsilon > 0 \mid \exists x \in [0,1]^2\ B(x,\varepsilon) \supseteq S\}$

Proof. $2 \leq_{\mathrm{W}} 3$ Trivial.

$3 \leq_{\mathrm{W}} 1$ Using $\mathrm{C}_{\mathbb{N}}$ we can guess a rational point q and a time $t \in \mathbb{N}$. If q gets not removed from S by time t, we reject. If ever a path from the boundary of $[0,1]^2$ to q gets entirely removed from S, we reject. For any rational q in the inside of the circle there exists a t that makes (q,t) a valid guess, and whenever (q,t) is a valid guess, then q is a rational point in the inside of the circle.

$4 \leq_{\mathrm{W}} 3$ The choice of a point in the inside of the circle is the only non-uniform argument in Miller's proof [10] that a c.e. set homeomorphic to a circle is computable. This lets us compute $S \in \mathcal{V}(\mathbb{R}^2)$. We know that $\mathbb{R}^2 \setminus S$ has two connectedness components, $(\mathfrak{D}S)^C$ and $(\mathfrak{D}S)^\circ$, we know a point in $(\mathfrak{D}S)^C$ by assumption, and we can use MiddlePoint to find a point in the latter. To compute $(\mathfrak{D}S)^C, (\mathfrak{D}S)^\circ \in \mathcal{O}(\mathbb{R}^2)$, we simply search for a path covered by S^C that links a reference point to the point at hand.

$5 \leq_{\mathrm{W}} 3$ If we have $(\mathfrak{D}S)^\circ \in \mathcal{O}(\mathbb{R}^2)$ and $S \in \mathcal{V}(\mathbb{R}^2)$, we can use the former to compute arbitrarily good lower bounds for InnerRadius(S), and the latter to compute arbitrarily good upper bounds. For the former, search for compact discs covered by $(\mathfrak{D}S)^\circ$. For the latter, check whether for some radius all open disks of that radius with center from a fine but finite grid intersect S.

$6 \leq_{\mathrm{W}} 5$ Trivial.

$1 \leq_{\mathrm{W}} 6$ We use the fact that $\mathrm{C}_{\mathbb{N}}$ is equivalent to $\textsc{UpperBound} :\subseteq \mathcal{O}(\mathbb{N}) \rightrightarrows \mathbb{N}$ that maps an enumeration of a finite set of natural numbers to a joint upper bound. Now we compute S as follows: Start by slowly removing everything but a contracting band around a big circle of radius 2^{-1}. If some number greater than 1 gets enumerated, then pick a small circle of radius 2^{-k} contained in the boundary band, and approximate this one next. If a number greater than k is enumerated, pick an even smaller circle, etc.

$1 \leq_{\mathrm{W}} 7$ The same argument as for $1 \leq_{\mathrm{W}} 6$ works, as the set we construct there has the same outer and inner radius.

$7 \leq_{\mathrm{W}} 2$ If we have a set $A \in (\mathcal{K} \wedge \mathcal{V})([0,1]^2)$ we can recognize both that $A \subseteq B(x,\varepsilon)$ and that $A \nsubseteq \overline{B}(x,\varepsilon)$. This is enough to search through balls with centers from a sufficiently fine grid and rational radii and find better and better upper and lower bounds for the outer radius of A.

□

Open Question 2. 1. What about the map $\text{Image}^{-1} : \mathcal{A}(\mathbb{R}^2)|_{\text{TC}} \rightrightarrows \mathcal{C}(\mathcal{S}_1, [0,1]^2)$ mapping S to some $f : \mathcal{S}_1 \to [0,1]^2$ with $f[\mathcal{S}_1] = S$?

2. How far can we generalize this? What about other ambient spaces than $\mathbb{R}^2$? What about Warsaw circles in place of circles?

3. What other interesting non-uniform arguments are there in this area?

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