Dialectica logical principles via free categorical constructions

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Abstract

In 1958 Gödel introduced the *Dialectica interpretation* [4] to prove the (relative) consistency of intuitionistic arithmetic and, over the years, this interpretation has been generalized from a categorical perspective by several authors, leading up to the notion of Dialectica category [2] (or more generally fibration [5]).

The most important clause of the Dialectica interpretation is the definition of the translation of the implication connective. It is well-explained in [1, 3] that the crucial point is that this translation validates two logical principles which are usually not acceptable from a constructive point of view, namely a variant of the principle of *independence of premises* (IP) and a variant of *Markov's principle* (MP).

The main purpose of this talk is to provide a categorical explanation of the validity of (IP) and (MP) in the Dialectica interpretation by using the language of Lawvere's doctrines.

To achieve our purpose we employ the tool of existential (and universal) completion introduced in [7] and further developed in [6] to define a proof-irrelevant categorification of the Dialectica interpretation given in [8] in terms of doctrines that we call *Gödel doctrines*.

Then, we show that the categorical notions of existential-free elements introduced in [6] and universal-free elements developed in [8] provide a categorical presentation of quantifier-free formulas and are the key ingredient to validate (IP) and (MP) underlying Gödel's Dialectica interpretation.

Finally, showing that every Gödel doctrine validates also the so-called *principle of Skolemization*, we can conclude the proof that the notion of Gödel doctrine provides a complete categorical account of the Dialectica interpretation and of the logical principles there involved.

References

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