On Envelopes and Backward Approximations

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When working over infinite data, such as real numbers, one frequently encounters computational problems that fail to be solvable exactly (in the sense of Exact Real Computation) for continuity reasons. A common remedy is to replace such problems with approximate formulations where a slightly perturbed problem instance is solved exactly. More precisely, let $f : X \to Y$ be a function between computable metric spaces. Let $\mathbb{Q}_{>0}$ denote the space of strictly positive rational numbers with the discrete topology. Consider the *backward approximation*

$$^{\dagger}f : X \times \mathbb{Q}_{>0} \rightsquigarrow Y,\ ^{\dagger}f(x,\varepsilon) = \{f(\widetilde{x}) \in Y \mid \widetilde{x} \in B(x,\varepsilon)\}. \tag{1}$$

This relaxation underlies for instance the non-deterministic inequality test for real numbers, the notion of "approximate solutions" of fixed point equations [Bro52, Sca67], and backwards stable algorithms in numerical analysis [TB97, Chapter III].

Observe that the function $^{\dagger}f$ is always continuous. Further, if f has a computable left inverse, then $^{\dagger}f$ is computable. The latter situation occurs frequently in practice, since discontinuous functions often arise as "inverse problems".

Backward approximations can be useful for computing quantities that depend continuously on the input data. For instance, the standard algorithms for computing the eigenvalues of a matrix proceed by first diagonalising the matrix using a backwards stable algorithm, and then reading the eigenvalues off the diagonal [TB97, Chapter V]. This yields good approximations of the eigenvalues, despite the base change matrices not depending continuously on the input matrix.

More generally, one can ask when it is possible to make an exact "idealised" algorithm that employs discontinuous functions as subroutines into a rigorous one by replacing the subroutines in question by backward approximations. We will give necessary and sufficient criteria with the help of continuous envelopes [Neu19].

Let us first consider the following question: Given functions $f_i : X_i \to X_{i+1}$ between computable metric spaces $X_1, \dots, X_{n+1}$ and $x \in X_1$, when do we have convergence

$$^{\dagger}f_n(\cdot,\delta) \circ \cdots \circ {}^{\dagger}f_1(\cdot,\delta)(x) \to f_n \circ \cdots \circ f_1(x) \qquad \text{as } \delta \to 0\,? \tag{2}$$

Here, $^{\dagger}f_i(\cdot,\delta) : X_i \rightsquigarrow X_{i+1}$ is the function which is obtained by binding the second parameter of $^{\dagger}f$ to δ.

For a computable metric space Y, let $\mathcal{K}_{\perp}(Y)$ denote the lattice of compact subsets of Y, ordered by reverse inclusion, with a bottom element added. Any function $f : X \to Y$ has a best continuous approximation $F : X \to \mathcal{K}_{\perp}(Y)$ in the following sense: For all $x \in X$ we have $f(x) \in F(x)$, and if $G : X \to \mathcal{K}_{\perp}(Y)$ satisfies $f(x) \in G(x)$ for all $x \in X$ then $F(x) \subseteq G(x)$ for all $x \in X$. We obtain the following convergence criterion:

Theorem 1. Let $X_1, \dots, X_{n+1}$ be computable metric spaces. Let $f_i : X_i \to X_{i+1}$, $i = 1, \dots, n$ be a family of functions. Let $F_i : X_i \to \mathcal{K}_{\perp}(X_{i+1})$ be the best continuous approximation of f_i. Assume that $F_i(x) \neq \perp$ for all $x \in X_i$. Then the following are equivalent:

1. There exists a total continuous multi-valued function $\omega : X_1 \times \mathbb{Q}_{>0} \rightsquigarrow \mathbb{Q}_{>0}$ such that for all $x \in X_1$, all $\varepsilon > 0$, all $\delta \in \omega(x,\varepsilon)$, and all $y \in {}^{\dagger}f_n(\cdot,\delta) \circ \cdots \circ {}^{\dagger}f_1(\cdot,\delta)(x)$ we have $d(y, f_n \circ \cdots \circ f_1(x)) < \varepsilon$.

2. We have $F_n \circ \cdots \circ F_1(x) = \{f_n \circ \cdots \circ f_1(x)\}$ for all $x \in X_1$. Here, the composition of the F_i's is taken in the Kleisli category of the monad $\mathscr{K}_\perp$.

In (2) we have chosen the same δ in each f_i depending only on ε and x. The notion of convergence can be weakened by allowing a different δ_i for each ${}^\dagger f_i$ that is allowed to depend on the value of ${}^\dagger f_{i-1}(x_{i-1}, \delta_{i-1})$.

Convergence in this sense can be characterised with the help of primary co-envelopes, introduced in [Neu21]. We will not give a full definition, see [Neu21, Section 5] for details. The *primary co-envelope* of $f_i \colon X_i \to X_{i+1}$ consists of a Σ-split injective space $\mathfrak{A}_{f_i}$ together with two continuous maps $\mathfrak{E}^\star_{f_i} \colon \mathfrak{A}_{f_i} \to \mathscr{O}(X_i)$ and $\pi_{\mathfrak{A}_{f_i}} \colon \mathfrak{A}_{f_i} \to \mathscr{O}(X_{i+1})$ satisfying $\mathfrak{E}^\star_{f_i} \leq f_i^\circ \circ \pi_{\mathfrak{A}_{f_i}}$ where $f^\circ \colon \mathscr{O}(X_{i+1}) \to \mathscr{O}(X_i)$ sends $U \in \mathscr{O}(X_{i-1})$ to the interior of the set $f^{-1}(U)$. The map $\pi_{\mathfrak{A}_{f_i}}$ preserves arbitrary joins and hence has an upper adjoint. This adjoint is continuous if and only if $\mathfrak{A}_{f_i}$ is isomorphic to $\mathscr{O}(X_{i+1})$. In this case $\mathfrak{E}^\star_{f_i}$ can be identified with the greatest continuous approximation of f_i with values in $\mathscr{K}_\perp(X_{i+1})$.

Theorem 2. Let $X_1, \ldots, X_{n+1}$ be computable metric spaces. Let $f_i \colon X_i \to X_{i+1}$, $i = 1, \ldots, n$ be a family of functions. For $i = 1, \ldots, n$, let $\mathfrak{E}^\star_i \colon \mathfrak{A}_{f_i} \to \mathscr{O}(X_i)$ be the primary co-envelope of f_i. Let $\rho_i \colon \mathscr{O}(X_{i+1}) \to \mathfrak{A}_{f_i}$ denote the upper adjoint of the projection $\pi_{\mathfrak{A}_{f_i}} \colon \mathfrak{A}_{f_i} \to \mathscr{O}(X_{i+1})$. The following are equivalent:

1. There exist continuous multi-valued functions $\omega_i \colon X_i \times \mathbb{Q}_{>0} \rightsquigarrow \mathbb{Q}_{>0}$, with ω_1 total and
$$(x_i, \varepsilon) \in \mathrm{dom}(\omega_i) \wedge x_{i+1} \in {}^\dagger f_i(x_i, \varepsilon) \rightarrow (x_{i+1}, \varepsilon) \in \mathrm{dom}(\omega_{i+1}),$$
such that for all sequences $x_1, \ldots, x_{n+1}$, $\delta_1 > 0, \ldots, \delta_n > 0$ satisfying $\delta_i \in \omega_i(x_i, \varepsilon)$ and $x_{i+1} \in {}^\dagger f_i(x_i, \delta_i)$ we have $d(x_{n+1}, f_n \circ \cdots \circ f_1(x)) < \varepsilon$.

2. We have
$$\mathfrak{E}^\star_1 \circ \rho_1 \circ \cdots \circ \mathfrak{E}^\star_{n-1} \circ \rho_{n-1} \circ \mathfrak{E}^\star_n \circ \rho_n(U) = (f_n \circ \cdots \circ f_1)^{-1}(U)$$
for all $U \in \mathscr{O}(X_{n+1})$.

When the envelopes of all functions $f_1, \ldots, f_n$ are known, checking, say, the equality
$$F_n \circ \cdots \circ F_1(x) = \{f_n \circ \cdots \circ f_1(x)\}$$
is arguably much simpler than proving convergence directly. It is worth mentioning that the proofs of Theorem 1 and 2 yield a method for calculating the moduli ω or ω_i with the help of the respective envelopes.

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