Radical theory of Scott-open predicates*

Peter Schuster[1], Daniel Wessel[2]

[1] *Dipartimento di Informatica, Università degli Studi di Verona*, *Strada le Grazie 15, 37134 Verona, Italy*

[2] *Mathematisches Institut der Universität München*, *Theresienstraße 39, 80333 München, Germany*

E-mail: [1]`peter.schuster@univr.it`, [2]`wessel@math.lmu.de`

The *Jacobson radical* $\mathrm{Jac}(\mathfrak{a})$ of an ideal $\mathfrak{a}$ of a commutative ring $\mathbf{A}$ with unit is usually defined as the intersection of the maximal ideals of $\mathbf{A}$ which contain $\mathfrak{a}$:

$$\mathrm{Jac}(\mathfrak{a}) = \bigcap \mathrm{Max}/\mathfrak{a}. \tag{1}$$

With the *Axiom of Choice* (AC), $\mathrm{Jac}(\mathfrak{a})$ has a well-known first-order description:

$$\mathrm{Jac}(\mathfrak{a}) = \{\, a \in \mathbf{A} \mid (\forall b \in \mathbf{A})(\, 1 \in \langle a, b\rangle \to 1 \in \langle \mathfrak{a}, b\rangle) \,\}\ . \tag{2}$$

In constructive algebra, the latter is taken [4] as definition of $\mathrm{Jac}(\mathfrak{a})$; whence (1), which we henceforth refer to as the *Jacobson Lemma* (JL), becomes a theorem that requires AC. With classical logic, in fact, JL is equivalent to Krull's *Maximal Ideal Theorem* (MIT) that every proper ideal is contained in a maximal one, which in turn is tantamount to AC. By analogy to (2), Jac can also be defined for distributive lattices [1, 2] and thus in propositional logic [3], where ideals are replaced by theories, and comaximality (i.e., generates 1) by inconsistency (i.e., proves $\bot$).

Viewed from the angle of syntax, JL shows a certain consequence relation complete with respect to maximal ideals, and thus helps to pin down the computational import of MIT [5]. This has prompted our motivating question: Can we find a syntactical counterpart to maximality principles even closer to AC, among which the *Teichmüller–Tukey Lemma* (TTL), in a manner similar to how JL relates to MIT? The resulting challenge thus is to first solve semantically

$$\frac{\mathrm{JL}}{\mathrm{MIT}} \sim \frac{\mathbf{?}}{\mathrm{TTL}} \tag{3}$$

and then to give a syntactical interpretation of the solution.

Abstracting from ideals and theories to elements of a complete lattice L, and from comaximality and inconsistency to a fixed but arbitrary Scott-open subset O of L, we are led to a closure operator

$$j : L \to L$$

generalising all the aforementioned Jacobson radicals. Special cases of j had appeared before [1,2].

In our general context some key features of j can be isolated by an inductive definition. For instance, j is the largest closure operator on L for which O consists of the j-dense elements of L; also, if L is distributive, then O is a filter precisely when j is a nucleus. With AC, moreover, we can prove the following statements for every $x \in L$, where $y \in L$ is *proper* if $\neg O(y)$, and $y \in L$ is *O-complete* if for every $z \in L$ either $z \leqslant y$ or $O(y \vee z)$:

(a) the radical jx is the meet of all proper O-complete $y \geqslant x$; and

(b) if x is proper, then there is a proper O-complete $y \geqslant x$.

*The present study was carried out within the project "Reducing complexity in algebra, logic, combinatorics - REDCOM" of the Fondazione Cariverona's programme "Ricerca Scientifica di Eccellenza 2018", and within GNSAGA of INdAM.

As among the ideals of a ring the proper O-complete ones are just the maximal ideals, (a) and (b) generalise JL and MIT, respectively. Moreover, if L is algebraic, then (b) generalises TTL.

With (a) we thus obtain the desired semantic solution of (3), and can focus on its syntactical interpretation. To this end we rather put (b) in classically equivalent contrapositive form:

(c) O consists of the $x \in L$ for which every O-complete $y \geqslant x$ belongs to O.

Adapting the recent syntactical treatments of prime ideal theorems [7] and of some fairly concrete maximality principles such as Hausdorff's maximal chain principle [6], we define inductively a class of finite binary trees labelled by elements of L, together with an appropriate termination concept for paths. All this allows us to prove constructively—in particular, without AC—the following syntactical counterpart of (c) whenever O is a filter, which normally is the case:

(d) $x \in L$ belongs to O iff there is a labelled finite binary tree with root labelled by x such that every branch of the tree terminates in O.

The feasibility of this syntactical characterisation is not obvious at all: unlike prime ideal theorems, which are of binary nature by the very form of the prime ideal axiom, abstract maximality principles such as (b) or (c) equivalent to full AC a priori fall short of lending themselves naturally to a computational simulation by finite binary trees. Our key idea to overcome this barrier, and in fact to get by with binary branching also in cases of AC proper such as TTL, is to complement every $a \in L$ by the O-variant $\overline{a}$ of the pseudo-complement of a.

In the spirit of dynamical algebra [4, 8], every tree t with root labelled by x represents the course of a dynamic argument *as if* a given $y \geqslant x$ were complete. Every complete $y \geqslant x$ gives indeed rise to a path through t: at each branching, corresponding to some $a \in L$, by completeness either $a \leqslant y$ or $O(a \vee y)$, according to which y leads in the direction to pursue: that is, a or $\overline{a}$.

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