Computing with Compact Sets: the Gray Code Case*

(extended abstract)

Dieter Spreen[♮] and Ulrich Berger[♭]

[♮]Department of Mathematics, University of Siegen, 57068 Siegen, Germany

[♭]Department of Computer Science, Swansea University, Swansea, SA1 8EN, UK

Infinite Gray code has been introduced by Tsuiki [Ts02] as a redundancy-free representation of the reals. In applications the signed digit representation is mostly used which has maximal redundancy. Tsuiki presented a functional program converting signed digit code into infinite Gray code. Moreover, he showed that infinite Gray code can effectively be converted into signed digit code, but the program needs to have some non-deterministic features (see also [TS05]). Berger and Tsuiki [BT21, BT?] reproved the result in a system of formal first-order intuitionistic logic (IFP) extended by inductive and co-inductive definitions, as well as some new logical connectives capturing possibly non-terminating and concurrent behaviour. The programs extracted from the proofs are exactly the ones given by Tsuiki. The main aim of the present paper is to do something similar for the non-empty compact subsets of the reals.

We restrict our consideration to the real interval $\mathbf{I\!I} \stackrel{\mathrm{Def}}{=} [-1, 1]$. Tsuiki's *infinite Gray code* for real numbers encodes a real number $x \in \mathbf{I\!I}$ by the itinerary of x along the *tent map*

$$\mathbf{t}\colon \mathbf{I\!I} \to \mathbf{I\!I}, \mathbf{t}(x) \stackrel{\mathrm{Def}}{=} 1 - 2|x|.$$

More precisely, x is encoded by the stream $a_0 : a_1 : a_2 : \ldots$, where the head of the stream, a_0, equals 0, 1 or ? (= undefined) depending on whether x is less, greater, or equal to 0, and the tail of the stream, $a_1 : a_2 : \ldots$, encodes $\mathbf{t}(x)$. Since $\mathbf{t}(0) = 1$ and $\mathbf{t}(1) = \mathbf{t}(-1) = -1$, at most one a_i can be undefined, and in that case $a_{i+1} = 1$ and $a_k = 0$ for all $k > i + 1$.

A *signed digit representation* of $x \in \mathbf{I\!I}$ is any stream $d_0 : d_1 : d_2 : \ldots$ of signed digits $d_i \in \mathbf{SD} \stackrel{\mathrm{Def}}{=} \{-1, 0, 1\}$ such that $x \in \mathbf{I\!I}_{d_0} \stackrel{\mathrm{Def}}{=} [d_0/2 - 1/2, d_0/2 + 1/2]$ and $d_1 : d_2 : \ldots$ is a signed digit representation of $2x - d_0$.

Let $\mathbf{S}$ and $\mathbf{G}$, respectively, be the coinductively largest subset of $\mathbf{I\!I}$ so that

$$\mathbf{S}(x) \to (\exists d \in \mathbf{SD})\, x \in \mathbf{I\!I}_d \wedge \mathbf{S}(2x - d)$$

and

$$\mathbf{G}(x) \to (x \neq 0 \to x \leq 0 \vee x \geq 0) \wedge \mathbf{G}(\mathbf{t}(x)).$$

By interpreting existential quantifiers and disjunctions constructively it follows that the signed digit representations of $x \in \mathbf{I\!I}$ are exactly the realisers of $\mathbf{S}(x)$. Similarly, the infinite Gray code of x realises $\mathbf{G}(x)$, and vice versa. Berger and Tsuiki [BT21] show in IFP that $\mathbf{S} \subseteq \mathbf{G}$. For the converse inclusion the logic needs to be extended by an additional logical connective $\|$, called *restriction*, and a concurrency modality $\downdownarrows$. The extension thus obtained is called *Concurrent Fixed Point Logic (CFP)*.

Let $\mathbf{S}_2$ be the coinductively largest subset of $\mathbf{I\!I}$ such that

$$\mathbf{S}_2(x) \to \downdownarrows((\exists d \in \mathbf{SD})\, x \in \mathbf{I\!I}_d \wedge \mathbf{S}_2(2x - d)).$$

Then Berger and Tsuiki [BT?] show in CFP that $\mathbf{G} \subseteq \mathbf{S}_2$.

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In [BS16], the present authors studied computability on the hyperspace $\mathcal{K}(X)$ of the non-empty compact subsets of an IFS (X, D) where X is a compact metric space. The hyperspace is endowed with the Hausdorff metric. A coinductive characterisation of the hyperspace was given and it was shown that the realisers obtained from the definition are the finitely branching labelled trees with only infinite paths and elements of D as labels

For $d \in \mathbf{SD}$, let $\mathbf{av}_d(x) \stackrel{\text{Def}}{=} (x-d)/2$. Moreover, let $\mathbf{S_K}$ be the coinductively largest subset of $\mathcal{K}(\mathbf{I\!I})$ such that

$$\mathbf{S_K}(K) \to (\exists E \in \mathbf{P_{fin}}(\mathbf{SD}))K \subseteq \bigcup_{d\in E} \mathbf{I\!I}_d \wedge (\forall d \in E)(K \cap \mathbf{I\!I}_d \neq \emptyset \wedge \mathbf{S_K}(\mathbf{av}_d^{-1}[K])).$$

Then $\mathbf{S_K} = \mathcal{K}(\mathbf{I\!I})$, classically.

A central requirement in [BS16] is that the IFS under investigation is *well-covering*, that is $X = \bigcup_{e\in D} \operatorname{int}(\operatorname{range}(e))$. In the case of $(\mathbf{I\!I}, \{\, \mathbf{av}_d \mid d \in \mathbf{SD}\,\})$ this condition is obviously satisfied. There is, however, no such IFS generating infinite Gray code. Thus, from [BS16] no recipe is obtained for a coinductive characterisation of $\mathcal{K}(\mathbf{I\!I})$ so that the realisers derived from the characterisation give us a redundancy-free representation of $\mathcal{K}(\mathbf{I\!I})$.

Let $\mathbf{GC} \stackrel{\text{Def}}{=} \{-1, 1\}$ and $\mathbf{G_K}$ be the largest subset of $\mathcal{K}(\mathbf{I\!I})$ such that

$$\mathbf{G_K}(K) \to \mathbf{G}(\min K) \wedge \mathbf{G}(\max K) \wedge (\forall d \in \mathbf{GC})\,(K \cap \mathbf{I\!I}_d \neq \emptyset \to \mathbf{G_K}(\mathbf{t}[K \cap \mathbf{I\!I}_d])).$$

Proposition 1. $\mathbf{S_K} \subseteq \mathbf{G_K}$ is derivable in IFP.

In order to derive the converse, again a concurrent version of the predicate $\mathbf{S_K}$ is needed. Note that in general, the concurrency modality $\downdownarrows$ is not a monad. We now have to turn it into a monad by considering its finite iterative closure $\overset{*}{\downdownarrows}$, that is, $\overset{*}{\downdownarrows}(A) \stackrel{\mu}{=} \downdownarrows(A \vee \overset{*}{\downdownarrows}(A))$.

Let $\mathbf{S^*_K}$ be the coinductively largest subset of $\mathcal{K}(\mathbf{I\!I})$ such that

$$\mathbf{S^*_K}(K) \to \overset{*}{\downdownarrows}((\exists E \in \mathbf{P_{fin}}(\mathbf{SD}))K \subseteq \bigcup_{d\in E} \mathbf{I\!I}_d \wedge (\forall d \in E)(K \cap \mathbf{I\!I}_d \neq \emptyset \wedge \mathbf{S^*_K}(\mathbf{av}_d^{-1}[K]))).$$

Proposition 2. $\mathbf{G_K} \subseteq \mathbf{S^*_K}$ is derivable in CFP.

So, we have that $\mathbf{S_K} \subseteq \mathbf{G_K} \subseteq \mathbf{S^*_K}$, which is quite unsatisfying as we want to compare the computational strength of both representations. Let $\mathbf{G}^*$ be the largest subset of $\mathbf{I\!I}$ so that

$$\mathbf{G}^*(x) \to (x \neq 0 \to \overset{*}{\downdownarrows}(x \leq 0 \vee x \geq 0)) \wedge \mathbf{G}^*(\mathbf{t}(x))$$

and $\mathbf{G^*_K}$ be the largest subset of $\mathcal{K}(\mathbf{I\!I})$ so that

$$\mathbf{G^*_K}(K) \to \mathbf{G}^*(\min K) \wedge \mathbf{G}^*(\max K) \wedge (\forall d \in \mathbf{GC})\,(K \cap \mathbf{I\!I}_d \neq \emptyset \to \mathbf{G^*_K}(\mathbf{t}[K \cap \mathbf{I\!I}_d])).$$

Theorem 1. $\mathbf{S^*_K} = \mathbf{G^*_K}$ is derivable in CFP.

References

[BS16] U. Berger, D. Spreen. A coinductive approach to computing with compact sets, *J. Logic & Analysis* 8 (3) (2016) 1–35; doi: 10.4115/jla.2016.8.3.

[BT21] U. Berger, H. Tsuiki, Intuitionistic fixed point logic, *Annals Pure Applied Logic* 172 (3) (2021), doi.org/10.1016/j.apal.2020.102903.

[BT?] U. Berger, H. Tsuiki, Extracting total Amb programs from proofs, submitted for publication, arXiv:2104.14669.

[Ts02] H. Tsuiki, Real number computation through Gray code embedding, *Theoretical Computer Science* 284(2) (2002) 467–485.

[TS05] H. Tsuiki, K. Sugihara, Streams with a bottom in functional languages, in M. Sagiv, editor, *ESOP 2005*, Lec. Notes Comput. Sci., vol. 3444, Springer-Verlag, Berlin, 2005, pages 201–216.