LAWVERE-TIERNEY TOPOLOGIES FOR COMPUTABILITY THEORISTS

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Abstract. In this talk, we study the lattice of Lawvere-Tierney topologies on Hyland's effective topos. For this purpose, we introduce a new computability-theoretic reducibility notion, which is a common extension of the notions of Turing reducibility and generalized Weihrauch reducibility. Based on the work by Lee and van Oosten, we utilize this reducibility notion for providing a concrete description of the lattice of the Lawvere-Tierney topologies on the effective topos. As an application, we solve several open problems proposed by Lee and van Oosten. For instance, we show that there exists no minimal Lawvere-Tierney topology which is strictly above the identity topology on the effective topos.

1. Introduction

1.1. **Summary.** Our goal in this talk is to accomplish a detailed analysis of the entire structure of "intermediate worlds" between "the world of computable mathematics" and "the world of set-theoretic mathematics." In [4], Hyland discovered the *effective topos* **Eff**, and proposed it as the *world of computable mathematics*. In topos theory, there is a notion called a *Lawvere-Tierney topology* (also known as a local operator or a geometric modality), and any topology j on a topos $\mathcal{E}$ yields a new subtopos $\mathcal{E}_j \hookrightarrow \mathcal{E}$ (see e.g. [7]). Indeed, Lawvere-Tierney topologies on $\mathcal{E}$ are in one-to-one correspondence with subtoposes of $\mathcal{E}$. The least topology is the identity topology `Id` that does not cause any change to the base topos. The largest topology is the topology that contracts all truth-values to a single value, and the resulting degenerated topos may be thought of as the *world of inconsistent mathematics*. The next largest topology is the double negation $\neg\neg$. In the effective topos, the new topos $\mathbf{Eff}_{\neg\neg}$ created from $\neg\neg$ is exactly the *world of set-theoretic mathematics*; that is, $\mathbf{Eff}_{\neg\neg} \simeq \mathbf{Set}$. What this suggests is that analyzing the intermediate topologies between `Id` and $\neg\neg$ on the effective topos may correspond to exploring the intermediate worlds between computable mathematics and set-theoretic mathematics.

Under this perspective, a topology on the effective topos is a kind of data that indicate how much non-computability to add to the world. In other words, a topology plays the same role as an *oracle*. Indeed, Hyland [4] noticed that each Turing degree $\mathbf{d}$ has a corresponding topology $j_{\mathbf{d}}$ on the effective topos, which yields the *world of* $\mathbf{d}$*-relatively computable mathematics*. However, this does not mean that we have exhausted all the

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topologies, and of course, there may be other topologies besides them. For instance, instead of a subset of $\mathbb{N}$ or a total function on $\mathbb{N}$, one can use a partial function as an oracle. Not only that, but even a partial multi-valued function can be used as an oracle, and has a corresponding topology on the effective topos. As another example, Pitts [8] found an intermediate topology that is not bounded by any Turing degree topology. This topology has properties that are far from any of the other topologies mentioned above. Remarkably, Lee-van Oosten [6] gave a concrete presentation of all topologies on the effective topos.

The first step of our work in this article is to capture the presentation of Lee-van Oosten [6] within the framework of *generalized Weihrauch reducibility* [3]. However, generalized Weihrauch reducibility (which involves a perfect information game) itself is insufficient to deal with all topologies, so we introduce an imperfect information game that incorporates some sort of nonuniform computation with advices. Coincidentally, it turns out that our notion is heavily related to another notion called *extended Weihrauch reducibility*, which is introduced in Bauer [1]. By viewing topologies in this way, it is possible, for example, to position the study of the structure of Lawvere-Tierney topologies on (equivalently, that of subtoposes of) a relative realizability topos as an extension of the Weihrauch-style analogue [2] of reverse mathematics.

By bringing the arguments on topologies/subtoposes into pure computability theory in this way, we solve some problems proposed in [5, 6]. For instance, we show that there exists no minimal Lawvere-Tierney topology which is strictly above the identity topology on the effective topos. Thus, in a certain sense, there is no world of non-computable mathematics which is closest to computable mathematics. We also discuss a few other topologies, which has not been studied in the past. One corresponds to the world of computable mathematics with error probability ε, and the other to computable mathematics with error density ε (in the sense of lower asymptotic density).

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