CONTINUITY FOR COMPUTABILITY

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Abstract. The domain theoretic notion of continuity requires actual infinite sets, whereas computation cannot deal with actual infinities. We will present a notion of continuity based on potential infinite sets.

§1. Introduction. Continuity needs infinite structures, and computability requires that infinity is read as potential infinity, considered as an unbounded, increasing finite. This consideration has the following consequences for infinite totalities:

1. There is no naive universal quantification over all elements.
2. There is no naive function application taking all elements as its domain.

In other words, universal quantification has to range over a finite set, similarly function application[1].

Let us look at domain theory. A domain is a special dcpo (directed complete partial order), i.e., every directed subset has a supremum. However, every *finite* directed subset automatically has a supremum, namely its greatest element. Thus, if the infinite is an indefinitely increasing finite, then directed completeness is tautological.

§2. Systems and Limits of Systems. To model a finite, dynamic set we use systems (as generalizations of a direct or inverse systems) and limits thereof. A system is a family $(\mathcal{M}_i)_{i\in\mathcal{I}}$ with directed index set $\mathcal{I}$ of approximation states, and finite sets $\mathcal{M}_i$. The objects in the different states $\mathcal{M}_{i'}$ and $\mathcal{M}_i$ for $i' \geq i$ are connected by a relation $a_{i'} \overset{p}{\mapsto} a_i$ if $a_i \in \mathcal{M}_i$ and $a_{i'} \in \mathcal{M}_{i'}$ with the intention that a_i is a *predecessor* of $a_{i'}$.

The key point is that the limits of a system are not necessarily absolute, outside the system and actual infinite. They can be regarded as relative, inside the system and finite. The stage $i \in \mathcal{I}$ of the limit set $\mathcal{M}_i$ depends on the context and outer factors to which the system is related. For instance, if the system is a model of a theory, the stage i depends on how many (finitely many) expressions of the potential infinite set of expressions are considered. In order that i can be regarded as indefinitely large, i.e., as index for a limit, we introduce the notion $C \ll i$, stating that i is sufficiently large, or indefinitely large, relative to an approximation context $C = (i_0, \ldots, i_{n-1})$.

[1] The point is that universal quantification and function application cannot be taken at face value. However, its naive use can be seen as an abbreviation for a reference to a sufficiently large finite set (see below).

The limit structure $\mathcal{M}$ is naturally endowed with a family $\approx_{\mathcal{I}} := (\approx_i)_{i\in\mathcal{I}}$ of PERs (partial equivalence relations), which we call PER-set. The PERs define increasingly finer versions of equality. We require a kind of density, more precisely, the existence of a filter $\mathfrak{D}$ on $\mathcal{I}$ such that $\{i \in \mathcal{I} \mid a \in [i]\} \in \mathfrak{D}$ for all $a \in \mathcal{M}$, whereby $[i] := \{a \in \mathcal{M} \mid a \approx_i a\}$.

§3. Continuity on PER-sets. Continuity occurs on the level of the limit sets, i.e., on PER-sets. Given PER-sets $(\mathcal{M}, \approx_{\mathcal{I}})$ and $(\mathcal{N}, \approx_{\mathcal{J}})$, then the equivalence relation of the function space $\approx_{\mathcal{I}\times\mathcal{J}}$ is defined as:

$$f \approx_{i\to j} g :\Longleftrightarrow a \approx_i b \text{ implies } f(a) \approx_j g(b) \text{ for all } a, b \in \mathcal{M}.$$

A function $f : \mathcal{M} \to \mathcal{N}$ is *i-j-continuous* iff $f \in [i \to j]$. A function $f : \mathcal{M} \to \mathcal{N}$ is *$\mathfrak{D}$-continuous* iff there are $\mathfrak{D}$-many indices $i \to j$ such that f is i-j-continuous. This notion of $\mathfrak{D}$-continuity is similar, but not equivalent to the notion of uniform continuity, it is not a topological notion and it depends on a filter $\mathfrak{D}$.

Systems together with PER-sets provide a potential infinite model for simple type theory (STT). Its relation to the hereditary total functionals is however an open question.

§4. Logic. This model is able to handle logic. Consider a STT with base types ι, including type *bool* with $[\![bool]\!] = \{true, false\}$, and the function space construction. The *positive* and *negative* types are defined as

$$\varrho^+ ::= \iota \,|\, (\varrho^- \to \varrho^+) \quad \text{and} \quad \varrho^- ::= bool \,|\, (\varrho^+ \to \varrho^-).$$

Positive types correspond to *objects* and to a direct limit construction, whereas negative types correspond to *properties* and an inverse limit construction. Let $[\![i]\!]$ denote the finite set at i (named $\mathcal{M}_i$ so far). We can define an interpretation of λ-terms on a fragment of STT (basically a restriction to positive and negative types), which however contains classical higher-order logic. Based on an approximation declaration $C \mid \mathsf{r} : i$, stating that within an approximation context C the term r has an approximation i, the universal quantifier can be defined in a finitistic way as follows:

$$[\![\forall_\varrho \mathsf{r}]\!]_{\boldsymbol{a}:C} := \bigwedge_{b\in[\![i]\!]} [\![\mathsf{r}]\!]^{i\to bool}_{\boldsymbol{a}:C}(b) \qquad \text{for } C \mid \mathsf{r} : i \to bool \text{ and } C \ll i.$$

Seeing infinity as an indefinitely large finite is a project presented in [1].

REFERENCES

[1] Matthias Eberl, *Infinity is not a size*, ***The Logica Yearbook 2020***, (2021), pp. 33–48.