From Daniell spaces to the integration spaces of Bishop and Cheng

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The most popular approach to classical measure and integration theory is to define integration through measure. Starting from a measure space $(X, \mathcal{A}, \mu)$, one defines simple and measurable functions (the latter through the Borel sets in $\mathbb{R}$). As a positive measurable function is the limit of an increasing sequence of positive, simple functions, the obviously defined integral of a simple function is extended to the integral of a positive, measurable function. The integral of a measurable function $f\colon X \to \mathbb{R}$ is then defined through the integrals of the positive, measurable functions f_+ and f_-. This approach can be characterised, roughly, as an approach "from sets to functions".

A less popular approach to classical measure and integration theory is to define measure through integration. It was introduced by Daniell [5], and it was taken further by Weil [12], Kolmogoroff [6], Carathéodory [4], and Segal [9], [10]. The starting point of this approach is the notion of a Daniell space $\big(X, L, \int\big)$, where L is a Riesz space of real-valued functions on X and $\int\colon L \to \mathbb{R}$ is a positive, linear functional that satisfies the Daniell property, a certain continuity condition. Using the Bolzano-Weierstrass theorem, one extends L to L^+, the set of functions $f\colon X \to \overline{\mathbb{R}}$ that are limits of increasing sequences in L, and $\int$ to $\int^+\colon L^+ \to \overline{\mathbb{R}}$ accordingly. The upper $\overline{\int} f$ and lower integral $\underline{\int} f$ of a function $f\colon X \to \overline{\mathbb{R}}$ are defined through the completeness axiom of real numbers, and f is integrable, or an element of L^1, if $\underline{\int} f = \int^+ f$. A function $f\colon X \to [0, +\infty]$ is called measurable, if it can be approximated by integrable functions, namely $\forall_{g \in L^1}\big(f \wedge g \in L^1\big)$. A subset A of X is measurable, if its characteristic function χ_A is measurable, while A is integrable, if $\chi_A \in L^1$. If A is measurable, a measure function $A \mapsto \mu(A)$ is defined through the the integral of χ_A, in case A is integrable. The so-called Daniell approach, presented e.g., in [7], [11], can be characterised, roughly, as an approach "from functions to sets".

As functions suit better to constructive study than sets, Bishop followed the Daniell approach both in [1], and, in a different and more uniform way, in [2], [3]. The starting point of a constructive development of the Daniell approach is the notion of an integration space, a constructive formulation of a Daniell space that was introduced by Bishop and Cheng in [2]. An integration space is a structure $\big(X, L, \int\big)$, where X is a Bishop set equipped with an inequality $\neq_X$ and L is a set of strongly extensional, real valued, *partial functions* on X that corresponds to the Riesz space of functions in a Daniell space. Moreover, the linear integral $\int\colon L \to \mathbb{R}$ satisfies three continuity properties, their relation of which to the Daniell property of an integral is not immediate to see.

In this talk we explain why the notion of an integration space of Bishop and Cheng is a natural, constructive counterpart to the classical notion of Daniell space. Firstly, we clarify the use of partial functions in the definition of an integration space. To curry out the aforementioned Daniell transition from functions to sets constructively, one needs to use partial functions from X to 2 that define *complemented subsets* of X. Secondly, we explain the strong extensionality of the elements of L. The characteristic function of a complemented subset of X is strongly extensional and strong extensionality is necessary to the proof of the equality between the proper classes of complemented subsets $\mathcal{P}^{\rrbracket\llbracket}(X)$ of X and of boolean-valued, strongly extensional, partial functions $F^{\mathrm{se}}(X, 2)$ on X. This equality is the constructive counterpart to the classical equipollence between the powerset $\mathcal{P}(X)$ of X and the set $\mathbb{F}(X, 2)$ of boolean-valued, total functions on X. Finally, we describe the relations between the three continuity properties in the definition of an integration space and the Daniell property of an integral.

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