INTERNAL NEIGHBOURHOOD STRUCTURES IV: COMPACT OBJECTS

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ABSTRACT. This paper develops further on internal preneighbourhood spaces [see 3, 5] in presenting the notion of *internal Hausdorff spaces* and the *Hausdorff reflection*. The development depends on the notions of proper morphisms, separated morphisms and separated objects, which are also developed in this paper. The Hausdorff reflection is described in three different ways: firstly as the largest subobject of the binary product whose components are indistinguishable by any internal Hausdorff space valued preneighbourhood morphism, secondly as the smallest effective equivalence relation whose quotient is an internal Hausdorff space and thirdly in admissibly well powered categories by transfinite induction on quotients by the diagonal.

1. **Preliminaries.** An *internal preneighbourhood space* in a context $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ is a pair (X, μ), where X is an object of $\mathbb{A}$ and $\mathsf{Sub}_\mathsf{M}(X)^{\mathrm{op}} \xrightarrow{\mu} \mathtt{Fil}X$ is an order reversing map from the complete lattice $\mathsf{Sub}_\mathsf{M}(X)$ of admissible subobjects of X to the complete lattice $\mathtt{Fil}X$ of filters in the lattice $\mathsf{Sub}_\mathsf{M}(X)$, such that $p \geq m$ whenever $p \in \mu(m)$. If further, for every $p \in \mu(m)$ there exists a $q \in \mu(m)$ with $p \in \mu(q)$ then μ is a *weak neighbourhood system* and (X, μ) is an *internal weak neighbourhood space*. Further if μ preserve arbitrary meets then it is a *neighbourhood system* and (X, μ) is an *internal neighbourhood space*. Given the internal preneighbourhood spaces (X, μ) and (Y, ϕ), a morphism $X \xrightarrow{f} Y$ is a *preneighbourhood morphism* if $p \in \phi(m)$ implies $f^{-1}p \in \mu(f^{-1}m)$, denoted $(X, \mu) \xrightarrow{f} (Y, \phi)$. A preneighbourhood morphism between internal neighbourhood spaces which further satisfies $f^{-1}(\bigvee T) = \bigvee_{t \in T} f^{-1}t$ for every $T \subseteq \mathsf{Sub}_\mathsf{M}(Y)$ is a *neighbourhood morphism*. The internal preneighbourhood spaces along with preneighbourhood morphisms make the category $\mathtt{pNbd}[\mathbb{A}]$ of internal preneighbourhood spaces; $\mathtt{wNbd}[\mathbb{A}]$ is the full subcategory of internal weak neighbourhood spaces and $\mathtt{Nbd}[\mathbb{A}]$ is the subcategory of internal neighbourhood spaces and neighbourhood morphisms. Contexts abound and so does internal preneighbourhood spaces, [see 3, for details].

The forgetful functor $\mathtt{pNbd}[\mathbb{A}] \xrightarrow{U} \mathbb{A}$ is topological [see 3, Theorem 4.8(a)]. Hence every limit (respectively, colimit) object is equipped with the smallest (respectively, largest) preneighbourhood system which makes each component of the limiting cone (respectively, colimiting cone) preneighbourhood morphisms. In particular, the terminal object $\mathbf{1}$ (respectively, initial object $\emptyset$) is provided with the smallest preneighbourhood system $\nabla_\mathbf{1}$ (respectively, largest preneighbourhood system $\uparrow_\emptyset$), where for any object X:

$$\nabla_X(m) = \begin{cases} \mathsf{Sub}_\mathsf{M}(X), & \text{if } m = \sigma_X \\ \{\mathbf{1}_X\}, & \text{if } m \neq \sigma_X \end{cases}, \quad \text{and} \quad \uparrow_X (m) = \big\{ x \in \mathsf{Sub}_\mathsf{M}(X) : m \leq x \big\}, \quad (1)$$

are the smallest and the largest preneighbourhood systems on any object X. The set $\mathtt{pnbd}[X]$ of all preneighbourhood systems on X is a complete lattice [see 3, for details].

Given an internal preneighbourhood space (X, μ), a closure operator $\mathsf{Sub}_\mathsf{M}(X) \xrightarrow{\mathrm{cl}_\mu} \mathsf{Sub}_\mathsf{M}(X)$ is defined by:

$$\mathrm{cl}_\mu x = \bigvee \big\{ p \in \mathsf{Sub}_\mathsf{M}(X)_{\neq 1} : u \in \mu(p) \Rightarrow u \wedge x \neq \sigma_X \big\}, \quad (2)$$

[see 4, §3 for details]. It is shown that cl_μ is grounded, extensional, transitive closure operator, additive if every filter of admissible subobjects is contained in a prime filter [see 4, Theorem 3.1]. An admissible subobject $M \xrightarrow{m} X$ is *closed* if $\mathrm{cl}_\mu m = m$, and $\mathfrak{C}_\mu$ denotes the (possibly large) set of closed admissible subobjects (also called *closed embeddings*) of

(X, μ). Furthermore, the closed embeddings are stable under pullbacks along continuous preneighbourhood morphisms [see 4, Definition 3.1 & Theorem 3.1(f)].

A preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a *closed morphism* if $m \in \mathfrak{C}_\mu$ implies $\exists_f m \in \mathfrak{C}_\phi$ [see 4, Definition 4.1]; evidently, for any $m \in \mathsf{M}$, m is a closed morphism if and only if $m \in \mathfrak{C}_\mu$. Stably closed morphisms are *proper morphisms* [see 4, Definition 6.1] and an internal preneighbourhood space (X, μ) is *compact* if the unique morphism $(X, \mu) \xrightarrow{\mathsf{t}_X} (1, \nabla_1)$ is proper [see 4, Definition 6.2]. The following results are established for proper morphisms and the full subcategory $\mathsf{K}[\mathbb{A}]$ of compact spaces.

(A) [see 4, Theorem 6.1 (a)] A preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a proper morphism if and only if for any internal preneighbourhood space (Z, ψ) every corestriction of $X \times Z \xrightarrow{f \times \mathbf{1}_Z} Y \times Z$ is a closed morphism.

(B) [see 4, Theorem 6.1(b)] The (possibly large) set $\mathbb{A}_{\mathsf{pr}}$ of proper morphisms is a pullback stable set, is closed under compositions and has the following properties:
  (a) If every preneighbourhood morphism is continuous then every closed embedding is a proper morphism.
  (b) If the composite $g \circ f$ is a proper morphism and g is a monomorphism then f is a proper morphism.
  (c) If the composite $g \circ f$ is a proper morphism and f is stably continuous and stably in E then g is a proper morphism.

(C) [see 4, Remark (K)] An internal preneighbourhood space (X, μ) is compact if and only if for each internal preneighbourhood space (Z, ψ) the product projection $X \times Z \xrightarrow{p_2} Z$ is a closed morphism.

(D) [see 4, Theorem 6.2(a)] If $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a proper morphism to a compact preneighbourhood space (Y, ϕ) then (X, μ) is compact.

(E) [see 4, Theorem 6.2(a)] If $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a preneighbourhood morphism with f stably in E and (X, μ) is compact then (Y, ϕ) is compact.

(F) [see 4, Theorem 6.2(c)] If every preneighbourhood morphism is continuous then the full subcategory $\mathsf{K}[\mathbb{A}]$ of compact preneighbourhood spaces is finitely productive and closed hereditary (i.e., if (X, μ) is a compact preneighbourhood space and $m \in \mathfrak{C}_\mu$ with M the domain of m then $\left(M, (\mu|_M)\right)$ is compact, where $(\mu|_M)$ is the smallest preneighbourhood system on M making m a preneighbourhood morphism).

(G) [see 5, Theorem 9.1] In an extensive context in which finite sum of closed morphisms is closed, a finite sum of proper morphisms is proper and $\mathsf{K}[\mathbb{A}]$ is closed under finite sums if and only if for each internal preneighbourhood space (X, μ) both the morphisms $\left(\emptyset, \uparrow_\emptyset\right) \xrightarrow{\mathsf{i}_X} (X, \mu)$ and $(X + X, \mu + \mu) \xrightarrow{[\![\mathbf{1}_X, \mathbf{1}_X]\!]} (X, \mu)$ are both proper.

2. **The Kuratowski-Mrówka Theorem.** Recall: a set A of closed embeddings of an internal preneighbourhood space (X, μ) has *finite meet property* if $\mathbf{1}_X \in A$ and for every natural number $n \geq 1$, $a_1, a_2, \ldots, a_n \in A$, $a_1 \wedge a_2 \wedge \cdots \wedge a_n \neq \sigma_X$; $B \subseteq \mathsf{Sub}_\mathsf{M}(X)$ has *nonzero meet* if $\bigwedge B \neq \sigma_X$. In this section it is shown in a lextensive context $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ with finite product projections in E and admissible subobjects closed under finite sums compactness of an internal preneighbourhood space is equivalent to every set of closed embeddings with finite meet property has nonzero meet. The following lemma encapusulates some computations required for the proof of this statement. The statement of Lemma 2.1(b) is proved in [4], [see 4, Lemma 3.3(c) for details].

Lemma 2.1.   *(a) Given the objects X and Y, in the diagram below*

where all the squares are pullback squares:

$$(v \times u) \wedge a = a{\circ}({a_X}^{-1}v \wedge {a_Y}^{-1}u), \tag{3}$$

whenever $a = (a_X, a_Y) \in \mathsf{Sub}_\mathsf{M}(X \times Y)$.

(b) If (X, μ) and (Y, μ) are internal preneighbourhood spaces in the context $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ with finite product projections in E, $X \times Y \xrightarrow{p_2} Y$ be the product projection to Y and $a = (a_X, a_Y) \in \mathfrak{C}_{\mu \times \phi}$ then for any $y \in \mathsf{Sub}_\mathsf{M}(Y)$:

$$a \wedge {p_2}^{-1}y = \sigma_{X \times Y} \Leftrightarrow y \wedge \exists_{p_2} a = \sigma_Y \Leftrightarrow y \not\leq \mathrm{cl}_\phi \exists_{p_2} a. \tag{4}$$

Theorem 2.2. *In a reflecting zero context $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ with finite product projections in E if in an internal preneighbourhood space every set of closed embeddings with finite meet property has a nonzero meet then the preneighbourhood space is compact.*

Further, if the context is lextensive and admissible subobjects are closed under finite sums then the converse holds too.

Proof. For the first statement, let (X, μ) be an internal preneighbourhood space such that every set of admissible closed subobjects of X with finite meet property has a nonzero meet. Choose and fix an $a \in \mathfrak{C}_{\mu \times \phi}$ and $\sigma_Y \neq y \leq \mathrm{cl}_\phi \exists_{p_2} a$. Consequently using Lemma 2.1(b) the set $A = \left\{ \mathrm{cl}_\mu \exists_{p_1}(a \wedge {p_2}^{-1}u) : u \in \phi(y) \right\}$ is a set of closed subobjects of X with finite meet property. By assumption $x = \bigwedge A \neq \sigma_X$. Since $x = \bigwedge_{u \in \phi(y)} \mathrm{cl}_\mu \exists_{p_1}(a \wedge {p_2}^{-1}u) \geq \mathrm{cl}_\mu\left(\bigwedge_{u \in \phi(y)} \exists_{p_1}(a \wedge {p_2}^{-1}u)\right)$, and the product projections are continuous:

$$
\begin{aligned}
a \wedge {p_1}^{-1}x &\geq a \wedge {p_1}^{-1}\mathrm{cl}_\mu\left(\textstyle\bigwedge_{u \in \phi(y)} \exists_{p_1}(a \wedge {p_2}^{-1}u)\right) \\
&\geq a \wedge \mathrm{cl}_{\mu \times \phi}{p_1}^{-1}\left(\textstyle\bigwedge_{u \in \phi(y)} \exists_{p_1}(a \wedge {p_2}^{-1}u)\right) &&\text{(continuity of } p_1) \\
&= a \wedge \mathrm{cl}_{\mu \times \phi}\textstyle\bigwedge_{u \in \phi(y)}\left({p_1}^{-1}\exists_{p_1}(a \wedge {p_2}^{-1}u)\right) \\
&\geq a \wedge \mathrm{cl}_{\mu \times \phi}(a \wedge {p_2}^{-1}y) \\
&\geq a \wedge {p_2}^{-1}y \neq \sigma_{X \times Y} &&\text{(equation (4))}.
\end{aligned}
$$

Hence, for each $v \in \mu(x)$ and $u \in \phi(y)$:

$$
\begin{aligned}
(v \times u) \wedge a &= a{\circ}({a_X}^{-1}v \wedge {a_Y}^{-1}u) &&\text{(equation (3))} \\
&= a{\circ}\left(a^{-1}(a \wedge {p_1}^{-1}v) \wedge a^{-1}(a \wedge {p_2}^{-1}u)\right) &&\text{(equation (3))} \\
&\geq a{\circ}\left(a^{-1}(a \wedge {p_1}^{-1}x) \wedge a^{-1}(a \wedge {p_2}^{-1}y)\right) \\
&\geq a{\circ}a^{-1}(a \wedge {p_2}^{-1}y) \neq \sigma_{X \times Y}.
\end{aligned}
$$

Hence $x \times y \leq \mathrm{cl}_{\mu \times \phi}a = a$ implying $y \leq \exists_{p_2} a$, using $p_2 \in \mathsf{E}$. Thus: $y \leq \mathrm{cl}_\phi \exists_{p_2} a \Rightarrow y \leq \exists_{p_2} a$ implies $\exists_{p_2} a$ is closed, proving p_2 is a closed morphism.

Towards a proof of the second statement, first some facts regarding an extensive context needs to be recalled [see 5, §2]. In a lextensive context, th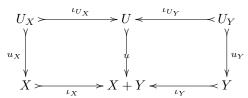e initial object $\emptyset$ is strict [see 1, §2]. Moreover in the coproduct $1 \xrightarrow{\iota_1} 1 + 1 \xleftarrow{\iota_2} 1$ the coproduct injections are split monomorphisms and hence are admissible monomorphisms. For any two objects X and Y, if $X \xrightarrow{\iota_X} X + Y \xleftarrow{\iota_Y} Y$ be their coproduct, then since in the diagram:

both the squares are pullback squares from extensivity, the coproduct injections are always admissible monomorphisms. Therefore every lextensive context is a quasi admissible context and hence every morphism reflects zero, or equivalently every preneighbourhood morphism is continuous [see 4, §9]. Hence, for each object Y, $\emptyset_Y \approx \emptyset$ so that $\mathtt{i}_Y$ is

the smallest admissible subobject of Y. Further, given any $u \in \mathtt{Sub}_\mathsf{M}(X + Y)$ from the diagram:

$$
\begin{array}{ccccc}
U_X & \xrightarrow{\ \iota_{U_X}\ } & U & \xleftarrow{\ \iota_{U_Y}\ } & U_Y \\
\downarrow{\scriptstyle u_X} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u_Y} \\
X & \xrightarrow[\ \iota_X\ ]{} & X + Y & \xleftarrow[\ \iota_Y\ ]{} & Y
\end{array}
$$

where u_X and u_Y are pullbacks of u along the coproduct injections ι_X and ι_Y respectively, extensivity ensures the top row is a coproduct too and hence $u = u_X + u_Y$. Thus, $u \leq \iota_X$ if and only if there exists a unique p such that $u = \iota_X \circ p$ and hence $u_X = \iota_X{}^{-1}u = p$ implying $U_X \approx U$ and hence from the coproduct in the top row, $u_Y = \emptyset$. Hence, if sum of admissible subobjects is an admissible subobject then for each $u_Y \in \mathtt{Sub}_\mathsf{M}(Y)$, $u_X \in \mathtt{Sub}_\mathsf{M}(X)$, if $u_y \neq \mathtt{i}_Y$ then $u_X + u_Y \not\leq \iota_X$.

Now assume (X, μ) is a compact preneighbourhood space in a lextensive context where sum of admissible subobjects is an admissible subobject. Let $A \subseteq \mathfrak{C}_\mu$ be a set of closed subobjects of X with finite meet property, $\widehat{A}$ be the set of all finite meets of elements of A. Evidently, $A \subseteq \widehat{A} \subseteq \mathfrak{C}_\mu$. Choose and fix any object Z of $\mathbb{A}$ and let their coproduct be $X \stackrel{\iota_X}{\rightarrowtail} X + Z \stackrel{\iota_Z}{\leftarrowtail} Z$. Define for each $y \in \mathtt{Sub}_\mathsf{M}(X + Z)$:

$$
\phi(y) = \begin{cases} \uparrow(\iota_X \circ x), & \text{if } (\exists x \in \mathtt{Sub}_\mathsf{M}(X))(y = \iota_X \circ x) \\ (\uparrow y) \cap \overrightarrow{A}, & \text{otherwise} \end{cases}, \tag{5}
$$

where $\overrightarrow{A} = \big\{ u \in \mathtt{Sub}_\mathsf{M}(X + Z) : (\exists k \in \widehat{A})\big(\iota_X{}^{-1}u \geq k\big) \big\} \in \mathtt{Fil}(X + Z)$ is the smallest filter containing each $\iota_X \circ k$, $k \in A$ and $\uparrow y = \big\{ z \in \mathtt{Sub}_\mathsf{M}(X + Z) : z \geq y \big\}$. Evidently (5) defines an internal preneighbourhood system $\mathtt{Sub}_\mathsf{M}(X + Z)^{\mathrm{op}} \xrightarrow{\phi} \mathtt{Fil}(X + Z)$ on $X + Z$. Now, for any $y \in \mathtt{Sub}_\mathsf{M}(X + Z)$, either $y \leq \iota_X$ or else $u \in \phi(y)$ implies $u \geq y$ and there exists a $k \in \widehat{A}$ such that $k \leq \iota_X{}^{-1}u \Leftrightarrow \iota_X \circ k \leq u$, implying $u \wedge \iota_X \geq \iota_X \wedge \iota_X \circ k = \iota_X \circ k \neq \sigma_{X+Z}$. Thus in either case $y \in \mathrm{cl}_\phi \iota_X$. Hence $\mathrm{cl}_\phi \iota_X = \mathbf{1}_{X+Z}$. Let $a = (a_X, a_{X+Z}) = \mathrm{cl}_{\mu \times \phi}(\mathbf{1}_X, \iota_X) \in \mathfrak{C}_{\mu \times \phi}$; since the projection $X \times (X + Z) \xrightarrow{p_2} X + Z$ is a closed morphism, $\exists_{p_2} a \in \mathfrak{C}_\phi$. On the other hand since p_2 is a closed morphism and is continuous, $\exists_{p_2} a = \exists_{p_2} \mathrm{cl}_{\mu \times \phi}(\mathbf{1}_X, \iota_X) = \mathrm{cl}_\phi \exists_{p_2}(\mathbf{1}_X, \iota_X) = \mathrm{cl}_\phi \iota_X = \mathbf{1}_{X+Z}$. Hence $a_{X+Z} \in \mathsf{E}$. Since $(\mathbf{1}_X, \iota_X) \leq a$, there exists a morphism p such that $a_X \circ p = \mathbf{1}_X$ and $a_Y \circ p = \iota_X$. Hence a_X is a split epimorphism, i.e., $a_X \in \mathsf{E}$, as a consequence of which $\exists_{p_1} a = \mathbf{1}_X$. Now choose and fix a $y \in \mathtt{Sub}_\mathsf{M}(X + Z)$ such that $\iota_X{}^{-1}y \neq \sigma_X$ and $y \not\leq \iota_X$ — the conditions on the context ensure the existence of such an admissible subobject. For such a y, firstly: $v \in \mu(\iota_X{}^{-1}y)$, $u \in \phi(y)$ implies from (3), $(v \times u) \wedge (\mathbf{1}_X, \iota_X) \geq \big((\iota_X{}^{-1}y) \times y\big) \wedge (\mathbf{1}_X, \iota_X) = (\mathbf{1}_X, \iota_X) \circ (\iota_X{}^{-1}y) = \big(\iota_X{}^{-1}y, \iota_X \circ (\iota_X{}^{-1}y)\big) \neq \sigma_{X \times (X+Z)}$, and hence $(\iota_X{}^{-1}y) \times y \leq a$. Since $y \not\leq \iota_X$, for each $k \in A$, $v \in \mu(\iota_X{}^{-1}y)$, $(v \times (\iota_X \circ k)) \wedge (\mathbf{1}_X, \iota_X) = (\mathbf{1}_X, \iota_X) \circ (v \wedge k) \neq \sigma_{X \times (X+Z)}$ implying $v \wedge k \neq \sigma_X$, i.e., $\iota_X{}^{-1}y \leq \mathrm{cl}_\mu k = k$. Hence $\iota_X{}^{-1}y \leq \bigwedge A$, i.e., A has a nonzero meet. A concrete choice for such a y is $y = \mathbf{1}_X + u$ for any $u \in \mathtt{Sub}_\mathsf{M}(Z)$, $u \neq \sigma_Z$, completing the proof. $\square$

Remark 2.3. In particular, in the proof of Theorem 2.7, one could consider $Z = \mathbf{1}$. The construction of the preneighbourhood system ϕ in (5) compares with the ones in topology or locales [see 2, Theorem 3.4], [see 6, Proposition VII.3.5].

Remark 2.4. Analysing the proof of Theorem 2.2, compactness of an internal preneighbourhood space implies every set of closed embeddings with finite meet property has a nonzero meet holds in any lextensive context in which the sum of admissible subobjects is an admissible subobject. The restriction on finite product projections is not required.

2.1. **In presence of complements.** Recall: a *pseudocomplement* x^* of an element x in a complete lattice L is the largest element disjoint from x, i.e., $x^* = \bigvee \{ y \in L : x \wedge y = 0 \}$ and $x \wedge x^* = 0$. A complement of x is an element x' such that $x \wedge x' = 0$ and $x \vee x' = 1$. In general, a complement need not be unique and $x' \leq x^*$. As for example, in the complete lattice of all subspaces of the real vector space $\mathbb{R}^2$, any two non-collinear lines passing through the origin are complements of each other, although the psudocomplement of any such line is $\mathbb{R}^2$. However, in a distributive lattice, a complement is the pseudocomplement, and hence unique. Furthermore, it is easy to see: if each element of L is pseudocomplemented then

$\left(\bigvee_{i\in I} x_i\right)^* = \bigwedge_{i\in I} x_i^*$. A lattice in which every element has a pseudocomplement is called a *pseudocomplemented lattice*.

Lemma 2.5. *For any internal preneighbourhood space (X,μ), if $p \in \mathfrak{C}_\mu$ (respectively, $p \in \mathfrak{D}_\mu$) has a pseudocomplement p^* in the lattice $\mathtt{Sub_M}(X)$, then $p^* \in \mathfrak{D}_\mu$ (respectively, $p^* \in \mathfrak{C}_\mu$).*

Proof. Assume $p \in \mathtt{Sub_M}(X)$ has a pseudocomplement p^* in $\mathtt{Sub_M}(X)$. Then
$$m \not\leq \mathrm{cl}_\mu p \Leftrightarrow (\exists u \in \mu(m))(u \wedge p = \sigma_X)$$
$$\Leftrightarrow (\exists u \in \mu(m))(u \leq p^*)$$
$$\Leftrightarrow p^* \in \mu(m).$$
Hence, if $p \in \mathfrak{C}_\mu$ then $m \not\leq p \Leftrightarrow p^* \in \mu(m)$. If $p^* \neq \sigma_X$ then $p^* \not\leq p$, and hence $p^* \in \mu(p^*)$, i.e., $p^* \in \mathfrak{D}_\mu$. On the other hand, if $p \in \mathfrak{D}_\mu$, then $m \not\leq \mathrm{cl}_\mu p^* \Leftrightarrow p^{**} \in \mu(m)$. Thus, for any $\sigma_X \neq m \leq p$, $p^{**} \in \mu(m)$, and such an $m \not\leq \mathrm{cl}_\mu p^*$. Consequently, $p \wedge \mathrm{cl}_\mu p^* = \sigma_X$. Hence $\mathrm{cl}_\mu p^* \leq p^*$, i.e., $p^* \in \mathfrak{C}_\mu$. $\square$

Proposition 2.6. *Let (X,μ) be an internal preneighbourhood space such that the lattice $\mathtt{Sub_M}(X)$ is a distributive pseudocomplemented complete lattice such that each $p \in \mathfrak{C}_\mu$ (respectively, $p \in \mathfrak{D}_\mu$) is complemented and for any $A \subseteq \mathfrak{C}_\mu$:*
$$\left(\bigwedge A\right)^* = \bigvee\{a^* : a \in A\}. \tag{6}$$
Then, for every $B \subseteq \mathfrak{D}_\mu$ with $\bigvee B = \mathbf{1}_X$ there exists a finite subset $C \subseteq B$ with $\bigvee C = \mathbf{1}_X$ if and only if every set of closed embeddings with finite meet property has a nonzero meet.

Proof. Towards the proof of *if* part, let $B \subseteq \mathfrak{D}_\mu$ with $\bigvee B = \mathbf{1}_X$. Without any loss of generality assume $\sigma_X \notin B$. If B has no finite subcover then since for $u, v \in B$, $u^* \wedge v^* \geq (u \vee v)^*$, using Lemma 2.5, $A = \{b^* : b \in B\}$ is a set of closed embeddings with finite meet property. Hence $\sigma_X \neq \bigwedge\{b^* : b \in B\} = (\bigvee B)^* = \mathbf{1}_X{}^* = \sigma_X$, a contradiction. Hence B must have a finite subcover. For the *only if* part, if A be a set of closed embeddings then $B = \{a^* : a \in A\} \subseteq \mathfrak{D}_\mu$ using Lemma 2.5. If $\bigwedge A = \sigma_X$ then from equation (6), $\bigvee B = \mathbf{1}_X$ and hence there exists a finite subcover, implying by (6) the existence of a finite subset of A with a zero meet. Hence, if A has finite meet property then it has nonzero meet. $\square$

As an immediate consequence of Proposition 2.6 and Theorem 2.7:

Theorem 2.7 (Kuratowski-Mrówka Theorem)**.** *In any lextensive context with finite product projections in E and admissible subobjects closed under finite sums, for any internal preneighbourhood space (X,μ) such that each $p \in \mathfrak{C}_\mu$ (respectively, $p \in \mathfrak{D}_\mu$) is complemented and for any $A \subseteq \mathfrak{C}_\mu$ the condition in (6) is true the following three statements are equivalent:*

(a) *(X,μ) is compact.*
(b) *Every set of closed embeddings with finite meet property has a nonzero meet.*
(c) *Every $\mathfrak{D}_\mu$-open cover of $\mathbf{1}_X$ has a finite subcover.*

3. On Vermeulen's Theorem on proper maps.

If $(X,\mu) \xrightarrow{f} (Y,\phi)$ is a preneighbourhood morphism, then given the (E,M)-factorisation $X \xrightarrow{f^\mathsf{E}} \mathbf{I}_f \xrightarrow{f^\mathsf{M}} Y$ of f in the context $\mathcal{A}$ there is the factorisation $(X,\mu) \xrightarrow{f^\mathsf{E}} \left(\mathbf{I}_f, (\phi|_{\mathbf{I}_f})\right) \xrightarrow{f^\mathsf{M}} (Y,\phi)$ of the preneighbourhood morphism f in $\mathtt{pNbd}[\mathbb{A}]$. The topologicity of the forgetful functor $\mathtt{pNbd}[\mathbb{A}] \xrightarrow{U} \mathbb{A}$ [see 3, Theorem 4.8(a)] ensures f^E is an epimorphism and f^M a monomorphism in $\mathtt{pNbd}[\mathbb{A}]$. The category $(\mathtt{pNbd}[\mathbb{A}] \downarrow (Y,\phi))$ has objects (X,μ,f), where X is an object of $\mathbb{A}$, $\mu \supseteq \overleftarrow{f}\phi(\exists_f)$ is a preneighbourhood system on X, $(X,\mu) \xrightarrow{f} (Y,\phi)$ is a preneighbourhood morphism, and its morphisms are $(X,\mu,f) \xrightarrow{h} (Z,\psi,g)$, where $(X,\mu) \xrightarrow{h} (Z,\psi)$ is a preneighbourhood morphism such that $f = g \circ h$. Thus, the pair:
$$\mathcal{E} = \left\{(X,\mu,f) \xrightarrow{e} (Z,\psi,g) : e \in \mathsf{E}, \mu \supseteq \overleftarrow{e}\psi(\exists_e)\right\},$$
$$\mathcal{M} = \left\{(X,\mu,f) \xrightarrow{m} (Z,\psi,g) : m \in \mathsf{M}, \mu = \overleftarrow{m}\psi(\exists_m) = (\psi|_X)\right\} \tag{7}$$
provides a proper $(\mathcal{E},\mathcal{M})$-factorisation system on $(\mathtt{pNbd}[\mathbb{A}] \downarrow (Y,\phi))$, $\mathtt{Sub}_\mathcal{M}((X,\mu,f))$ is a complete lattice and $(\mathcal{A} \downarrow (Y,\phi)) = ((\mathtt{pNbd}[\mathbb{A}] \downarrow (Y,\phi)), \mathcal{E}, \mathcal{M})$ is a context. An internal preneighbourhood system on the object (X,μ,f) is $\mathtt{Sub}_\mathcal{M}((X,\mu,f))^{\mathrm{op}} \xrightarrow{\mu'} \mathtt{Fil}(X,\mu,f)$

an order preserving map such that $p \in \mu'(m) \Rightarrow p \geq m$. Essentially this is given by a preneighbourhood system μ' on X such that $\mu' \vee \overleftarrow{f}\phi(\exists_f) \neq 1$. An internal preneighbourhood space of $(\mathcal{A} \downarrow (Y, \phi))$ is denoted by (X, μ, f, μ'), and a preneighbourhood morphism $(X, \mu, f, \mu') \xrightarrow{h} (Z, \psi, g, \psi')$ is a morphism $X \xrightarrow{h} Z$ of $\mathbb{A}$ such that $f = g \circ h$ and $(X, \mu) \xrightarrow{h} (Z, \psi)$, $(X, \mu') \xrightarrow{h} (Z, \psi')$ are both preneighbourhood morphisms. Hence the complete lattice $\mathtt{pnbd}[(X, \mu, f)]$ of all internal preneighbourhood systems on (X, μ, f) has $\overleftarrow{f}\phi(\exists_f)$ as its smallest element, i.e., $\nabla_{(X, \mu, f)} = \overleftarrow{f}\phi(\exists_f)$.

Lemma 3.1. *A preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ is proper if and only if the internal preneighbourhood space (X, μ, f, μ) in the context $(\mathcal{A} \downarrow (Y, \phi))$ is compact.*

Proof. The preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ is proper if and only if for every preneighbourhood morphism $(Z, \psi) \xrightarrow{g} (Y, \phi)$ the pullback $(X \times_Y Z, \mu \times_\phi \psi) \xrightarrow{f_g} (Z, \psi)$ of f along g is a closed morphism. Since in $(\mathtt{pNbd}[\mathbb{A}] \downarrow (Y, \phi))$, $(X, \mu, f) \times (Z, \psi, g) = (X \times_Y Z, \mu \times_\phi \psi, g \circ f_g)$ with projections $(X \times_Y Z, \mu \times_\phi \psi) \xrightarrow{f_g} (Z, \psi)$, $(X \times_Y Z, \mu \times_\phi \psi) \xrightarrow{g_f} (X, \mu)$ the pullbacks of f along g, g along f respectively, the first statement is equivalent to the projection $(X \times_Y Z, \mu \times_\phi \psi, g \circ f_g, \mu \times_\phi \psi) \xrightarrow{f_g} (Z, \psi, g, \psi)$ is closed in the context $(\mathcal{A} \downarrow (Y, \phi))$, and hence equivalent to compactness of (X, μ, f, μ) in $(\mathcal{A} \downarrow (Y, \phi))$. $\square$

Theorem 3.2. *In a lextensive context with finite product projections in E a preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ with f stably in E is proper if and only if the order preserving map $\mathfrak{C}_\mu \xrightarrow{\exists_f} \mathfrak{C}_\phi$ preserves filtered meets.*

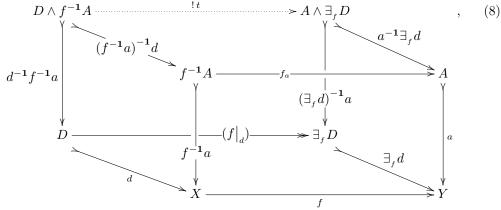*Proof.* Firstly, for an internal preneighbourhood space (X, μ) with $(X, \mu) \xrightarrow{\mathsf{t}_X} (1, \nabla_1)$ closed, the order preserving map $\mathfrak{C}_\mu \xrightarrow{\exists_{\mathsf{t}_X}} \mathfrak{C}_{\nabla_1}$ preserve filtered meets if and only if every filtered set of closed embeddings of (X, μ) has a zero meet if and only if σ_X is a member of the filtered set.

Let $(X, \mu) \xrightarrow{f} (Y, \phi)$ be a preneighbourhood morphism with f stably in E. Using Lemma 3.1, the preneighbourhood morphism f is proper if and only if (X, μ, f, μ) is compact in the context $(\mathcal{A} \downarrow (Y, \phi))$. Since every lextensive category is locally lextensive [see 1, Proposition 4.8], from Theorem 2.2, (X, μ, f, μ) is compact in $(\mathcal{A} \downarrow (Y, \phi))$ if and only if every set of closed embeddings of (X, μ, f, μ) with finite intersection property has a non-zero meet. Since closed embeddings of (X, μ, f, μ) are none else than elements of $\mathfrak{C}_\mu$, $\nabla_{(Y, \phi, \mathbf{1}_Y)} = \phi$ and $\mathsf{t}_{(X, \mu, f)} = f$, it follows that (X, μ, f, μ) is compact in $(\mathcal{A} \downarrow (Y, \phi))$ if and only if $\exists_f$ preserves filtered meet. This completes the proof. $\square$

Lemma 3.3. *Assume $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a closed preneighbourhood morphism in a reflecting zero context, $a \in \mathsf{Sub}_{\mathsf{M}}(Y)$ and $d \in \mathfrak{C}_\mu$.*

    (a) If $\exists_f (d \wedge f^{-1}a) = a \wedge \exists_f d$ then the corestriction f_a of f along a is a closed preneighbourhood morphism.

    (b) If the corestriction f_a of f along a is a closed preneighbourhood morphism and $d \wedge f^{-1}a \neq \sigma_X$ then $\exists_f (d \wedge f^{-1}a) = a \wedge \exists_f d$.

Proof. Consider the diagram

$$\tag{8}$$

in which the vertical front and the extreme squares are pullbacks and the base square is the (E, M)-factorisation of $f \circ d$. Since $a \circ f_a \circ ((f^{-1}a)^{-1}d) = f \circ (f^{-1}a) \circ ((f^{-1}a)^{-1}d) = f \circ d \circ (d^{-1}f^{-1}a) = (\exists_f d) \circ (f|_d) \circ (d^{-1}f^{-1}a)$, there exists the unique morphism t such that all the squares commute; in particular, $(f|_d)^{-1}((\exists_f d)^{-1}a) = d^{-1}f^{-1}a$, $((f|_d))_{(\exists_f d)^{-1}a} = t$.

If $f_a \circ ((f^{-1}a)^{-1}d) = (\exists_{f_a}((f^{-1}a)^{-1}d)) \circ (f_a|_{(f^{-1}a)^{-1}d})$ be its (E, M)-factorisation, then there exists a unique morphism $\exists_{f_a} D \wedge f^{-1}A \xrightarrow{u} A \wedge \exists_f D$ such that $\exists_{f_a}((f^{-1}a)^{-1}d) = (a^{-1}\exists_f d) \circ u$ and $t = u \circ (f_a|_{(f^{-1}a)^{-1}d})$. Hence $a \circ (\exists_{f_a}((f^{-1}a)^{-1}d)) \circ (f_a|_{(f^{-1}a)^{-1}d}) = a \circ (a^{-1}\exists_f d) \circ t = (a \wedge \exists_f d) \circ t = f \circ (d \wedge f^{-1}a) = (\exists_f (d \wedge f^{-1}a)) \circ (f|_{d \wedge f^{-1}a})$, the last one being the (E, M)-factorisation of $\exists_f (d \wedge f^{-1}a)$. Consequently from the uniqueness of (E, M)-factorisation, $\exists_f (d \wedge f^{-1}a) = a \circ (\exists_{f_a}((f^{-1}a)^{-1}d))$ and $(f_a|_{(f^{-1}a)^{-1}d}) = (f|_{d \wedge f^{-1}a})$. Thus: $\exists_f (d \wedge f^{-1}a) = a \wedge \exists_f d$ if and only if u is an isomorphism, which in turn is equivalent to $t \in \mathsf{E}$.

Towards the proof of (a), consider $p \in \mathfrak{C}_{(\mu|_{d \wedge f^{-1}a})}$. Hence, taking $d = \mathrm{cl}_\mu (f^{-1}a) \circ p \in \mathfrak{C}_\mu$,

$p = \mathrm{cl}_{(\mu|_{d \wedge f^{-1}a})} p = (f^{-1}a)^{-1}\mathrm{cl}_\mu (f^{-1}a) \circ p = (f^{-1}a)^{-1}d$. Then $a \circ (\exists_{f_a} p) = \exists_f ((f^{-1}a) \circ p) = \exists_f (d \wedge f^{-1}a) = a \wedge \exists_f d = a \wedge \exists_f \mathrm{cl}_\mu (f^{-1}a) \circ p = a \wedge \mathrm{cl}_\phi \exists_f ((f^{-1}a) \circ p) = a \wedge \mathrm{cl}_\phi (a \circ (\exists_{f_a} p)) = a \circ \mathrm{cl}_{(\phi|_a)} \exists_{f_a} p$, implying $\exists_{f_a} p = \mathrm{cl}_{(\phi|_a)} \exists_{f_a} p$, i.e., f_a is closed.

Towards the proof of (b), if $\mathbf{1}_Y \neq x \in \phi(a \wedge \exists_f d)$ then since $f^{-1}x \wedge d \wedge f^{-1}a \geq d \wedge f^{-1}a \neq \sigma_X$ and f reflects zero, $x \wedge \exists_f (d \wedge f^{-1}a) \neq \sigma_Y$. Hence $a \wedge \exists_f d \leq \mathrm{cl}_\phi \exists_f (d \wedge f^{-1}a) = \exists_f \mathrm{cl}_\mu (d \wedge f^{-1}a)$. Since $d \in \mathfrak{C}_\mu$ and for each $u \in \mu(d)$, $u \wedge d \wedge f^{-1}a \geq d \wedge f^{-1}a \neq \sigma_X$, $d \leq \mathrm{cl}_\mu (d \wedge f^{-1}a)$; consequently, $d = \mathrm{cl}_\mu (d \wedge f^{-1}a)$. Since f_a is closed and $(f^{-1}a)^{-1}d \in (\mu|_{f^{-1}a})$, $\exists_{f_a} (f^{-1}a)^{-1}d \in (\phi|_a)$. Hence

$$\exists_f (d \wedge f^{-1}a) = a \circ (\exists_{f_a}((f^{-1}a)^{-1}d))$$
$$= a \wedge \mathrm{cl}_\phi (a \circ (\exists_{f_a}((f^{-1}a)^{-1}d)))$$
$$= a \wedge \mathrm{cl}_\phi \exists_f (d \wedge f^{-1}a)$$
$$= a \wedge \mathrm{cl}_\phi \exists_f (d \wedge f^{-1}a)$$
$$= a \wedge \exists_f \mathrm{cl}_\mu (d \wedge f^{-1}a)$$
$$= a \wedge \exists_f d.$$

$\square$

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