Homotopy Type Theory as a logical framework for the Minimalist Foundation

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Homotopy Type Theory (HoTT) [7] is a well-established foundation for constructive mathematics with interesting applications both in mathematics and in computer science. It is an extension of Martin-Löf's type theory with Voevodsky's Univalence Axiom implying the remarkable property that 'isomorphic' sets are propositionally equal. One of the main problems regarding HoTT has been to find a *computational* interpretation proving that its proofs are terminating programs as it is the case for Martin-Löf's Type Theory. The most promising proposal for solving this problem is the interpretation of HoTT in a cubical type theory [1], which recently has been shown to enjoy a normalization result [6].

Our long-term research aim is to understand how this computational interpretation can be adapted to the Minimalist Foundation (MF) and its extensions. MF, first conceived in [5] and then fully formalized in [3], is a two-level type theory consisting of an intensional level, an extensional one and an interpretation of the latter in the first. It was designed to be interpretable in the most relevant existing constructive foundations by preserving the meaning of logical and set-theoretical constructors. Therefore, MF enjoys a lot of different interpretations, including the computational interpretations given in [2] and [4].

As a first step of our research program, we want to show that HoTT can be thought of as a logical framework of MF where to interpret both levels of MF in order to provide a homotopical computational interpretation for each of them. This is contrary to what happens in intensional Martin-Löf's Type Theory which interprets only the intensional level of MF as has been shown in [3] . In fact, the striking feature of HoTT with higher inductive types is to have enough constructors to interpret both levels of MF by preserving the meaning of logical and set-theoretical constructors.

The main difficulty is to interpret the extensional level of MF and in particular definitional equalities among its types and terms since these include extensional equalities about quotient sets and proof-irrelevance of propositions.

References

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