

On a Categorical Framework for Coalgebraic Modal Logic

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Abstract

A category of one-step semantics is introduced to unify different approaches to coalgebraic logic parametric in a contravariant functor that assigns to the state space its collection of predicates with propositional connectives. Modular constructions of coalgebraic logic are identified as colimits, limits, and tensor products, extending known results for predicate liftings. Generalised predicate liftings as modalities are introduced. Under common assumptions, the logic of all predicate liftings together with a complete axiomatisation exists for any type of coalgebras, and it is one-step expressive for finitary functors. Colimits and compositions of one-step expressive coalgebraic logics are shown to remain one-step expressive.

1 Introduction

Two syntax-oriented approaches to coalgebraic modal logic — Moss' cover modality [23] and Pattinson's predicate liftings [24, 25, 26] — are successful in producing a wide range of modal logics parametric in a \mathbf{Set} functor. Subsequently, the algebraic semantics of the logics of predicate liftings was formulated elegantly as a particular form of natural transformations using Stone duality [15]. To explain it, let \mathbf{BA} denote the category of Boolean algebras and $\mathcal{Q}: \mathbf{Set} \rightarrow \mathbf{BA}$ the contravariant powerset functor. For an endofunctor T , a family of sets Λ_n of n -ary

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predicate liftings indexed by \mathbb{N} amounts to a natural transformation δ to UQT from the coproduct indexed by all n -ary predicate liftings λ of n -fold product of UQ , i.e. $\coprod_{n \in \mathbb{N}} \coprod_{\lambda \in \Lambda_n} UQ^n$ where U is the forgetful functor, so by adjunction this family (Λ_n) gives rise to an *interpretation of modalities* $\delta: LQ \Rightarrow QT$ for some functor L ; it introduced what we call *one-step semantics* (Definition 3.1), an expression first coined by Cirstea and Pattinson [7] in a different but equivalent form. Later Moss' cover modality was also formulated in this way [21].

This abstract functorial framework was further developed in [17, 4, 14, 19, 18] with the aim of finding suitable modal logics for various coalgebras, e.g. [5, 11, 13, 20], replacing Q with other contravariant functors. For example, Markov processes are coalgebras of the Giry monad, and propositional connectives for measurable spaces can be specified by the contravariant functor $\mathbb{S}: \text{Meas} \rightarrow \text{MSL}$ mapping a space to its σ -algebra, considered as a meet semilattice.

Adequacy and soundness of the functorial framework follow from its very formulation as shown by Kurz [17], and a sufficient condition of *expressiveness* was first established by Klin [14] for finitary type functors on locally finitely presentable categories (to be presented in Subsection 5.2 below). For example, Boolean logic extended with the possibility modality is expressive for all image-finite Kripke frames, i.e. coalgebras for the finite powerset functor \mathbb{P}_ω . The restriction to finitary functors is not necessary, however. Multi-modal logic is expressive for image-finite *A-labelled* Kripke frames, but Klin's condition does not cover this case, since its corresponding type functor, the A -fold product \mathbb{P}^A , is not necessarily finitary.

Another important line of research investigated *modularity* of predicate liftings [7, 27]: not only expressiveness but also completeness are stable under certain constructions. With regards to the example above, expressiveness of multi-modal logic for \mathbb{P}_ω^A -coalgebras is *inherited* from modal logic for \mathbb{P}_ω -coalgebras. The idea has since been incorporated into the functorial framework over Stone duality in [19], where a subset of constructions is considered, focusing on completeness. In [18], Kurz and Leal show how to translate Moss' cover modality into predicate liftings and vice versa; thus making it amenable to their completeness analysis.

In the present paper, we put forward a *fully categorical treatment in a syntax-independent fashion beyond Stone duality*, so that existing results and concepts can be further applied without further effort to richer structures such as ordered, topological, or measurable spaces. Our running examples will be mostly over Set , however, in the hope that the reader will be able to make a direct comparison with known results and concepts.

In Section 3, we introduce a category CoLog of one-step semantics, which includes Pattinson's predicate liftings and Moss' cover modalities as objects, and exhibit its rich structure. Modularity of coalgebraic modal logics are recognised as standard categorical constructions in this category, avoiding any syntactic bookkeeping. A "full logic" for each type of coalgebras, using the basic properties of adjunction, is identified: every other logic for the same type can be (uniquely) translated to it. In Section 4, we then focus on *equational* one-step semantics, which are found to be isomorphic to those determined by predicate liftings. Notions of modalities and equations are derived naturally from the analysis

of categorical structures, as is the characterisation of the full equational logic. Logical properties of one-step semantics are discussed in Section 5. Klin’s condition of expressiveness is adapted. The categorical viewpoint allows us to formulate general preservation principles for coalgebraic *expressiveness* that apply to *all* logics in CoLog.

The framework is parametric in a contravariant functor mapping state spaces to “algebras” for the base logic about which very little needs to be assumed to cover a large variety of examples. Indeed, we would like to suggest that our work provides the right level of abstraction for understanding coalgebraic modal logic.

This paper summarises the first author’s PhD thesis [6], to which we point for most of the proofs.

2 Preliminaries

We follow Mac Lane’s terminology of category theory [22] except that for us a (co)reflective subcategory is defined to be *full*. For an endofunctor T , a **coalgebra** for T is a morphism $\xi: X \rightarrow TX$; a **coalgebra homomorphism** from (X, ξ) to (Y, γ) is a morphism $f: X \rightarrow Y$ satisfying $Tf \circ \xi = \gamma \circ f$. The category of T -coalgebras is denoted by \mathcal{X}_T .

2.1 Dual adjunctions

Definition 2.1. A contravariant functor $P: \mathcal{X} \rightarrow \mathcal{A}$ is said to be **dual** to a contravariant functor S if together they form an adjunction $S \dashv P: \mathcal{X}^{\text{op}} \rightarrow \mathcal{A}$ with unit η and counit ϵ .

We use dual adjunctions to set up a link between “state spaces”, the objects of \mathcal{X} , and (algebras of) “logics”, the objects of \mathcal{A} . This is a particular instance of “Stone duality”; for a general introduction see [12].

- Example 2.2.**
1. Consider the powerset 2^- as a contravariant functor from **Set** to **Set**; it is dual to itself. Alternatively, consider the powerset as a Boolean algebra; we obtain \mathcal{Q} , a contravariant functor from **Set** to **BA**. The natural dual to \mathcal{Q} is the ultrafilter functor \mathcal{F} , equivalently described as $\text{BA}(-, \{\perp \leq \top\})$. The pair $(\mathcal{Q}, \mathcal{F})$ is our leading example.
 2. When \mathcal{X} is the category of posets, the upset functor $\mathcal{U}: \text{Pos} \rightarrow \text{DLat}$ mapping a poset to the distributive lattice of upsets is dual to the prime filter functor, naturally isomorphic to $\text{DLat}(-, \{\perp \leq \top\})$.
 3. The contravariant functor $\mathbb{S}: \text{Meas} \rightarrow \text{MSL}$ mentioned above is dual to the filter functor \mathcal{F} which maps a semi-lattice A to its collection of filters with the σ -algebra generated by the units $(\eta(a))_{a \in A}$.

A wide range of state-based systems can be formulated as **Set** coalgebras, and beyond **Set** we have further examples, including *a*) descriptive Kripke frames as Stone coalgebras of the Vietoris topology [16]; *b*) positive Kripke frames as **Pos**

coalgebras of the convex powerset functor [13]; and *c*) Markov processes as Meas coalgebras of the Giry monad [9].

2.2 Factorisation systems

Definition 2.3. (see [3]) Given two classes \mathcal{E} , \mathcal{M} of morphisms of a category \mathcal{X} , we say that $(\mathcal{E}, \mathcal{M})$ is a **factorisation system** if

1. every morphism f has an $(\mathcal{E}, \mathcal{M})$ -factorisation;
2. \mathcal{E} and \mathcal{M} contain all isomorphisms and are closed under composition.
3. it has the **diagonal fill-in property**; i.e., for each equation $g \circ e = m \circ f$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ there is a unique morphism d such that the equations $d \circ e = f$ and $m \circ d = g$ hold.

We say that a factorisation system $(\mathcal{E}, \mathcal{M})$ is **proper** if \mathcal{E} is contained in the class of epimorphisms and \mathcal{M} in the class of monomorphisms.

Proposition 2.4. *For any factorisation system $(\mathcal{E}, \mathcal{M})$ on \mathcal{X} , the following statements are true:*

1. \mathcal{E} -morphisms are preserved by pushouts;
2. \mathcal{M} is closed under limits in the arrow category $\mathcal{X}^{\rightarrow}$;
3. if $f \circ g$ and f are \mathcal{M} -morphisms, then so is g .

2.3 Equationally presentable functors

Let \mathcal{A} be a finitary variety with a left adjoint F to the forgetful functor $U: \mathcal{A} \rightarrow \mathbf{Set}$, and J the full inclusion of the subcategory \mathcal{A}_{ω}^f of \mathcal{A} on free algebras Fn for $n \in \mathbb{N}$.

Definition 2.5. (see [4, 29, 6]) An endofunctor L of \mathcal{A} is **finitely based** if it satisfies one of the following equivalent statements:

1. L is finitary and preserves canonical presentations;
2. L is of the form $\text{Lan}_J LJ$, a left Kan extension of LJ along J .

Let $\text{FinB}[\mathcal{A}, \mathcal{A}]$ denote the full subcategory of the functor category $[\mathcal{A}, \mathcal{A}]$ on finitely based functors.

The notion of finitely based functors plays an important role in our theory of one-step semantics, since they are precisely the *equationally presentable* ones. Some elementary facts are given first.

Proposition 2.6. *$\text{FinB}[\mathcal{A}, \mathcal{A}]$ is equivalent to $[\mathcal{A}_{\omega}^f, \mathcal{A}]$.*

Every variety \mathcal{A} is locally presentable and (effectively) regular so the same holds for $\text{FinB}[\mathcal{A}, \mathcal{A}]$ by this equivalence and \mathcal{A}_ω^f being small. Another useful fact about $\text{FinB}[\mathcal{A}, \mathcal{A}]$ is that every functor has a finitely based coreflection:

Lemma 2.7. $\text{FinB}[\mathcal{A}, \mathcal{A}]$ is a coreflective subcategory of $[\mathcal{A}, \mathcal{A}]$ with the right adjoint $\text{Lan}_J(- \circ J)$.

It follows that $\text{FinB}[\mathcal{A}, \mathcal{A}]$ is closed under colimits of $[\mathcal{A}, \mathcal{A}]$ since the inclusion is a left adjoint. Moreover, it can be shown further that the coreflector sends finite products to $\text{FinB}[\mathcal{A}, \mathcal{A}]$ unchanged:

Proposition 2.8. $\text{FinB}[\mathcal{A}, \mathcal{A}]$ is closed under finite products of $[\mathcal{A}, \mathcal{A}]$.

The finitely based coreflection of L is written as $\rho_L: \widehat{L}_\omega \Rightarrow L$ with $\widehat{L}_\omega = \text{Lan}_J L J$.

The slogan ‘every finitely based functor is equationally presentable’ is justified as follows. A **signature** is a functor Σ from the discrete category \mathbb{N} of natural numbers to Set . Every signature defines a finitely based endofunctor on \mathcal{A} by

$$\widehat{H}_\Sigma := F H_\Sigma U \quad \text{where} \quad H_\Sigma X := \prod_{n \in \mathbb{N}} \Sigma_n \times X^n. \quad (1)$$

Theorem 2.9. The functor $\text{FinB}[\mathcal{A}, \mathcal{A}] \rightarrow [\mathbb{N}, \text{Set}]$ defined by mapping L to $(n \mapsto ULF)$, is finitary and monadic with a left adjoint defined by (1).

By monadicity, every finitely based functor L has a *canonical presentation*, i.e. a coequaliser $\widehat{H}_\Gamma \rightrightarrows \widehat{H}_\Sigma \rightarrow L$ where $\Sigma = ULF$ and $\Gamma = U\widehat{H}_\Sigma F$ are signatures. By adjunction, the two parallel morphisms correspond to

$$\Gamma \rightrightarrows U\widehat{H}_\Sigma F = UFH_\Sigma UF, \quad (2)$$

so H_Σ and the parallel morphisms are the *generator* and the *equation* of the presentation, respectively.

Remark 2.10. For $n \in \mathbb{N}$, an element t of $U\widehat{H}_\Sigma F n$ is to be understood as a term in \mathcal{A} consisting of at most one layer of operations in Σ at terms of n variables, called a **rank-1 Σ -term**:

$$U\widehat{H}_\Sigma F n = U \quad F \quad \overbrace{H_\Sigma \quad U \quad F n}^{\Sigma\text{-operations}} \\ \text{\scriptsize } n\text{-ary terms}$$

The two parallel morphisms (2) can be presented by a family of sets $\mathcal{E}_n \subseteq (U\widehat{H}_\Sigma F n)^2$ of **rank-1 equations**, indexed by $n \in \mathbb{N}$.

Example 2.11. Modal algebras can be characterised as algebras for a finitely based functor \mathbb{M} , defined by the following BA-presentation¹

$$\mathbb{M}A := \text{BA} \langle \{\blacklozenge a\}_{a \in A} \mid \blacklozenge \perp = \perp, \blacklozenge(a \vee b) = \blacklozenge a \vee \blacklozenge b \rangle \quad (3)$$

¹A BA-presentation $\text{BA}(\mathcal{G} \mid \mathcal{E})$ indicates an algebra generated by \mathcal{G} subject to equations in \mathcal{E} .

and $\mathbb{M}f(\blacklozenge a) := \blacklozenge fa$ for each homomorphism f . This functor has an equational presentation consisting of a signature $\Sigma_1 := \{\blacklozenge\}$ and $\Sigma_n := \emptyset$ otherwise; and equations \mathcal{E}_n by

$$\mathcal{E}_n := \begin{cases} \{\blacklozenge \perp \sim \perp\} & \text{if } n = 0 \\ \{\blacklozenge(a \vee b) \sim \blacklozenge a \vee \blacklozenge b\} & \text{if } n = 2 \\ \emptyset & \text{otherwise} \end{cases}$$

where 2 is presented as $\{a, b\}$.

Remark 2.12. In categories such as **Set** and the category \mathbf{Vec}_K of vector spaces over a field K , every finitely generated algebra is free, so finitary functors are finitely based. Moreover, every finitary endofunctor of **BA** is naturally isomorphic to a finitely based functor on non-trivial Boolean algebras [20]. However, this coincidence does not hold in general, and a counterexample can be found in [29, Example 3.12].

3 Categories of One-Step Semantics

We study properties of the category of all one-step semantics, including colimits, limits, a monoidal structure, coreflections, and its opposite category. For each type T of coalgebras, we explore the category of one-step semantics of T and show that there is always a ‘full’ one-step semantics to which every one-step semantics can be translated in a *unique way*.



In the following context, $P: \mathcal{X} \rightarrow \mathcal{A}$ always denotes a contravariant functor. It is suppressed if an ambiguity is unlikely.

3.1 The category of all one-step semantics

Definition 3.1. A **one-step semantics over P** consists of an endofunctor T of \mathcal{X} , called the *type of one-step semantics*; an endofunctor L of \mathcal{A} , called the *syntax of modalities*; a natural transformation $\delta: LP \Rightarrow PT$, called the *interpretation of modalities*, denoted by (L, T, δ) .

Proposition 3.2. *Every one-step semantics (L, T, δ) defines a functor P^δ from the category of T -coalgebras to the category of L -algebras, called the *lifting of P* .*

Example 3.3. (Classical modal logic) Kripke semantics for modal logic with its algebraic semantics defines a one-step semantics $(\mathbb{M}, \mathbb{P}, \delta)$ over \mathcal{Q} as follows where \mathbb{M} is given in (3) and \mathbb{P} is the covariant powerset functor.

Define a natural transformation $\blacklozenge: 2^- \Rightarrow 2^\mathbb{P}$ by $S \mapsto \{U \in \mathbb{P}X \mid U \cap S \neq \emptyset\}$ for each subset $S \subseteq X$. Then, \blacklozenge defines a natural transformation δ from $\mathbb{M}\mathcal{Q}$ to $\mathcal{Q}\mathbb{P}$ by mapping $\blacklozenge S$ to $\blacklozenge_X S$, since \blacklozenge_X satisfies the above two equations in (3) for every X .

The lifting given by the one-step semantics $(\mathbb{M}, \mathbb{P}, \delta)$ is the functor mapping every \mathbb{P} -coalgebra $\xi: X \rightarrow \mathbb{P}X$ to the \mathbb{M} -algebra $\mathcal{Q}\xi \circ \delta_X: \mathbb{M}\mathcal{Q}X \rightarrow \mathcal{Q}X$ which

is equivalent to the *complex algebra* of (X, ξ) . The unique algebra homomorphism $\llbracket - \rrbracket: \Phi \rightarrow \mathcal{Q}X$ from the initial \mathbb{M} -algebra (Φ, α) to the complex algebra interprets every element in Φ to a predicate on X . The semantics of possibility, $\llbracket \blacklozenge \varphi \rrbracket = \{x \in X \mid \xi(x) \cap \llbracket \varphi \rrbracket \neq \emptyset\}$, follows from the commutativity of homomorphisms.

Polyadic predicate liftings for a **Set** functor T , i.e. natural transformations $\lambda: (2^-)^n \Rightarrow 2^T$ for the contravariant powerset functor 2^- , also provide a class of examples:

Example 3.4. (Predicate liftings) Every set Λ of polyadic predicate liftings for T gives rise to a one-step semantics $(L^\Lambda, T, \delta^\Lambda)$ over \mathcal{Q} as follows. Let Λ_n denote the set of n -ary predicate liftings in Λ so that Λ is a signature $\Lambda: \mathbb{N} \rightarrow \mathbf{Set}$ and F the left adjoint to the forgetful functor $U: \mathbf{BA} \rightarrow \mathbf{Set}$. Define the syntax functor L^Λ by the signature Λ without equations, and the interpretation $\delta^\Lambda: L^\Lambda \mathcal{Q} \Rightarrow \mathcal{Q}T$ on generators by $(\lambda, S) \mapsto \lambda_X(S)$ for $\lambda \in \Lambda_n$ and each n -tuple $S = (S_i \subseteq X)_{i \in n}$.

Example 3.5. (Cover modality [18]) The cover modality for a finitary and weak-pullback preserving **Set** functor T with Boolean logic also defines a one-step semantics. Let \in_X denote the membership relation on a set X . Define a function $\nabla_X^T: T2^X \rightarrow 2^{TX}$ by

$$\alpha \mapsto \{t \in TX \mid (t, \alpha) \in \overline{T}(\in_X)\}$$

where $(t, \alpha) \in \overline{T}(\in_X)$ if there is $w \in T(\in_X)$ with $T\pi_1(w) = t$ and $T\pi_2(w) = \alpha$. ∇_X^T is natural in X because T preserves weak pullbacks. By adjunction and $2^- = U\mathcal{Q}$, the transpose of ∇^T is a natural transformation $\overline{\nabla}^T$ from $FTU\mathcal{Q}$ to $\mathcal{Q}T$. That is, the cover modality defines a one-step semantics $(FTU, T, \overline{\nabla}^T)$.

Definition 3.6. The **category \mathbf{CoLog}^P of one-step semantics** over P is defined to be the comma category $(P^* \downarrow P_*)$ from the precomposition $P^*: L \mapsto LP$ to the postcomposition $P_*: T \mapsto PT$. (Following our convention, we will usually suppress the superscript P .)

That is, the objects of **CoLog** are one-step semantics (L, T, δ) and a morphism from (L, T, δ) to (L', T', δ') is a pair of natural transformations $(\tau: L \Rightarrow L', \nu: T' \Rightarrow T)$ satisfying $P\nu \circ \delta = \delta' \circ \tau P$. The natural transformation τ is intuitively understood as a **translation** from syntax L to syntax L' . We will justify this intuition in Section 4. We denote the projection $(L, T, \delta) \mapsto L$ with U_L and the contravariant $(L, T, \delta) \mapsto T$ with U_R .

Proposition 3.7 (Coreflection). *Given a coreflective subcategory \mathcal{B} of the functor category $[\mathcal{A}, \mathcal{A}]$, the pullback of the full inclusion functor $J: \mathcal{B} \hookrightarrow [\mathcal{A}, \mathcal{A}]$ along the forgetful functor $U_L: \mathbf{CoLog} \rightarrow [\mathcal{A}, \mathcal{A}]$ is also coreflective.*

Note that the pullback category is the full subcategory of **CoLog** of those one-step semantics whose syntax functor lies in \mathcal{B} .

Proof. Proof sketch Let $\rho_L: L^\dagger \Rightarrow L$ denote the coreflection of the functor L . Then, the coreflection of a one-step semantics (L, T, δ) is $(L^\dagger, T, \delta \circ \rho_L)$. \square

3.1.1 Colimits and limits

A colimit of a J -indexed diagram D in \mathbf{CoLog} can be constructed by a pointwise colimit $(\tau_i : L_i \Rightarrow L)_{i \in J}$ of $U_L D$ and a pointwise limit $(\nu : T \Rightarrow T_i)_{i \in J}$ of $(U_R D)^{\text{op}}$ using the universal property:

$$\begin{array}{ccccc}
 L_i P & \xrightarrow{(U_L D f) P} & L_j P & & \\
 \downarrow \iota_i P & \searrow \delta_i & \downarrow \delta_j & & \\
 & & P T_i & \xrightarrow{P(U_R D f)} & P T_j \\
 & & \downarrow P \pi_i & & \downarrow P \pi_j \\
 L P & \xrightarrow{\delta} & P T & &
 \end{array}$$

since by assumption LPX is a colimit for each component X .

Theorem 3.8 (Colimit). *The pair (U_L, U_R) of projections of \mathbf{CoLog} creates pointwise colimits.*

A pointwise colimit means that it is constructed by a pointwise colimit in $[\mathcal{A}, \mathcal{A}]$ and a pointwise limit in $[\mathcal{X}, \mathcal{X}]$.

Example 3.9. (Multi-modal logic) For a set A of labels, the A -fold coproduct of classical modal logic $(\mathbb{M}, \mathbb{P}, \delta)$ is a multi-modal logic for A -labelled Kripke frames.

Limits in \mathbf{CoLog} can be constructed similarly as

$$\text{Lim } L_i P \xrightarrow{\delta = \text{Lim } \delta_i} \text{Lim } P T_i \cong P(\text{Colim } T_i)$$

and $\text{Lim } \delta_i$ is a pointwise limit in the arrow category, provided that P maps a colimit to a limit:

Theorem 3.10 (Limit). *Suppose that P has a dual adjoint S . Then, the pair (U_L, U_R) of projections creates pointwise limits.*

Example 3.11. An **alternating system** over a set A of actions [10] is a coalgebra for the functor $\mathcal{D} + \mathbb{P}^A$ (where \mathcal{D} is the probability distribution functor). For such systems, a one-step semantics can be obtained as a product of a one-step semantics of type \mathbb{P}^A and one of type \mathcal{D} . For the former we may take multi-modal logic, and the latter can be probabilistic modal logic induced by predicate liftings $\langle p \rangle$ for \mathcal{D} , indexed by $p \in \mathbb{Q} \cap [0, 1]$:

$$S \mapsto \{ \mu \in \mathcal{D} X \mid \sum \mu(S) \geq p \}$$

for each subset $S \subseteq X$.

3.1.2 Composition

The **composition** \otimes of one-step semantics is defined by *pasting*:

$$\left[\begin{array}{c} \xleftarrow{L_1} \\ \Downarrow \delta_1 \\ \xleftarrow{T_1} \end{array} \right] \otimes \left[\begin{array}{c} \xleftarrow{L_2} \\ \Downarrow \delta_2 \\ \xleftarrow{T_2} \end{array} \right] := \left[\begin{array}{c} \xleftarrow{L_1 L_2} \\ \Downarrow \delta_1 \otimes \delta_2 \\ \xleftarrow{T_1 T_2} \end{array} \right]$$

i.e. $\delta_1 \otimes \delta_2 := \delta_1 T_2 \circ L_1 \delta_2$.

Lemma 3.12. *The composition \otimes of one-step semantics is a bifunctor, mapping each pair of morphisms $(\tau_o, \nu_o): \delta_1 \rightarrow \delta_3$ and $(\tau_e, \nu_e): \delta_2 \rightarrow \delta_4$ to a morphism from $\delta_1 \otimes \delta_2$ to $\delta_3 \otimes \delta_4$ and defined by the horizontal composites*

$$\left(\left(\begin{array}{c} \xrightarrow{L_2} \\ \Downarrow \tau_e \\ \xrightarrow{L_4} \end{array} \right), \left(\begin{array}{c} \xrightarrow{L_1} \\ \Downarrow \tau_o \\ \xrightarrow{L_3} \end{array} \right), \left(\begin{array}{c} \xrightarrow{T_2^{\text{op}}} \\ \Downarrow \nu_e \\ \xrightarrow{T_4^{\text{op}}} \end{array} \right), \left(\begin{array}{c} \xrightarrow{T_1^{\text{op}}} \\ \Downarrow \nu_o \\ \xrightarrow{T_3^{\text{op}}} \end{array} \right) \right)$$

Theorem 3.13 (Monoidal structure). *The composition \otimes of one-step semantics with the identity semantics $(\mathcal{I}, \mathcal{I}, \text{id}_P)$ is a strict monoidal structure on CoLog .*

Example 3.14. A **simple Segala system** over a set A of actions [28] is a coalgebra for the composite $\mathbb{P}^A \circ \mathcal{D}$. Thus, a one-step semantics for simple Segala systems can be obtained as the composite of the A -fold coproduct of $(\mathbb{M}, \mathbb{P}, \delta)$ and the one-step semantics of probabilistic modal logic.

3.1.3 Mate correspondence

To finish our study on CoLog , we study the mate correspondence of one-step semantics, a tool used first by Klin [14] to analyse one-step expressiveness of coalgebraic logic in a categorical approach:

Definition 3.15. Suppose that P has a dual adjoint S . The **mate** δ^* of a natural transformation $\delta: LP \Rightarrow PT$ is a natural transformation from TS to SL defined by the pasting diagram

$$\begin{array}{ccc} & \xrightarrow{L} & \xrightarrow{S} \\ \mathcal{I} \nearrow & \Downarrow \eta & \Downarrow \delta & \Downarrow \epsilon & \searrow \mathcal{I} \\ & P & \Downarrow \delta & P & \\ & \xrightarrow{S} & \xrightarrow{T^{\text{op}}} & \end{array}$$

in the opposite of \mathcal{X} , that is, $\delta^* = SL\eta \circ S\delta S \circ \epsilon TS$.

The mate operation maps a one-step semantics to an object of the comma category $\text{CoLog}^* := (S^* \downarrow S^*)$. Furthermore:

Proposition 3.16. *CoLog is dually isomorphic to CoLog^* where a one-step semantics (L, T, δ) is mapped to its mate (T, L, δ^*) .*

Remark 3.17. By this isomorphism, colimits and limits in \mathbf{CoLog} are transposed to their duals in \mathbf{CoLog}^* , but more specifically, it can be shown that every (pointwise) colimit in \mathbf{CoLog} can be constructed as a pointwise limit in the arrow category, to be used in Theorem 5.10. A bifunctor \oplus defined by $\theta_1 \oplus \theta_2 := \theta_1 L_2 \circ T_1 \theta_2$ with the identity $id: S \Rightarrow S$ defines a strict monoidal structure on \mathbf{CoLog}^* , which is the image of (\otimes, id) under the mate correspondence.

3.2 Fibre categories of \mathbf{CoLog}

Definition 3.18. For every endofunctor T of \mathcal{X} , the category \mathbf{CoLog}_T is defined to be the fibre category over T . More precisely, the objects of \mathbf{CoLog}_T are one-step semantics of type T , denoted (L, δ) . A morphism τ from (L, δ) to (L', δ') is a natural transformation satisfying $\delta = \delta' \circ \tau P$ (i.e. a syntax translation).

The type functor being fixed, we focus on the syntax projection functor which maps $U_L: (L, \delta) \mapsto L$ and $\tau \mapsto \tau$.

Proposition 3.19. *The following statements hold:*

1. U_L reflects isomorphisms.
2. For every coreflective subcategory \mathcal{C} of the functor category $[\mathcal{A}, \mathcal{A}]$, the pullback of the full inclusion $i: \mathcal{C} \hookrightarrow [\mathcal{A}, \mathcal{A}]$ along $U_L: \mathbf{CoLog}_T \rightarrow [\mathcal{A}, \mathcal{A}]$ is coreflective.
3. U_L creates pointwise colimits.

We are ready to establish a fundamental theorem for coalgebraic modal logics representable as one-step semantics:

Theorem 3.20. *Suppose that P has a dual adjoint S . Then every fibre \mathbf{CoLog}_T has a terminal object*

$$(PTS, PT\epsilon: PTS P \Rightarrow PT)$$

where $\epsilon: \mathcal{I} \rightarrow SP$ is the counit of the dual adjunction.

In every fibre category \mathbf{CoLog}_T , this terminal object is called the **full one-step semantics** for T . It may be too elusive for practical purposes, but conceptually it explains that every collection of modalities may be viewed as a (uniquely determined) fragment of this canonical one-step semantics.

4 Equational One-Step Semantics

We now focus on one-step semantics whose syntax functor is defined by operations and equations, called *equational*. To work with equational one-step semantics, we use finitely based functors (as introduced in Section 2) as syntax functors. Equational one-step semantics are characterised as (generalised) predicate liftings subject to equations. In particular, a full equational one-step semantics exists and is the logic of all predicate liftings subject to a complete axiomatisation *up to isomorphism*.

4.1 The category of equational one-step semantics

Definition 4.1. A one-step semantics $(L: \mathcal{A} \rightarrow \mathcal{A}, T, \delta)$ is (finitary) **equational** if \mathcal{A} is a variety and L is finitely based. **ECoLog** is the corresponding full subcategory of **CoLog**.

The examples given in Section 3 are equational, since **BA** is a variety and \mathbb{M} , L^Λ for a set Λ of predicate liftings, and L^{∇^T} for a finitary and weak-pullback preserving functor T , are all finitely based.

It is not hard to show that the composite $L_1 \circ L_2$ of finitely based functors remains finitely based, so every composite of equational one-step semantics remains equational.

Proposition 4.2. *The composition \otimes on **ECoLog** with the identity semantics $(\mathcal{I}, \mathcal{I}, id_P)$ is a strict monoidal structure on **ECoLog**.*

By applying Proposition 3.7 to the finitely based coreflection $\rho_L: \widehat{L}_\omega \Rightarrow L$, every one-step semantics has an equational coreflection:

Proposition 4.3. ***ECoLog** is a coreflexive subcategory of **CoLog**. Therefore, **ECoLog** is closed under colimits of **CoLog**.*

As we are actually interested in *equational* one-step semantics rather than all one-step semantics, the coreflection ensures that colimits are still constructed as in **CoLog**. By Proposition 2.8, this is also true for finite products:

Proposition 4.4. *Suppose that P has a dual adjoint. Pointwise finite products in **ECoLog** coincide with products in **CoLog**.*

Corollary 4.5. *The category of equational one-step semantics over the contravariant 2^- , \mathcal{Q} , \mathcal{U} , and \mathbb{S} in Example 2.2, respectively, has colimits and finite products constructed pointwise.*

For example, the one-step semantics for alternating systems in Example 3.11 is indeed a product in **ECoLog**.

The universal property of (co)limits hints at certain optimal conditions. For instance, the fusion of predicate liftings [8], known as the smallest conservative extension of two given logics of predicate liftings, is in fact the coproduct in **ECoLog** of the corresponding equational one-step semantics:

Example 4.6. (Binary coproduct as fusion) Given two sets Λ_1 and Λ_2 of polyadic predicate liftings for **Set** functors T_1 and T_2 , respectively, the coproduct of the one-step semantics induced by $(\Lambda_i)_{i=1,2}$ consists of $T_1 \times T_2$, as its type, and $L := L^{\Lambda_1} + L^{\Lambda_2} \cong \widehat{H}_\Lambda$ with $\Lambda := \Lambda_1 + \Lambda_2$, as its syntax, and as its interpretation the natural transformation $\delta: L\mathcal{Q} \Rightarrow \mathcal{Q}T$ defined for each set X on the generators of L by

$$(\lambda, S) \mapsto (\pi_i^{-1} \circ \lambda_X)(S)$$

for each $\lambda \in \Lambda_{i,n}$ and n -tuple $S = (S_j)_{j \in n}$.

4.2 Fibre categories of ECoLog

Definition 4.7. For any endofunctor T of \mathcal{X} , ECoLog_T is defined to be the fibre category of ECoLog over T .

By Proposition 3.19, each ECoLog_T is a coreflective subcategory of CoLog_T . Now the equational version of Theorem 3.20 follows, as the coreflector preserves limits, including the terminal object:

Corollary 4.8. *Suppose that P has a dual adjoint S . Every fibre ECoLog_T has a terminal object*

$$\left(\widehat{PTS}_\omega, PT\epsilon \circ \rho_P: \widehat{PTS}_\omega P \rightarrow PT \right)$$

where ρ is the finitely based coreflection $\widehat{PTS}_\omega \Rightarrow PTS$ and ϵ is the counit of the dual adjunction.

Again, this terminal object is called the **full equational one-step semantics** for T , and every equational one-step semantics for T is a fragment of it.

The remaining part of this section is used to describe equational one-step semantics as logics of predicate liftings subject to rank-1 equations.

4.2.1 Predicate liftings

Definition 4.9. Let $U: \mathcal{A} \rightarrow \text{Set}$ be a functor and T an endofunctor of \mathcal{X} . A (finitary) **predicate lifting** for T is a natural transformation from $(UP)^n$ to UPT where $n \in \mathbb{N}$.

When $P = \mathcal{Q}$, this definition boils down to the usual definition of polyadic predicate liftings, since the underlying set $U\mathcal{Q}X$ is the powerset 2^X .

Example 4.10. When $P = \mathbb{S}: \text{Meas} \rightarrow \text{MSL}$, a predicate lifting for T maps a measurable set of a measurable space $X = (X, \mathbb{S}_X)$ to a measurable set on TX natural in X . Take the Giry monad \mathcal{G} , for example; \mathcal{G} maps a space (X, \mathbb{S}_X) to the collection of subprobability distributions $\mu: \mathbb{S}_X \rightarrow [0, 1]$ satisfying

$$\mu(\emptyset) = 0 \quad \text{and} \quad \mu\left(\bigcup_i M_i\right) = \sum_i \mu(M_i)$$

for countable unions of pairwise disjoint measurable sets M_i (the σ -algebra on $\mathcal{G}X$ is ignored). Then a predicate lifting for \mathcal{G} can be defined for each $p \in [0, 1]$ by

$$M \mapsto \{ \mu \in \mathcal{G}X \mid \mu(M) \geq p \}$$

for $M \in \mathbb{S}_X$, which is exactly the modality in [11, Section 4.3].

Assume that \mathcal{A} is a variety and F denotes the left adjoint to the forgetful functor U . Every n -ary predicate lifting λ for T determines a one-step semantics

(FU^n, λ^*) , called the **unimodal logic** of λ , where $\lambda^*: F(UP)^n \Rightarrow PT$ is the transpose of λ by adjunction. A set Λ of predicate liftings for T determines a one-step semantics, called the **logic of predicate liftings**, consisting of

$$\widehat{H}_\Lambda = F \left(\prod_{n \in \mathbb{N}} \Lambda_n \times (U-)^n \right) \quad \text{and} \quad \delta_X^\Lambda: \widehat{H}_\Lambda PX \Rightarrow PTX,$$

mapping generators (λ, S) to $\lambda_X(S)$ for $\lambda \in \Lambda_n$ and $S = (S_i \in UPX)_{i \in n}$.

Moreover, every finitely based functor L is a coequaliser of parallel morphisms $\widehat{H}_\mathcal{E} \rightrightarrows \widehat{H}_\Sigma \rightarrow L$, so characterisations follow readily:

Corollary 4.11. *For every endofunctor T of \mathcal{X} ,*

1. *every logic of predicate liftings for T is a coproduct of unimodal logics;*
2. *every object in \mathbf{ECoLog}_T is a coequaliser of a logic of predicate liftings.*

4.2.2 Translations between equational one-step semantics

By Corollary 4.11, a morphism from (L', δ') to (L, δ) in \mathbf{ECoLog}_T boils down to a family of translations from a unimodal logic to (L, δ) , indexed by some set Λ of predicate liftings λ :

$$\begin{array}{ccc} F(UPX)^n & \xrightarrow{\tau_{PX}} & LPX \\ & \searrow \bar{\lambda}_X & \swarrow \bar{\delta}_X \\ & & PTX. \end{array}$$

Commutativity implies that a translation is not only a syntactic translation but also preserves the interpretation.

Example 4.12. (Continuing Example 3.5) Possibility \diamond and necessity \square can be translated to the cover modality ∇ by setting

$$\square\varphi := \nabla\{\varphi\} \vee \nabla\emptyset \quad \text{and} \quad \diamond\varphi := \nabla\{\varphi, \top\}.$$

The syntactic translation defines a morphism τ from the one-step semantics (\mathbb{M}, δ) to the one-step semantics $(F\mathbb{P}_\omega U, \bar{\nabla}^{\mathbb{P}_\omega})$ of the cover modality, i.e. $\bar{\nabla} \circ \tau \mathcal{Q} = \delta$.²

4.2.3 Equations valid under an interpretation

Let Λ be a set of predicate liftings. A rank-1 Λ -term in n variables can be interpreted by a function that maps any n -tuple of predicates on X to a predicate on TX as follows. By adjunction, every n -tuple $S = (S_i \in PX)$ is presented

²For consistency, we restrict to the finitary powerset functor \mathbb{P}_ω , otherwise FPU is not finitely based.

as a morphism $(Fn \xrightarrow{a} PX)$. Define the interpretation $\llbracket - \rrbracket_{\Lambda, X}^S$ for terms with n variables as the composite

$$U\widehat{H}_\Lambda Fn \xrightarrow{U\widehat{H}_\Lambda a} U\widehat{H}_\Lambda PX \xrightarrow{U\delta_X^\Lambda} UPTX$$

where δ_X^Λ is the logic of predicate liftings in Λ .

Definition 4.13. Given a set Λ of predicate liftings for T , a rank-1 equation $t \sim t'$ of Λ is **valid under the interpretation** of Λ if $\llbracket t \rrbracket_{\Lambda, X}^S = \llbracket t' \rrbracket_{\Lambda, X}^S$ for any X and n -tuple $(S_i \in UPX)$.

The universal quantifiers can be simplified to a pair of parallel morphisms:

Theorem 4.14. *Given a set Λ of predicate liftings for T and families of rank-1 Λ -equations \mathcal{E}_n with n variables indexed by $n \in \mathbb{N}$, the following statements are equivalent:*

1. *For each n , every equation $t \sim t' \in \mathcal{E}_n$ is valid under the interpretation of Λ .*
2. *The following diagram commutes*

$$\widehat{H}_\mathcal{E}P \begin{array}{c} \xrightarrow{\pi_1 P} \\ \xrightarrow{\pi_2 P} \end{array} \widehat{H}_\Lambda P \xrightarrow{\delta^\Lambda} PT$$

where $\pi_1, \pi_2: \widehat{H}_\mathcal{E} \rightrightarrows \widehat{H}_\Lambda$ are the parallel morphisms induced by $\mathcal{E} = (\mathcal{E}_n)_{n \in \mathbb{N}}$.

Therefore, a regular quotient of a one-step semantics in ECoLog_T retains the set of modalities but they are subject to more equations. Moreover, every translation factors through a regular quotient:

Theorem 4.15. *For every T , the category ECoLog_T has $(\text{RegEpi}, U^{-1}\text{Mono})$ as a factorisation system where U is the forgetful functor $U: \text{ECoLog}_T \rightarrow \text{FinB}[\mathcal{A}, \mathcal{A}]$.*

This factorisation system also leads us to an important notion:

Definition 4.16. An equational one-step semantics for T is said to have a **complete axiomatisation** if it has no proper regular quotient in ECoLog_T .

Informally, a one-step semantics (L, δ) has a complete axiomatisation if every rank-1 equation valid under the interpretation δ is derivable from the presentation of L . Note that this is not model-theoretic completeness.

Example 4.17. The one-step semantics (\mathbb{M}, δ) of classical modal logic has a complete axiomatisation. Any proper regular quotient of \mathbb{M} would identify a rank-1 equation which is not derivable from $\blacklozenge \perp = \perp$ and $\blacklozenge(a \vee b) = \blacklozenge a \vee \blacklozenge b$. However, classical modal logic is complete with respect to the class of all Kripke frames, so there exists an instance refuting the equation, a contradiction. Thus (\mathbb{M}, δ) has a complete axiomatisation.

4.2.4 Objects of predicate liftings

Using the Yoneda Lemma, Schöder observes [26] that n -ary predicate liftings in the case where $P = Q$ are in bijection with the subsets of $T2^n$, because $UQ = 2^-$ is naturally isomorphic to $\text{hom}(-, 2)$. We generalise and combine this with Klin's *objects of T -modalities* [14], again employing the Yoneda Lemma and the *dual adjunction*:

Lemma 4.18. *Suppose that P has a dual adjoint S and \mathcal{A} is a variety with a left adjoint F to the forgetful functor U . For any endofunctor T of \mathcal{X} , the following statements hold:*

1. *For any $n \in \mathbb{N}$, the set $U(PTS)Fn$ is in bijection with the collection of n -ary predicate liftings.*
2. *The bijection is natural in objects n in the Kleisli category of the induced monad UF .*

That is, $PTSFn$ is precisely the *object of n -ary predicate liftings*. This ensures that the collection of all finitary predicate liftings is small, so a coproduct of all unimodal logics exists. Naturality means that for any function $f: n \rightarrow UFm$ between free algebras, the diagram

$$\begin{array}{ccc} UPTSFn & \xrightarrow{[-]} & \text{Nat}(UP^n, UPT) \\ UPTS\bar{f} \downarrow & & \downarrow \hat{f}^* \\ UPTSFm & \xrightarrow{[-]} & \text{Nat}(UP^m, UPT) \end{array}$$

commutes, where $[-]$ indicates the bijection from $UPTSFn$ to the collection of n -ary predicate liftings, \bar{f} is the transpose of f , and \hat{f}^* is pre-composition with

$$\hat{f}: (m \xrightarrow{a} UP) \mapsto (n \xrightarrow{f} UFm \xrightarrow{Ua} UFUP \xrightarrow{U\epsilon P} UP).$$

Remark 4.19. Note that the forgetful functor $U: \mathcal{A} \rightarrow \text{Set}$ is naturally isomorphic to $\text{hom}(F1, -)$ by adjunction, so the composite UP is naturally isomorphic to $\text{hom}(-, SF1)$ by dual adjunction. Let Ω be $SF1$. Then, an n -ary predicate lifting is a natural transformation from $\text{hom}(-, \Omega)^n$ to $\text{hom}(T-, \Omega)$, which is in a more familiar form. Specifically, the underlying set of $SF1$ is a two-element set of *truth values* for the dual adjoints in Example 2.2.

4.2.5 Characterising the full equational one-step semantics

The finitely based part \widehat{PTS}_ω of PTS has a canonical presentation, so the full equational one-step semantics is a regular quotient of the one-step semantics consisting of

$$F \left(\prod_{n \in \mathbb{N}} UPTSFn \times U^n \right) \quad \text{and} \quad (PT\epsilon \circ \rho P) \circ e \quad (4)$$

where $\rho: \widehat{PTS}_\omega \Rightarrow PTS$ is the coreflection and e the regular quotient. We call (4) the **logic of all predicate liftings** for T .

Lemma 4.20. *The following statements hold:*

1. *The logic of all predicate liftings is a coproduct of all unimodal logics.*
2. *The full equational one-step semantics has a complete axiomatisation.*

Theorem 4.21. *An equational one-step semantics (L, δ) of type T is isomorphic to the full equational one-step semantics iff (L, δ) has a complete axiomatisation and every unimodal logic has a translation to (L, δ) .*

Proof. Proof sketch By the previous lemmas, the ‘only if’ part follows. For the converse, let (L, δ) be a one-step semantics with a complete axiomatisation such that every unimodal logic has a translation to it. By assumption, there exists a mediating morphism τ from the logic $(L^\Lambda, \delta^\Lambda)$ of all predicate liftings to it, since $(L^\Lambda, \delta^\Lambda)$ is a coproduct of all unimodal logics. Consider the pushout of τ and the regular quotient e from $(L^\Lambda, \delta^\Lambda)$ to the full equational one-step semantics:

$$\begin{array}{ccc}
 (L^\Lambda, \delta^\Lambda) & \xrightarrow{e} & (\widehat{PTS}_\omega, PT\epsilon \circ \rho P) \\
 \downarrow \tau & & \downarrow \\
 (L, \delta) & \longrightarrow & (L', \delta').
 \end{array}$$

Using the factorisation system on \mathbf{ECoLog}_T and the fact that the full semantics is terminal, the statement follows. \square

5 Logic of a One-Step Semantics

We remind the reader how a one-step semantics provides a coalgebraic modal logic, and refine Klin’s expressiveness condition [14] for finitary functors. We also address the modularity problem of expressiveness.

\diamond For the sake of brevity, we restrict our discussion to the case of a one-step semantics which has a language (see below). Also, we assume that P has a dual adjoint S .

5.1 Logical setup

Definition 5.1. The **language** of (L, T, δ) is the initial L -algebra, denoted (Φ_L, α_L) .

The initial L -algebra can be constructed by the *initial sequence* [1]. For a finitary functor L , the following ω -sequence starting from the initial object $\mathbf{0}$:

$$\mathbf{0} \xrightarrow{!} L\mathbf{0} \xrightarrow{L!} \dots \longrightarrow L^i\mathbf{0} \xrightarrow{L^i!} \dots$$

has a colimit $(f_i: L^i \mathbf{0} \rightarrow L^\omega \mathbf{0})_{i \in \omega}$. Then, by L being finitary, it is not hard to conclude that there exists an isomorphism $\beta: L^\omega \mathbf{0} \rightarrow LL^\omega \mathbf{0}$, so we can set $\Phi_L := L^\omega \mathbf{0}$ and $\alpha_L := \beta^{-1}$ for the initial L -algebra.

Definition 5.2. The **semantic interpretation** $\llbracket - \rrbracket_{(X, \xi)}$ of a language (Φ_L, α_L) in a T -coalgebra (X, ξ) is the unique L -algebra homomorphism from (Φ_L, α_L) to the L -algebra $(PX, P\xi \circ \delta)$.

The **logic** of (L, T, δ) refers to the language of L and its semantic interpretation. The interpretation $\llbracket - \rrbracket$ maps a formula φ , as an element of Φ , to the subset of X consisting of those states that satisfy φ . For example, if the syntax functor is induced by a set $\lambda \in \Lambda$ of (unary) predicate liftings, then $\alpha(\lambda, \varphi)$ represents the modal formula $\lambda\varphi$. The semantics $\llbracket \lambda\varphi \rrbracket$ on a T -coalgebra (X, ξ) is given by the diagram

$$\begin{array}{ccccc} L\Phi & \xrightarrow{\alpha} & \Phi & & \\ \downarrow L[-] & & \downarrow \llbracket - \rrbracket & & \\ LPX & \xrightarrow{\delta_X} & PTX & \xrightarrow{P\xi} & PX \end{array}$$

so that $\llbracket \lambda\varphi \rrbracket = (P\xi \circ \delta_X \circ L[-])(\lambda, \varphi) = P\xi(\lambda_X \llbracket \varphi \rrbracket)$. By definition, *soundness* is easy to see, since any rank-1 equation encoded in the syntax functor L is valid in the language (Φ_L, α_L) .

Definition 5.3. The **theory map** for a T -coalgebra (X, ξ) is the transpose $th: X \rightarrow S\Phi$ of the semantic interpretation $\llbracket - \rrbracket$ under the dual adjunction $S \dashv P$.

Intuitively, the theory map simply maps every state x to the collection of formulae satisfied by x , which is indeed the case for, say, the dual adjoints in Example 2.2.

Assuming that \mathcal{X} has kernel pairs, define **logical equivalence** of a theory map th_ξ to be the kernel of th_ξ ; a logic of (L, T, δ) may then be said to be **expressive** if the logical equivalence is contained in some kernel of a coalgebra homomorphism regarded as an \mathcal{X} -morphism. For concrete categories, such as **Set** and **Meas**, two elements x and y of a coalgebra (X, ξ) are logically equivalent if $th_\xi(x) = th_\xi(y)$; two elements are behaviourally equivalent if there exists a coalgebra homomorphism f with $fx = fy$, and thus a logic is expressive if logically equivalent elements are also behaviourally equivalent.³

It is known that behavioural equivalence implies logical equivalence, i.e., *adequacy* holds. The converse, *expressiveness*, is more interesting, and we turn to it next.

5.2 Expressiveness

We have shown that every equational one-step semantics has a unique translation to the full equational one-step semantics (Corollary 4.8), which is in fact the

³The justification of this point-free formulation may be found in [6, Section 4.1.5] (it assumes the existence of a proper factorisation system). In the following we will simply assume that \mathcal{X} is concrete though this does not necessarily imply the existence of kernel pairs.

most expressive logic, so its expressiveness is equivalent to the existence of an expressive logic of some one-step semantics:

Theorem 5.4. *Every morphism in CoLog_T preserves expressiveness.*

Recall Klin’s general condition for expressiveness in the functorial framework (in the formulation by Jacobs and Sokolova [11]):

Theorem 5.5 (see [14, 11]). *Suppose that \mathcal{X} has a proper factorisation system $(\mathcal{E}, \mathcal{M})$. Then, if a) T preserves \mathcal{M} -morphisms and b) the mate δ^* is a pointwise \mathcal{M} -morphism, then $th(x) = th(y)$ implies that x and y are behaviourally equivalent.*

Note that this result only gives sufficient conditions for expressiveness, but on the positive side, these conditions are particularly suitable for further generalisation. Hence, in the presence of a proper factorisation system, we define an equational one-step semantics of T to be **one-step expressive** if a) T preserves \mathcal{M} -morphisms, and b) the mate δ^* is a pointwise \mathcal{M} -morphism.

5.3 Expressiveness for finitary functors

We restrict attention to strongly locally finitely presentable categories:

Definition 5.6. (see [2]) A locally finitely presentable category is **strongly locally finitely presentable** if for every cofiltered limit $(\sigma_i: Y \rightarrow Y_i)_{i \in I}$ and every monomorphism $f: X \hookrightarrow Y$ with X finitely generated, there is $i \in I$ such that the composite $\sigma_i \circ f$ is monic.

For example, Set , Pos , and Vec_K are strongly locally finitely presentable. Klin showed [14] that when \mathcal{A} is a locally finitely presentable category the full *finitary* one-step semantics⁴ of a finitary functor on a strongly locally presentable category is one-step expressive if the counit $\epsilon: \mathcal{I} \rightarrow \mathcal{SP}$ is pointwise monic. We adapt Klin’s argument and apply it to the full *equational* one-step semantics in the case that \mathcal{A} is a *variety*. By Theorem 5.4, the equational version also recovers the finitary one.

Theorem 5.7 (c.f. [14, Theorem 4.4]). *Let \mathcal{X} be a strongly locally presentable category, \mathcal{A} a variety, and $T: \mathcal{X} \rightarrow \mathcal{X}$ a finitary and monomorphism-preserving functor. If the counit $\epsilon: \mathcal{I} \rightarrow \mathcal{SP}$ is pointwise monic, then the full equational one-step semantics of T is one-step expressive.*

Proof. Proof sketch Klin’s theorem for full finitary one-step semantics is established in two steps. Let \mathcal{A}_ω be the full (small) subcategory on finitely presentable objects. First it is shown that if for every A the source

$$\{TSf: TSA \rightarrow TSA_i\}_{f \in (\mathcal{A}_\omega \downarrow A)} \quad (5)$$

⁴This terminology is defined analogously: A one-step semantics is **finitary** if its syntax functor is finitary on a locally finitely presentable category. The **full finitary one-step semantics** for T is the terminal object in the category of finitary one-step semantics for T by Proposition 3.19.

is jointly monic, then the mate δ^* is pointwise monic where $(\mathcal{A}_\omega \downarrow A)$ is the comma category from \mathcal{A}_ω to A . Second, the family (5) is shown to be jointly monic by the strong local presentability.

For the same reason, to show that δ^* is pointwise monic, it suffices to show that the source

$$\{Tsg: TSA \rightarrow TSFn\}_{g \in (\mathcal{A}_\omega \downarrow A)} \quad (6)$$

is jointly monic. However, every morphism $(Fn \xrightarrow{g} A)$ factors through a regular epimorphism $e: Fn \rightarrow B$ with B finitely presentable, and TSe is a monomorphism by the dual adjunction and assumption. It is easy to see that (6) is jointly monic if and only if (5) is jointly monic. Then Klin's second step completes this proof. \square

5.4 Modularity of one-step expressiveness

As we discussed colimits, finite products, and compositions on \mathbf{CoLog} , it is of interest to know if one-step expressiveness is stable under these constructions at this level of generality. Surprisingly, compositions and colimits preserve one-step expressiveness in a straightforward way:

Theorem 5.8. *The composite $\delta_1 \otimes \delta_2$ of one-step expressive semantics (L_1, T_1, δ_1) and (L_2, T_2, δ_2) is one-step expressive for $T_1 T_2$.*

Proof. By Remark 3.17, the mate of $\delta_1 \otimes \delta_2$ is equal to $\delta_1^* L_2 \circ T_1 \delta_2^*$. By assumption, T_i preserves \mathcal{M} -morphisms and δ_i^* is a pointwise \mathcal{M} -morphism for $i = 1, 2$. Hence the composite $\delta_1^* L_2 \circ T_1 \delta_2^*$ is a pointwise \mathcal{M} -morphism. By assumption, $T_1 T_2$ preserves \mathcal{M} -morphisms. \square

Example 5.9. The double finite powerset functor $\mathbb{P}_\omega \circ \mathbb{P}_\omega$ does not have a separating set of unary predicate liftings [26]. However, we may simply self-compose the usual one-step semantics for \mathbb{P}_ω to obtain a one-step expressive logic.

Theorem 5.10. *The pointwise colimit of one-step expressive semantics is one-step expressive.*

Proof. Let D be a diagram in \mathbf{CoLog} . The colimit of D is a one-step semantics of type $\text{Lim}_i T_i$. By assumption, each T_i preserves \mathcal{M} -morphisms and by Proposition 3.19, \mathcal{M} -morphisms are closed under limits in $\mathcal{X}^{\rightarrow}$, so $\text{Lim} T_i$ preserves \mathcal{M} -morphisms.

The mate of $(\text{Colim} D)$ is a pointwise \mathcal{M} -morphism: Every D_i^* is a pointwise \mathcal{M} -morphism by assumption, so the limit of D_i^* in the arrow category $\mathcal{X}^{\rightarrow}$ is also a pointwise \mathcal{M} -morphism. By Remark 3.17, $\text{Lim}(D_i^*)$ is isomorphic to $(\text{Colim} D_i)^*$ and the latter is a pointwise \mathcal{M} -morphism since \mathcal{M} contains isomorphisms. \square

Example 5.11. (Labelling T^A) Suppose that \mathcal{X} and \mathcal{A} have products and coproducts, respectively. Let A be a set of labels. Every coalgebra $\xi: X \rightarrow T^A X$

for the A -fold product of T corresponds to a family $(\xi_a)_{a \in A}$ of T -coalgebras, i.e. an A -labelled T -coalgebra, and the A -fold coproduct of a one-step semantics (L, T, δ) defines a one-step semantics for T^A . Moreover, the coproduct is one-step expressive if and only if (L, T, δ) is one-step expressive. The result applies immediately to \mathbb{P} , \mathcal{D} , the convex powerset functor $\hat{\mathbb{P}}$, and the Giry monad \mathcal{G} , to name but a few.

As for finite products, we are encouraged by the result of Cîrstea's and Pattinson [7] that one-step expressiveness is preserved by finite products for one-step semantics over 2^- , but we do not have a general proof at this point.

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