

Bitopology and four-valued logic

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Abstract

Bilattices and d-frames are two different kinds of structures with a four-valued interpretation. Whereas d-frames were introduced with their topological semantics in mind, the theory of bilattices has a closer connection with logic. We consider a common generalisation of both structures and show that this not only still has a clear bitopological semantics, but that it also preserves most of the original bilattice logic. Moreover, we also obtain a new bitopological interpretation for the connectives of four-valued logic.

Keywords: Bilattices, d-frames, nd-frames, bitopological spaces, four-valued logic.

1 Introduction

In 1977, Nuel D. Belnap [5] gave a philosophical justification for distinguishing between two orders when studying information systems: the information order and the logical order. He also suggested that in addition to the classical logical values

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true and false (denoted tt and ff) it would also be useful to have the values \top and \perp for information-wise maximum and minimum, corresponding to the situation when there is contradicting information and no information, respectively.

Belnap's insights and then Ginsberg's application of those ideas to inference systems [9] motivated Arieli and Avron to develop a formal logical system and to analyse its algebraic models – bilattices [3]. Bilattices proved to be also useful in other areas such as in logic programming [7], algebra [11], and abstract algebraic logic [14].

Later on Jung and Moshier studied bitopological spaces (or *bispaces* for short) to clarify the interplay of various topologies arising in domain theory. They discovered that bispaces actually also give a very natural semantics to a four-valued logic. The fact that the first topology can represent the observably true predicates and the second the observably false ones gives a good four-valued interpretation/reading and this is even more transparent in the algebraic duals of bitopological spaces, dubbed *d-frames* in [10].

Whereas d-frames were introduced with their topological semantics in mind, the theory of bilattices has a closer connection with logic. Because of this, one might not expect many similarities between both theories but the discovery of the so-called *twist-representation* of bilattices [6] shows that quite the opposite is the case.

In this paper, we are trying to tackle the obvious question of whether there is a reasonable generalisation of both theories that would give us some better insight into the similarities and differences between bilattices and d-frames. We claim that the answer to this question is yes. As a starting point we take d-frames and we will show that they can be very naturally extended into a new structure which we call *nd-frames*. It seems that this way we get the best from both worlds: We have a clear bitopological semantics while still preserving most of the original logic of bilattices. Moreover, in the nd-frame context the negation of four-valued logic has a new and clear bitopological realisation via interior operators.

This paper contributes to both the study of bilattices and the study of d-frames. For the former, it shows how to generalise bilattices to get four-valued structures where the components are not isomorphic. Contributions to d-frame theory are by giving an explanation of proof-theoretic negation and, moreover, extending this negation to the whole d-frame. By this we also show another connection between geometry (interior operators) and proof theory (cut rules). Moreover, nd-frames allow a finer distinction of bispaces as the class of their spectra is broader than the class of spectra of d-frames. Having a generalisation of both structures allows us to compare partial implication in d-frames with the implication of bilattices and to show that the former is much stronger than latter.

2 Preliminaries

Below we give very brief presentations of the two types of structures that this paper aims to combine, *bilattices* and *d-frames*. As will soon become clear, much of the underlying structure is *symmetric* with respect to a positive and a negative part; we will respect this when stating a definition or a proposition but will generally restrict proofs to one variant without further comment.

2.1 Bilattices

Bilattices are the algebraic manifestation of Belnap’s “useful four-valued logic”, [5]. One key feature of this logic is *paraconsistency*, [13], which means that it is *not* possible to derive an arbitrary proposition from a contradiction. Secondly, the logic is *truth-functional* in that every connective is characterised by its behaviour on the set of (four) truth values.

Traditionally, *bilattices* are presented as structures of the type $(A; \wedge, \vee, \sqcap, \sqcup, ff, tt, \perp, \top, \neg, \supset)$ satisfying a list of axioms. However, a decomposition theorem can be shown for them (see [4,14,6]) and it is more straightforward to approach the subject from the characterisation that results from it.⁵

Let $H = (H; \wedge, \vee, 1, 0, \rightarrow)$ be a Heyting algebra. On $H \times H$ one defines the bilattice operations by setting, for $\alpha = (\alpha_+, \alpha_-)$, $\beta = (\beta_+, \beta_-) \in H \times H$:⁶

$$\begin{aligned} \alpha \vee \beta &\stackrel{\text{def}}{=} (\alpha_+ \vee \beta_+, \alpha_- \wedge \beta_-), & \alpha \wedge \beta &\stackrel{\text{def}}{=} (\alpha_+ \wedge \beta_+, \alpha_- \vee \beta_-), \\ \alpha \sqcup \beta &\stackrel{\text{def}}{=} (\alpha_+ \vee \beta_+, \alpha_- \vee \beta_-), & \alpha \sqcap \beta &\stackrel{\text{def}}{=} (\alpha_+ \wedge \beta_+, \alpha_- \wedge \beta_-), \\ ff &\stackrel{\text{def}}{=} (0, 1), & tt &\stackrel{\text{def}}{=} (1, 0), & \perp &\stackrel{\text{def}}{=} (0, 0), & \top &\stackrel{\text{def}}{=} (1, 1). \end{aligned}$$

The two final operations deserve to be highlighted: *Negation* \neg is defined purely by the exchange of components:

$$\neg \alpha \stackrel{\text{def}}{=} (\alpha_-, \alpha_+)$$

and without reference to the internal logical structure of the component Heyting algebra. *Weak implication* \supset is the only *non-symmetric* operation in the signature and it is in fact a remarkable feature of bilattice logic that it can be given in the following way at all:

$$\alpha \supset \beta \stackrel{\text{def}}{=} (\alpha_+ \rightarrow \beta_+, \alpha_+ \wedge \beta_-)$$

The set $H \times H$ together with the four constants and six operations defined above is called the *twist-construction* over H and denoted by H^{\bowtie} . As we said above, the characterisation theorem states that, up to isomorphism, every bilattice arises in this way.

Notice that the “logical” reduct $(H \times H; \wedge, \vee, ff, tt)$ and the “informational” reduct $(H \times H; \sqcap, \sqcup, \perp, \top)$ are automatically bounded distributive lattices. The associated orders, \leq and \sqsubseteq , however, are not the same; they may be (loosely) said to be “at 90° to each other”, and this helps to explain that negation \neg is antitone w.r.t. the logical order and monotone w.r.t. the informational one.

2.2 D-frames

The motivation for d-frames comes from semantics, in particular, from the observation that “domains” (in the sense of Scott) carry two topologies which are loosely

⁵ The decomposition theorem is surprisingly robust; very little of the structure of bilattices is required. While this is an intriguing aspect of the theory, it has also led to a proliferation of terminology, not all of which is universally accepted. Our choice of “bilattice” for the purposes this paper is really just for brevity and simplicity as we will have no need to consider any variations in the axiomatisation.

⁶ Note the overloading of \wedge and \vee as operations on both the Heyting algebra and the bilattice. We hope that the context will always make clear what we are referring to.

connected and in some sense complementary to each other, the *Scott-topology* and the *weak lower topology*, [2,8]. Smyth, [15], proposed to interpret open sets as propositions of an “observational logic”, and Abramsky fully developed this programme in his celebrated “Domain Theory in Logical Form”, [1], but both these works focus entirely on the Scott-topology which begged the question what the logical status of the other topology might be. Taking a step back from domain theory, [10] began the exploration of bitopological spaces under Smyth’s interpretation. D-frames are the result of this investigation.

A *d-frame*⁷ is a structure $\mathcal{L} = (L_+ \times L_-; \text{con}, \text{tot})$ where $(L_+; \vee, \wedge, 0, 1)$ and $(L_-; \vee, \wedge, 0, 1)$ are frames⁸ and the *consistency* $\text{con} \subseteq L_+ \times L_-$ and *totality* $\text{tot} \subseteq L_+ \times L_-$ predicates satisfy the following axioms, for all $\alpha, \beta \in L_+ \times L_-$:

$$\begin{aligned}
 (\text{con-}\downarrow) \quad & \alpha \in \text{con} \text{ and } \beta \sqsubseteq \alpha \implies \beta \in \text{con}, \\
 (\text{tot-}\uparrow) \quad & \alpha \in \text{tot} \text{ and } \beta \sqsupseteq \alpha \implies \beta \in \text{tot}, \\
 (\text{con-}\wedge, \vee) \quad & \alpha, \beta \in \text{con} \implies \alpha \vee \beta \in \text{con} \text{ and } \alpha \wedge \beta \in \text{con}, \\
 (\text{tot-}\wedge, \vee) \quad & \alpha, \beta \in \text{tot} \implies \alpha \vee \beta \in \text{tot} \text{ and } \alpha \wedge \beta \in \text{tot}, \\
 (\text{con}, \text{tot-}tt, ff) \quad & tt \in \text{con} \text{ and } tt \in \text{tot}, \quad ff \in \text{con} \text{ and } ff \in \text{tot}, \\
 (\text{con-tot}) \quad & \alpha \in \text{con}, \beta \in \text{tot} \text{ and } (\alpha \sqcup \beta = \alpha \wedge \beta \text{ or } \alpha \sqcup \beta = \alpha \vee \beta) \implies \alpha \sqsubseteq \beta \\
 (\text{con-}\sqcup\uparrow) \quad & A \subseteq \text{con} \text{ and } A \text{ is } \sqsubseteq\text{-directed} \implies \sqcup\uparrow A \in \text{con}.
 \end{aligned}$$

where $\wedge, \vee, \sqcap, \sqcup, ff, tt, \perp, \top$ and the induced logical order \leq and information order \sqsubseteq are defined the same way as in bilattices. In fact, the similarity with bilattices (presented as twist structures) is obvious and it may therefore be helpful to highlight the *differences*:

- In d-frames, the two component lattices may be different, in bilattices they are identical;
- (consequently) it is not possible to define negation or weak implication on d-frames in the same way as it is done for bilattices;
- frames are *complete* Heyting algebras (but frame homomorphisms may not preserve Heyting implication);
- the two predicates *con* and *tot* are *relational*, not *algebraic* structure.

These differences are also apparent in the definition of *d-frame homomorphism* which we take to be a pair of frame homomorphisms $h_+ : L_+ \rightarrow M_+, h_- : L_- \rightarrow M_-$ such that $h_+ \times h_- [\text{con}_{\mathcal{L}}] \subseteq \text{con}_{\mathcal{M}}$ and $h_+ \times h_- [\text{tot}_{\mathcal{L}}] \subseteq \text{tot}_{\mathcal{M}}$. We denote the category of d-frames and d-frame homomorphisms by **d-Frm**.

As we said before, d-frames arose from consideration of bitopological spaces and indeed, it is straightforward to adapt the open-set functor from spaces to frames to one from the category **biTop** of bispaces to **d-Frm**. To this end, we set $\Omega_d(X) = (\tau_+, \tau_-; \text{tot}_X, \text{con}_X)$ for a bispace $(X; \tau_+, \tau_-)$ where, for $U \in \tau_+$ and $V \in \tau_-$,

$$(U, V) \in \text{con}_X \stackrel{\text{def}}{\equiv} U \cap V = \emptyset \quad \text{and} \quad (U, V) \in \text{tot}_X \stackrel{\text{def}}{\equiv} U \cup V = X.$$

⁷ This definition of d-frames agrees with the definition of reasonable d-frames in [10].

⁸ *Frames* are complete lattices satisfying the equation: $b \wedge (\bigvee_i a_i) = \bigvee_i (b \wedge a_i)$. Frame homomorphisms are monotone maps distributing over all joins and finite meets. For more information see [12].

Example 2.1 The dual of the one-point bispaces $\mathbf{1} = (\{*\}; \tau, \tau)$ has exactly four elements: $\perp = (\emptyset, \emptyset)$, $\text{ff} = (\emptyset, \{*\})$, $\text{tt} = (\{*\}, \emptyset)$, and $\top = (\{*\}, \{*\})$. These are the truth values of bilattice logic and in that context the structure is usually denoted *FOUR*. What we are saying here is that *FOUR* is also a canonical d-frame with component frames $L_+ = L_- = \mathbf{2} = \{0 < 1\}$.

3 Nd-frames

We saw in the Preliminaries that, in order to define negation and implication for bilattices represented as twist-structures, we heavily use the fact that the carrier is the product of a Heyting algebra with itself. Therefore, we can freely send elements from one component of the product to the other.

Similarly to bilattices, d-frames are also formed of two components but those do not have to be the same and it seems that there are no natural order-preserving mappings between them (as required by the definition of \neg and \supset). For example, taking pseudocomplements⁹ is antitone and it could be used to define an operation sometimes called *conflation*, but this is known to be different from negation.

However, looking at the semantic counterparts of d-frames, i.e. bitopological spaces, suggests that we have very natural candidates for maps between both frames of open sets. Let $(X; \tau_+, \tau_-)$ be a bispaces; then assigning for every τ_+ -open (or τ_- -open) set its interior with respect to the other topology is a monotone map. Moreover, it also distributes over intersections. Let us denote those maps by $m: \tau_+ \rightarrow \tau_-$ and $p: \tau_- \rightarrow \tau_+$; to wit:

$$m: U \in \tau_+ \mapsto U^{\circ\tau_-} \in \tau_- \quad \text{and} \quad p: V \in \tau_- \mapsto V^{\circ\tau_+} \in \tau_+.$$

When translated to the language of d-frames, one can postulate the existence of maps $m: L_+ \rightarrow L_-$ and $p: L_- \rightarrow L_+$ satisfying the following axioms:

- (pm-1) $m(a \wedge b) = m(a) \wedge m(b)$, $p(a \wedge b) = p(a) \wedge p(b)$,
- (pm-2) $m(1) = 1$, $p(1) = 1$,
- (pm-3) $m(0) = 0$, $p(0) = 0$,
- (pm-4) $p \circ m \leq \text{id}$, $m \circ p \leq \text{id}$.

Also, the intuition of p and m being the interiors with respect to the other topology justifies the following axioms involving *con* and *tot*:

$$\frac{(a \wedge b, c) \in \text{con}}{(a, m(b) \wedge c) \in \text{con}} \text{ (con-}m\text{)} \quad \frac{(a, b \wedge c) \in \text{con}}{(a \wedge p(b), c) \in \text{con}} \text{ (con-}p\text{)}$$

$$\frac{(a, m(b) \vee c) \in \text{tot}}{(a \vee b, c) \in \text{tot}} \text{ (tot-}m\text{)} \quad \frac{(a \vee p(b), c) \in \text{tot}}{(a, b \vee c) \in \text{tot}} \text{ (tot-}p\text{)}$$

Definition 3.1 $(L_+ \times L_-; \text{con}, \text{tot}; p, m)$ is an *nd-frame* if $(L_+, L_-; \text{con}, \text{tot})$ is a d-frame and all axioms for (p, m) mentioned above, i.e. (pm-1), (pm-2), (pm-3), (pm-4), (con- m), (con- p), (tot- m) and (tot- p), are satisfied.

⁹ See Section 6 for the definition.

Let us recall a known fact about continuous maps:

Lemma 3.2 *Let $f: X \rightarrow Y$ be a continuous map and let $M \subseteq Y$, then $f^{-1}[M^\circ] \subseteq (f^{-1}[M])^\circ$.*

This motivates the following definition:

Definition 3.3 A d-frame homomorphism $h: \mathcal{L} \rightarrow \mathcal{M}$ between two nd-frames is an *nd-frame homomorphism* if $h_+ \circ p \leq p \circ h_-$ and $h_- \circ m \leq m \circ h_+$.

Since the component maps are monotone, it follows that nd-frame homomorphisms are closed under composition. We denote the resulting category with **nd-Frm**.

Example 3.4 We have seen before that the bilattice \mathcal{FOUR} may alternately be viewed as a d-frame. The axioms (pm-1) – (pm-4) ensure that the identity function is the unique choice for both p and m so that $\mathcal{FOUR} = \mathbf{2} \times \mathbf{2}$ also qualifies as an nd-frame. Whether we view it as a bilattice, d-frame, or nd-frame, we always denote it with \mathcal{FOUR} .

The theory of d-frames works best when the two topologies complement each other, in the sense of the closed sets of one approximating the opens of the other. Examples of this are given by the Scott-topology and the weak lower topology considered in domain theory. If this is not the case, then the two relations **con** and **tot** tend to be trivial, by which we mean that $(a, b) \in \mathbf{con}$ iff one of two elements equals 0, and $(a, b) \in \mathbf{tot}$ iff one of them equals 1. For the new structure m and p , the situation is exactly reversed. To see this, consider the following two bispaces:

- (i) $X_1 = (\{a, b\}, \tau, \tau)$ where the only non-trivial open of τ is $\{a\}$.
- (ii) $X_2 = (\{aa, ab, ba, bb\}, \tau_+, \tau_-)$ where the only non-trivial open of τ_+ is $\{aa, ab\}$ and that of τ_- is $\{aa, ba\}$.

In both cases, **con** and **tot** are trivial, which means that $\Omega_d(X_1)$ and $\Omega_d(X_2)$ are isomorphic. On the other hand, the interior operators on X_1 are the identity whereas on X_2 they are trivial. Both bispaces will turn out to be nd-sober, whereas only X_2 is d-sober.

4 Logic of nd-frames

We introduced p and m to be able to define negation and implication for d-frames similarly to how they are defined for bilattices in their twist-structures representation. Let $(L_+ \times L_-; \mathbf{con}, \mathbf{tot}; p, m)$ be an nd-frame and define

$$\neg\varphi \stackrel{\text{def}}{=} (p(\varphi_-), m(\varphi_+)) \quad \text{and} \quad \varphi \supset \psi \stackrel{\text{def}}{=} (\varphi_+ \rightarrow \psi_+, m(\varphi_+) \wedge \psi_-).$$

Now any nd-frame \mathcal{L} gives rise to the same signature as bilattices: $(L_+ \times L_-; \wedge, \vee, \sqcap, \sqcup, \mathit{ff}, \mathit{tt}, \perp, \top, \neg, \supset)$. Let **Ln** be the language of bilattices and let **Fm(Ln)** be the term algebra of **Ln** (generated by countably many variables). Valuations are the **Ln**-homomorphisms $\mathbf{Fm}(\mathbf{Ln}) \rightarrow \mathcal{L}$. We can define (algebraic) semantic validity the same way as it was defined for bilattices [14]; we say φ *holds in* (or, *is*

valid in) \mathcal{L} , and write $\mathcal{L} \models \varphi$ iff

$$v(\varphi) = v(\varphi \supset \varphi) \text{ for all valuations } v: \text{Fm}(\mathbf{Ln}) \rightarrow \mathcal{L},$$

Before we get to the axioms let us first prove the following lemma.

Lemma 4.1 *The following holds in all nd-frames \mathcal{L} :*

- (L1) $\mathcal{L} \models \varphi$ iff $v(\varphi) \sqsupseteq \mathbf{tt}$ (or equivalently: $v(\varphi)_+ = 1$) for all valuations v into \mathcal{L} ;
- (L2) $\mathcal{L} \models \varphi \supset \psi$ iff $v(\varphi)_+ \leq v(\psi)_+$ for all valuations v into \mathcal{L} ; and
- (L3) $\mathcal{L} \models \varphi \equiv \psi$ iff $v(\varphi)_+ = v(\psi)_+$ for all valuations v into \mathcal{L} ,
where $\varphi \equiv \psi$ is a shorthand for $(\varphi \supset \beta) \wedge (\beta \supset \alpha)$.

Proof.

- (L1) Right-to-left implication: If $v(\varphi) = (1, a)$ then $v(\varphi \supset \varphi) = v(\varphi) \supset v(\varphi) = (1 \rightarrow 1, m(1) \wedge a) = (1, a) = v(\varphi)$. Reverse direction: $v(\varphi) = v(\varphi \supset \varphi) = v(\varphi) \supset v(\varphi)$ implies that the positive parts are equal and therefore we have $v(\varphi)_+ = v(\varphi \supset \varphi)_+ = 1$ and $v(\varphi) \sqsupseteq \mathbf{tt}$.
- (L2) From (L1) we know that $\mathcal{L} \models \varphi \supset \psi$ iff $v(\varphi \supset \psi)_+ = v(\varphi)_+ \rightarrow v(\psi)_+$ is equal to 1 for all valuations v , and this is true if and only if $v(\varphi)_+ \leq v(\psi)_+$.
- (L3) Follows from (L2) and from the fact that $\mathcal{L} \models v(\varphi \wedge \psi) \sqsupseteq \mathbf{tt}$ iff $\mathcal{L} \models v(\varphi) \sqsupseteq \mathbf{tt}$ and $\mathcal{L} \models v(\psi) \sqsupseteq \mathbf{tt}$. Then, $\mathcal{L} \models v(\varphi \equiv \psi)$ iff $\mathcal{L} \models \varphi \supset \psi$ and $\mathcal{L} \models \psi \supset \varphi$ iff $v(\varphi)_+ \leq v(\psi)_+$ and $v(\psi)_+ \leq v(\varphi)_+$ for all valuations v .

□

Arieli and Avron, [3], gave a Hilbert-style axiomatisation of a four-valued logic which is sound and complete with respect to bilattices. Here we show that a large part of their logic is still valid in nd-frames.

Theorem 4.2 *The following axioms of four-valued logic are valid in any nd-frame: (Weak implication)*

$$\begin{array}{ll}
 (\supset 1) & \varphi \supset (\psi \supset \varphi) \\
 (\supset 2) & (\varphi \supset (\psi \supset \gamma)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \gamma)) \\
 (\neg\neg R) & \neg\neg\varphi \supset \varphi \qquad\qquad\qquad (\star A)
 \end{array}$$

(Logical conjunction and disjunction)

$$\begin{array}{ll}
 (\wedge \supset) & (\varphi \wedge \psi) \supset \varphi \text{ and } (\varphi \wedge \psi) \supset \psi \\
 (\supset \wedge) & \varphi \supset (\psi \supset (\varphi \wedge \psi)) \\
 (\supset \mathbf{tt}) & \varphi \supset \mathbf{tt} \\
 (\supset \vee) & \varphi \supset (\varphi \vee \psi) \text{ and } \psi \supset (\varphi \vee \psi) \\
 (\vee \supset) & (\varphi \supset \gamma) \supset ((\psi \supset \gamma) \supset ((\varphi \vee \psi) \supset \gamma)) \\
 (\supset \mathbf{ff}) & \mathbf{ff} \supset \varphi
 \end{array}$$

(Informational conjunction and disjunction)

$$\begin{array}{ll}
 (\sqcap \supset) & (\varphi \sqcap \psi) \supset \varphi \text{ and } (\varphi \sqcap \psi) \supset \psi \\
 (\supset \sqcap) & \varphi \supset (\psi \supset (\varphi \sqcap \psi))
 \end{array}$$

$$\begin{aligned}
 (\supset \top) \quad & \varphi \supset \top \\
 (\supset \sqcup) \quad & \varphi \supset (\varphi \sqcup \psi) \text{ and } \psi \supset (\varphi \sqcup \psi) \\
 (\sqcup \supset) \quad & (\varphi \supset \gamma) \supset ((\psi \supset \gamma) \supset ((\varphi \sqcup \psi) \supset \gamma)) \\
 (\supset \perp) \quad & \perp \supset \varphi
 \end{aligned}$$

(Negation)

$$\begin{aligned}
 (\neg \wedge L) \quad & \neg(\varphi \wedge \psi) \subset \neg\varphi \vee \neg\psi & (\star B) \\
 (\neg \vee) \quad & \neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi \\
 (\neg \sqcap) \quad & \neg(\varphi \sqcap \psi) \equiv \neg\varphi \sqcap \neg\psi \\
 (\neg \sqcup L) \quad & \neg(\varphi \sqcup \psi) \subset \neg\varphi \sqcup \neg\psi & (\star B) \\
 (\neg \supset R) \quad & \neg(\varphi \supset \psi) \supset \varphi \wedge \neg\psi & (\star A)
 \end{aligned}$$

Furthermore, the rule of Modus Ponens is sound:

$$(MP) \quad \varphi, (\varphi \supset \psi) \vdash \psi$$

Proof. The axioms $(\supset 1)$, $(\supset 2)$, all the logical conjunction and disjunction axioms and the informational conjunction and disjunction axioms hold for the same reason. We know from Lemma 4.1 (L1) that only the first coordinate determines their validity. Moreover, those axioms do not contain negation and so they hold simply because, when projected to the first coordinate, they hold in all Heyting algebras (and therefore also in frames).

From (L2) we know that $(\neg \wedge L)$ is equivalent to $p(v(\varphi)_- \vee v(\psi)_-) \geq p(v(\varphi)_-) \vee p(v(\psi)_-)$ for all valuations v which is true since p is monotone. The same argument applies for $(\neg \sqcup L)$. From (L3) we know that $(\neg \vee)$ and $(\neg \sqcap)$ are equivalent to $p(v(\varphi)_- \wedge v(\psi)_-) = p(v(\varphi)_-) \wedge p(v(\psi)_-)$ for all valuations v and this is true simply because p preserves finite infima.

(L2) implies that $(\neg \supset R)$ is equivalent to $v(\neg(\varphi \supset \psi))_+ \leq v(\varphi \wedge \neg\psi)_+$ by expanding the definitions we get $(\neg v(\varphi \supset \psi))_+ = p(v(\varphi \supset \psi)_-) = p(m(v(\varphi)_+) \wedge v(\psi)_-) = p(m(v(\varphi)_+)) \wedge p(v(\psi)_-)$ which, by $p \circ m \leq \text{id}$, is less or equal to $v(\varphi)_+ \wedge p(v(\psi)_-) = v(\varphi \wedge \neg\psi)_+$ as we wanted.

For Modus Ponens, no matter if we interpret comma as \wedge or as \sqcap we get the requirement that $v(\varphi)_+ \wedge (v(\varphi)_+ \rightarrow v(\psi)_+) = 1$ should imply $v(\psi)_+ = 1$ which is again true for all Heyting algebras. \square

Remark 4.3 The axioms denoted by $(\star A)$ or $(\star B)$ are the only axioms that differ from the original axioms of bilattices because they are expressed as implications, whereas the original axioms are equivalences. Requiring equivalence instead of implication in the axioms marked by $(\star A)$ is equivalent to requiring that $p \circ m = \text{id}$ ¹⁰ and requiring equivalences for axioms marked by $(\star B)$ is the same as requiring that p preserves finite suprema. Also, in some presentations of bilattices, [3], the following axiom, called Peirce's law, is added

$$(\supset 3) \quad ((\varphi \supset \psi) \supset \varphi) \supset \varphi$$

Assuming this to hold is equivalent to assuming that L_+ is a Boolean frame.

¹⁰Notice that assuming $p \circ m = \text{id}$ implies that m is a one-one frame homomorphism and p is its right adjoint.

4.1 Implications and a cut rule

One of the nice properties of d-frames is that one can restrict one's attention to the set \mathbf{con} of consistent predicates without losing any expressivity, see [10, Proposition 7.4]. We can think of this structure as a semantics for predicates *without contradictions*. Moreover, there is a binary relation between the elements of \mathbf{con} which is in many ways similar to a consequence relation (see Section 7 of [10]). Define, for all $\alpha, \beta \in \mathbf{con}$,

$$\alpha \prec \beta \stackrel{\text{def}}{=} (\beta_+, \alpha_-) \in \mathbf{tot}.$$

Given the many similarities between d-frames and bilattices one wonders what exactly the relationship between \prec and \supset is. We would like to suggest that the right place to answer this question is within nd-frames as they generalise both notions. Indeed, we will see below that \prec is, from this perspective, much stronger than \supset .

Theorem 4.4 *Let \mathcal{L} be an nd-frame and let $\alpha, \beta \in \mathbf{con}$ such that $\alpha \prec \beta$. Then the following also hold*

- (i) $\alpha' \supset \beta' \sqsupseteq \mathbf{tt}$ for all $\alpha', \beta' \in \mathbf{con}$ with $\alpha' \sqsupseteq \alpha$ and $\beta' \sqsupseteq \beta$;
- (ii) $\neg\beta' \supset \neg\alpha' \sqsupseteq \mathbf{tt}$ for all $\alpha', \beta' \in \mathbf{con}$ with $\alpha' \sqsupseteq \alpha$ and $\beta' \sqsupseteq \beta$;

Remark: The logical conjunction of $\alpha \supset \beta$ with $\neg\beta \supset \neg\alpha$ is called *strong implication* in the bilattice logic literature.

Proof. We know from [10, Proposition 7.1(4)] that $\alpha \prec \beta$ implies $\alpha' \prec \beta'$ whenever $\alpha' \sqsupseteq \alpha$ and $\beta' \sqsupseteq \beta$ in \mathbf{con} , so it suffices to show the statements for α and β : By Lemma 4.1 (L2), $(\alpha \supset \beta) \sqsupseteq \mathbf{tt}$ iff $\alpha_+ \leq \beta_+$ and this follows, by (con-tot), from $(\beta_+, \alpha_-) \in \mathbf{tot}$ and $\alpha \in \mathbf{con}$. Similarly, since $(\neg\beta \supset \neg\alpha) = (p(\beta_-) \rightarrow p(\alpha_-), mp(\beta_-) \wedge m(\alpha_+))$, then $(\neg\beta \supset \neg\alpha) \sqsupseteq \mathbf{tt}$ iff $p(\beta_-) \rightarrow p(\alpha_-) = 1$ which is equivalent to $p(\beta_-) \leq p(\alpha_-)$. However, we know that $(\beta_+, \alpha_-) \in \mathbf{tot}$ and $\beta \in \mathbf{con}$, therefore, by (con-tot), $\beta_- \leq \alpha_-$ and, since p is monotone, also $p(\beta_-) \leq p(\alpha_-)$. \square

The (tot- m) and (tot- p) axioms give us immediately the following two rules combining strong implication and negation:

$$\frac{\alpha \prec \neg\beta \vee \gamma}{\alpha \wedge \beta \prec \gamma} \quad \text{and} \quad \frac{\alpha \wedge \neg\beta \prec \gamma}{\alpha \prec \beta \vee \gamma}$$

and from these the following cut rule follows by the transitivity of \prec [10, Proposition 7.1 (3)]:

$$\frac{\alpha \prec \neg\neg\beta \vee \gamma \quad \gamma \wedge \neg\neg\alpha' \prec \beta'}{\alpha \wedge \alpha' \prec \beta \vee \beta'}$$

5 Stone duality for nd-frames

5.1 Spectra of nd-frames

In this section we define a spectrum functor $\Sigma: \mathbf{nd-Frm} \rightarrow \mathbf{biTop}$ by extending the definition of the spectrum functor for d-frames $\Sigma_d: \mathbf{d-Frm} \rightarrow \mathbf{biTop}$ as presented

in [10]. Let $\mathcal{L} = (L_+ \times L_-; \mathbf{con}, \mathbf{tot}; p, m)$ be an nd-frame. Define $\Sigma(\mathcal{L})$ to be the bispace $(\Sigma(\mathcal{L}); \Phi_+[L_+], \Phi_-[L_-])$ where the underlying set $\Sigma(\mathcal{L})$ is the set of *nd-points*, that is, the pairs (F_+, F_-) where F_+ and F_- are complete prime filters of L_+ and L_- , respectively, such that, for all $\alpha \in L_+ \times L_-$,

$$\begin{aligned} (\mathbf{dp}_{\mathbf{con}}) \quad & \alpha \in \mathbf{con} \implies \alpha_+ \notin F_+ \text{ or } \alpha_- \notin F_-; \\ (\mathbf{dp}_{\mathbf{tot}}) \quad & \alpha \in \mathbf{tot} \implies \alpha_+ \in F_+ \text{ or } \alpha_- \in F_-; \\ (\mathbf{dp}_p) \quad & p(\alpha_-) \in F_+ \implies \alpha_- \in F_-; \\ (\mathbf{dp}_m) \quad & m(\alpha_+) \in F_- \implies \alpha_+ \in F_+. \end{aligned}$$

Equivalently, we can define the underlying set of $\Sigma(\mathcal{L})$ to be the set of all nd-frame homomorphisms from \mathcal{L} to \mathcal{FOUR} . The topologies of $\Sigma(\mathcal{L})$ are defined the same way as for spectra of d-frames. That is, $\Phi_+[L_+] = \{\Phi_+(a) : a \in L_+\}$ and $\Phi_-[L_-] = \{\Phi_-(b) : b \in L_-\}$ where

$$\Phi_+(a) = \{(F_+, F_-) \mid a \in F_+\} \quad \text{and} \quad \Phi_-(b) = \{(F_+, F_-) \mid b \in F_-\}.$$

Also, similarly to the d-frame spectrum functor, for every nd-frame homomorphism $h: \mathcal{L} \rightarrow \mathcal{M}$, set $\Sigma(h): \Sigma(\mathcal{M}) \rightarrow \Sigma(\mathcal{L})$ to be the map

$$\Sigma(h): (F_+, F_-) \longmapsto (h_+^{-1}[F_+], h_-^{-1}[F_-]).$$

Proposition 5.1 Σ is a contravariant functor from **nd-Frm** to **biTop**.

Proof. Σ is well defined on objects for the same reason as the corresponding functor for d-frames [10]. When we think of the nd-points of an nd-frame \mathcal{L} as nd-frame homomorphisms $\mathcal{L} \rightarrow \mathcal{FOUR}$ we see that Σ is also well defined on **nd-Frm** morphisms simply because nd-frame homomorphisms are closed under composition. \square

5.2 Nd-frames from bispaces

Let $X = (X; \tau_+, \tau_-)$ be a bispace. Set $\Omega(X) = (\tau_+, \tau_-; \mathbf{con}_X, \mathbf{tot}_X; p_X, m_X)$ where \mathbf{con}_X and \mathbf{tot}_X are as before and

$$m_X: U_+ \longmapsto U_+^{\circ\tau_-} \quad \text{and} \quad p_X: U_- \longmapsto U_-^{\circ\tau_+}.$$

Again, Ω acts on morphisms the same way as the d-frames analogue does, i.e. for a bicontinuous map $f: X \rightarrow Y$ set $\Omega(f): \Omega Y \rightarrow \Omega X$ to be the map

$$\Omega(f): (U_+, U_-) \longmapsto (f^{-1}[U_+], f^{-1}[U_-]).$$

Proposition 5.2 Ω is a contravariant functor from **biTop** to **nd-Frm**.

Proof. Ω is clearly well defined on objects. From the duality for d-frames, we know that the Ω -image of a bicontinuous map is a d-frame homomorphism. The fact that it is also an nd-frame homomorphism, that is $f(-\alpha) \sqsubseteq \neg(f\alpha)$ for all α , follows directly from Lemma 3.2. \square

5.3 Sobriety, spatiality and the adjunction

We say that a bispace X is *nd-sober* if there exists an nd-frame \mathcal{L} such that $X \cong \Sigma(\mathcal{L})$. We also have the usual embedding into the *sobrification* of a space (the *unit* of adjunction) $\eta_X: X \rightarrow \Sigma\Omega(X)$ defined as $x \mapsto (\mathcal{U}_+(x), \mathcal{U}_-(x))$ where $\mathcal{U}_+(x)$ and $\mathcal{U}_-(x)$ are the neighbourhood filters in τ_+ and τ_- , respectively. For the same reason as in the case of d-frames, η_X is natural in X .

Theorem 5.3 *For a bitopological space X , the following are equivalent:*

- (i) X is nd-sober;
- (ii) X is bihomeomorphic to $\Sigma\Omega(X)$;
- (iii) The unit map η_X is a bihomeomorphism;
- (iv) The unit map η_X is a bijection.

The reader may now check that the two examples we gave earlier (at the end of Section 3) are indeed nd-sober.

We say that an nd-frame \mathcal{L} is *spatial* if there exists a bitopological space X such that $\mathcal{L} \cong \Omega(X)$. Again, similarly to d-frame theory, we have the (*co-unit*) map $\epsilon_{\mathcal{L}}: \mathcal{L} \rightarrow \Omega\Sigma(\mathcal{L})$ defined as $(a, b) \mapsto (\Phi_+(a), \Phi_-(b))$. This, again, is natural in \mathcal{L} .

Theorem 5.4 *For an nd-frame \mathcal{L} , the following are equivalent:*

- (i) \mathcal{L} is spatial.
- (ii) $\mathcal{L} \cong \Omega\Sigma(\mathcal{L})$.
- (iii) The co-unit $\epsilon_{\mathcal{L}}$ is an isomorphism.
- (iv) The co-unit $\epsilon_{\mathcal{L}}$ is injective, reflects **con** and **tot**, and $\epsilon_{\mathcal{L}}^{-1}(-\alpha) \sqsubseteq -\epsilon_{\mathcal{L}}^{-1}(\alpha)$ for all $\alpha \in \Omega\Sigma(\mathcal{L})$.
- (v) \mathcal{L} satisfies the following conditions:

$$\begin{aligned}
 (s_+) \quad & \forall x \not\leq x' \in L_+ \exists (F_+, F_-) \in \Sigma(\mathcal{L}). x \in F_+, x' \notin F_+; \\
 (s_-) \quad & \forall y \not\leq y' \in L_- \exists (F_+, F_-) \in \Sigma(\mathcal{L}). y \in F_-, y' \notin F_-; \\
 (s_{\text{con}}) \quad & \forall \alpha \notin \text{con} \exists (F_+, F_-) \in \Sigma(\mathcal{L}). \alpha_+ \in F_+, \alpha_- \in F_-; \\
 (s_{\text{tot}}) \quad & \forall \alpha \notin \text{tot} \exists (F_+, F_-) \in \Sigma(\mathcal{L}). \alpha_+ \notin F_+, \alpha_- \notin F_-; \\
 (s_p) \quad & \forall a \not\leq p(x) \in L_+ \exists (F_+, F_-) \in \Sigma(\mathcal{L}). a \in F_+, x \notin F_-; \\
 (s_m) \quad & \forall b \not\leq m(y) \in L_- \exists (F_+, F_-) \in \Sigma(\mathcal{L}). y \notin F_+, b \in F_-.
 \end{aligned}$$

Corollary 5.5 *Ω and Σ form a (dual) adjunction with η and ϵ being the unit and co-unit, respectively. Moreover, the restriction of Σ and Ω to spatial nd-frames and nd-sober bispaces, respectively, forms a duality of categories.*

It is a good sign that extending Stone duality for d-frames to nd-frames is quite straightforward as it points towards the robustness of the theory. All the additional assumptions make sense topologically given that p and m should correspond to interior operators. The only difference with d-frame duality is that we added the conditions (dp_p) and (dp_m) . Similarly, in the characterisation of spatial nd-frames we needed to assume (s_p) and (s_m) in addition to the original conditions for spatial d-frames.

On the other hand, the language of nd-frames is definitely more expressive in

the sense that more bispaces are nd-sober than d-sober. We gave an example of this at the end of Section 3.

6 Canonical (p, m)

Every d-frame can be turned into an nd-frame in a trivial way, just augment $(L_+ \times L_-; \text{con}, \text{tot})$ with p^{triv} and m^{triv} where p^{triv} and m^{triv} are trivial in the sense that they send 1 to 1 and everything else to 0. It is easy to see that this construction provides a left adjoint to the forgetful functor from **nd-Frm** to **d-Frm** that erases p and m .

The purpose of this section is to demonstrate that under mild conditions on a d-frame, a more interesting choice for p and m is available, one which interacts well with Stone duality. For motivation we begin by reviewing the notion of *regularity* for d-frames.

For a d-frame $\mathcal{L} = (L_+ \times L_-; \text{con}, \text{tot})$ and for $c, a \in L_+$ we say that c is *well-inside* a (and write $c \triangleleft_+ a$) if there exists a $d \in L_-$ such that $(c, d) \in \text{con}$ and $(a, d) \in \text{tot}$. We define $c \triangleleft_- a$ for $c, a \in L_-$ dually, that is, with the roles of L_+ and L_- switched. We say that \mathcal{L} is *d-regular* if

$$a = \bigvee \{c \in L_+ \mid c \triangleleft_+ a\} \quad \text{and} \quad b = \bigvee \{c \in L_- \mid c \triangleleft_- b\}$$

for all $a \in L_+$ and $b \in L_-$. Finally, we say that a bitopological space X is *d-regular* if $\Omega_d(X)$ is. Note that the well-inside relation has a clear bitopological reading. For $U, V \in \tau_+$, $U \triangleleft_+ V$ just means that τ_- -closure of U is a subset of V .

We can express the interior operations of d-regular bispaces explicitly in the language of d-frames. Indeed, let $X = (X; \tau_+, \tau_-)$ be d-regular. Then, for a $U \in \tau_+$,

$$U^{\circ\tau_-} = \bigcup \{V \in \tau_- \mid \exists V' \in \tau_+. V \cap V' = \emptyset \text{ and } V' \cup U = X\}$$

Let $\mathcal{L} = \Omega_d(X)$, then the term above becomes, for an $a \in L_+$,

$$m^r(a) = \bigvee \{x_- \in L_- \mid \exists x_+ \in L_+. (x_+, x_-) \in \text{con} \text{ and } x_+ \vee a = 1\}.$$

Notice the similarity of the relationship between x_- and a in this definition, and the well-inside relation defined above, except that here it is between elements from the two *different* components of \mathcal{L} . Also note that the definition of $m^r(a)$ does not presuppose regularity of the underlying d-frame.

To simplify the definition of m^r a bit further, recall for any $x \in L_-$ the *pseudo-complement* x^* of x is defined as $\bigvee \{c \in L_+ \mid (c, x) \in \text{con}\}$. This allows us to define our candidate interior operators as follows

$$m^r(a) = \bigvee \{x \in L_- \mid x^* \vee a = 1\} \quad \text{and} \quad p^r(b) = \bigvee \{x \in L_+ \mid x^* \vee b = 1\}$$

and to prove some of the required properties. To begin we see that $m^r(1) = \bigvee \{x \mid x^* \vee 1 = 1\} \geq \bigvee \{1\} = 1$. For the preservation of 0 recall that $x^* = 1$ implies $x = 0$ because $(x^*, x) = (1, x) \in \text{con}$ and $(1, 0) \in \text{tot}$ gives, by (con-tot), $x \leq 0$. Therefore $m^r(0) = \bigvee \{x \mid x^* \vee 0 = 1\} = \bigvee \{0\} = 0$.

It is clear that m^r is monotone, so in order to show that it preserves binary meets, it suffices to check that $m^r(a \wedge a') \geq m^r(a) \wedge m^r(a')$ for all $a, a' \in L_+$. Assume $x^* \vee a = 1$ and $x'^* \vee a' = 1$, then because pseudocomplement is antitone, we have for $x'' = x \wedge x'$, $x''^* \vee a = 1$ and $x''^* \vee a' = 1$ from which it follows that $x''^* \vee (a \wedge a') = 1$ and hence $x'' \leq m^r(a \wedge a')$. Frame distributivity now allows us to conclude the desired inequality.

We are also able to show (con- m). For this let $(a \wedge b, c) \in \text{con}$ and let $x \in L_-$ be such that $x^* \vee b = 1$. Since, $(x^*, x) \in \text{con}$, by (con- \vee), we have that $(a \wedge b, c) \vee (x^*, x) \in \text{con}$. Therefore, since con is \sqsubseteq -downwards closed,

$$(a \wedge b, c) \vee (x^*, x) = ((a \vee x^*) \wedge (b \vee x^*), c \wedge x) \sqsupseteq (a, c \wedge x) \in \text{con}$$

where the inequality follows from $x^* \vee b = 1$. Since $(a, c \wedge x) \in \text{con}$ for all x such that $x^* \vee b = 1$, then by (con- \bigsqcup^\uparrow) and frame distributivity we get that also $(a, c \wedge m^r(b)) = (a, c \wedge \bigvee\{x \mid x^* \vee b = 1\}) \in \text{con}$.

However, we can show neither (pm-4) nor (tot- p) at this level of generality. To make progress, recall the following two *infinitary cut rules* for d-frames (already discussed in [10], but not part of the definition of d-frames):

$$\frac{(x, y \vee \bigvee_{i \in I} b_i) \in \text{tot}, \forall i \in I. (x \vee a_i, y) \in \text{tot}, (a_i, b_i) \in \text{con}}{(x, y) \in \text{tot}} \text{ (CUT}_r\text{)}$$

$$\frac{(x \vee \bigvee_{i \in I} a_i, y) \in \text{tot}, \forall i \in I. (x, y \vee b_i) \in \text{tot}, (a_i, b_i) \in \text{con}}{(x, y) \in \text{tot}} \text{ (CUT}_l\text{)}$$

These two rules are precisely what we need to complete our construction:

Proposition 6.1 *Let $\mathcal{L} = (L_+ \times L_-; \text{con}, \text{tot})$ be a d-frame satisfying the infinitary cut rules. Then, $(\mathcal{L}; p^r, m^r) = (L_+ \times L_-; \text{con}, \text{tot}; p^r, m^r)$ is an nd-frame.*

Proof. Only (pm-4) and (tot- p) remain to be shown. The former says that $m^r p^r(b) \leq b$ for all $b \in L_-$. Since

$$m^r p^r(b) = \bigvee\{y \in L_- \mid y^* \vee p^r(b) = 1\},$$

it is enough to show that every $y \in L_-$, such that $y^* \vee p^r(b) = 1$, is less or equal to b . From the definition of p^r we have $(1, 0) = (y^* \vee \bigvee\{x \mid x^* \vee b = 1\}, 0) \in \text{tot}$, therefore, from (tot- \uparrow), we get

- (1a) $(y^* \vee \bigvee\{x \mid x^* \vee b = 1\}, b) \in \text{tot}$
- (2a) for all x such that $x^* \vee b = 1$: $(y^*, x^* \vee b) \in \text{tot}$
- (3a) $(x, x^*) \in \text{con}$.

By applying (CUT $_l$) to (1a), (2a) and (3a) we obtain $(y^*, b) \in \text{tot}$, and from (con-tot) that $y \leq b$ as we wanted.

To show (tot- p), let $(a \vee p^r(b), c) \in \text{tot}$. Again, by unwrapping the definitions we get $(a \vee \bigvee\{x \mid x^* \vee b = 1\}, c) \in \text{tot}$ and this, by (tot- \uparrow), gives us

- (1b) $(a \vee \bigvee\{x \mid x^* \vee b = 1\}, b \vee c) \in \text{tot}$
- (2b) for all x such that $x^* \vee b = 1$: $(a, x^* \vee b \vee c) \in \text{tot}$.

$$(3b) \quad (x, x^*) \in \mathbf{con}.$$

Therefore, (CUT_I) applied to (1b), (2b) and (3b) gives us that $(a, b \vee c) \in \mathbf{tot}$. \square

Proposition 6.2 *The mapping $\mathbf{N}: \mathbf{d}\text{-Frm}_{\text{CUT}} \rightarrow \mathbf{nd}\text{-Frm}$ assigning $\mathcal{L} \mapsto (\mathcal{L}; p^r, m^r)$ is functorial, where $\mathbf{d}\text{-Frm}_{\text{CUT}}$ is the category of d-frames satisfying the infinitary cut rules.*

Proof. \mathbf{N} is well defined on objects by Proposition 6.1. For morphisms, let $h: \mathcal{L} \rightarrow \mathcal{M}$ be a d-frame homomorphism between two d-frames that satisfy the infinitary cut rules. We need to show that $h(-\alpha) \sqsubseteq \neg h(\alpha)$ for all $\alpha \in L_+ \times L_-$. From the definition we see that the corresponding plus coordinates are computed as follows:

$$(h(-\alpha))_+ = h_+(p^r(\alpha_-)) = h_+(\bigvee\{x \mid x^* \vee \alpha_- = 1\}) = \bigvee\{h_+(x) \mid x^* \vee \alpha_- = 1\}$$

and

$$(\neg h(\alpha))_+ = p^r(h_-(\alpha_-)) = \bigvee\{w \mid w^* \vee h_-(\alpha_-) = 1\}.$$

It is sufficient to show that $x^* \vee \alpha_- = 1$ implies $h_+(x)^* \vee h_-(\alpha_-) = 1$. This is true because from $(x, x^*) \in \mathbf{con}$ we get that $(h_+(x), h_-(x^*)) \in \mathbf{con}$ and hence $h_+(x)^* \geq h_-(x^*)$. Therefore, by applying the frame homomorphism h_- to $x^* \vee \alpha_- = 1$ we obtain $h_-(x^*) \vee h_-(\alpha_-) = 1$ and this implies $h_+(x)^* \vee h_-(\alpha_-) = 1$. \square

Remark 6.3 The d-frame of truth values \mathcal{FOUR} satisfies the cut rules (as it is spatial, for example) so we can also apply the functor \mathbf{N} to equip it with interior operators. However, only the identity maps $\mathbf{2} \rightarrow \mathbf{2}$ are available, so this is what \mathbf{N} will produce.

6.1 Spectra and comparison with the interior operations

We are now ready to show that the spectrum of a (p^r, m^r) enriched d-frame is the same as the spectrum of the original d-frame.

Proposition 6.4 *Let \mathcal{L} be a d-frame satisfying the infinitary cut rules. Then, the spectra of \mathcal{L} and $(\mathcal{L}; p^r, m^r)$ are the same; that is*

$$\Sigma_d(\mathcal{L}) = \Sigma(\mathcal{L}; p^r, m^r).$$

Proof. This follows from the functoriality of \mathbf{N} as proved in Proposition 6.2: Every d-point of \mathcal{L} viewed as a d-frame homomorphism $p: \mathcal{L} \rightarrow \mathcal{FOUR}$ is also an nd-point $\mathbf{N}(p): (\mathcal{L}; p^r, m^r) \rightarrow \mathcal{FOUR}$. The converse inclusion is immediate. \square

From the fact that spatial d-frames satisfy the infinitary cut rules (Lemma 5.10 and Corollary 5.13 in [10]), we have:

Corollary 6.5 *Let \mathcal{L} be a spatial d-frame. Then $(\mathcal{L}; p^r, m^r)$ is an nd-frame and, moreover, the spectra of \mathcal{L} and $(\mathcal{L}; p^r, m^r)$ are the same.*

Note that this still does not mean that, for a spatial d-frame \mathcal{L} , p^r and m^r are the interior operations of the corresponding bspace, but in the d-regular case everything works out:

Proposition 6.6 *Let \mathcal{L} be a spatial d-regular d-frame. Then the nd-frame $(\mathcal{L}; p^r, m^r)$ is spatial.*

Proof. We need to prove that the conditions (s_p) and (s_m) hold in $(\mathcal{L}; p^r, m^r)$. For (s_p) , assume $a \not\leq p^r(b)$ for some $a \in L_+$ and $b \in L_-$. From d-regularity, there exists $c \in L_+$ such that $(a, c^*) \in \text{tot}$ and $c \not\leq p^r(b)$. Since $p^r(b) = \bigvee \{x \mid x^* \vee b = 1\}$, the latter condition on c implies that $c^* \vee b \neq 1$ or, in other words, $(0, c^* \vee b) \notin \text{tot}$. From spatiality of \mathcal{L} , there exists a point (F_+, F_-) such that $c^* \vee b \notin F_-$ and, therefore, also $c^* \notin F_-$ and $b \notin F_-$. Finally, we know that $(a, c^*) \in \text{tot}$, therefore it has to be the case that $a \in F_+$. \square

6.2 Maximality of (p^r, m^r)

We saw before that d-regularity is enough to ensure that p^r and m^r correspond to the interior operations on the corresponding bispace. Here we show (assuming just d-regularity) that (p^r, m^r) is “larger” than any other (p, m) pair:

Proposition 6.7 *Let \mathcal{L} be a d-regular d-frame and let (p, m) be such that $(\mathcal{L}; p, m)$ is an nd-frame. Then, $p \leq p^r$ and $m \leq m^r$ in the pointwise order.*

Proof. Let $b \in L_-$. Since \mathcal{L} is d-regular, $p(b) = \bigvee \{c \mid (p(b), c^*) \in \text{tot}\}$. But, any time $(p(b), c^*) \in \text{tot}$, from $(\text{tot}-p)$, we also have that $(0, c^* \vee b) \in \text{tot}$ and this is equivalent to $c^* \vee b = 1$. Therefore, $p \leq p^r$. \square

The fact that (p^r, m^r) is maximal says that it frame-theoretically mimics the interior operations as closely as possible. Indeed, if (p°, m°) were the interior operators of a (spatial) d-regular d-frame then, since (p°, m°) satisfies the (p, m) axioms, the previous proposition says that (p°, m°) is pointwise smaller than (p^r, m^r) . On the other hand, $p^r(b)$ is computed as a join of τ_+ -open elements well-inside b , whereas $p^\circ(b)$ is computed as the join of *all* τ_+ -open subsets of b . Therefore, (p^r, m^r) is also pointwise smaller than (p°, m°) . This means that in the spatial case (p^r, m^r) and (p°, m°) coincide.

6.3 Proof-theoretic negation

Assuming the Gentzen cut rule in the original paper [10] allowed Jung and Moshier to give a proof-theoretic characterisation of negation. For a $\gamma \in \mathcal{L}$, let $\bar{I}\gamma = \{\varphi \in \text{con} \mid \varphi \wedge \gamma \prec \text{ff}\}$ and $\bar{F}\gamma = \{\psi \in \text{con} \mid \text{tt} \prec \gamma \vee \psi\}$. Then, define the *proof-theoretic* negation of γ as

$$\bar{\gamma} \stackrel{\text{def}}{=} \left(\bigvee_{\varphi \in \bar{I}\gamma} \varphi_+, \bigvee_{\psi \in \bar{F}\gamma} \psi_- \right) \quad (\dagger)$$

We can now observe that this negation is actually exactly the same as the one obtained from the canonical (p^r, m^r) :

Theorem 6.8 *Let \mathcal{L} be a d-frame. Then, $\bar{\gamma} = (p^r(\gamma_-), m^r(\gamma_+))$ for all $\gamma \in L_+ \times L_-$.*

Proof. For “ \sqsubseteq ”, let $\varphi \in \bar{I}\gamma$. Notice that $\varphi \wedge \gamma \prec \text{ff}$ is equivalent to $(0, \varphi_- \vee \gamma_-) \in \text{tot}$ which is the same as $\varphi_- \vee \gamma_- = 1$. Since $\varphi \in \text{con}$, $(\varphi_+)^* \geq \varphi_-$ and so $(\varphi_+)^* \vee \gamma_- = 1$,

therefore $\varphi_+ \leq p^r(\gamma_-)$. Dually also $\psi \in \overline{F}\gamma$ implies $\psi_- \leq m^r(\gamma_+)$. For “ \sqsupseteq ”, let $\varphi_+ \in L_+$ be such that $(\varphi_+)^* \vee \gamma_- = 1$. Define

$$\chi \stackrel{\text{def}}{=} (\varphi_+, (\varphi_+)^*).$$

Obviously $\chi \in \mathbf{con}$ and $\chi \wedge \gamma \prec \mathbf{ff}$ (this is exactly the condition $\chi_- \vee \gamma_- = 1$). Therefore, $\chi \in \overline{I}\gamma$ and $\varphi_+ = \chi_+ \leq \overline{\gamma}_+$. Dually, for every $\varphi_- \in L_-$ such that $(\varphi_-)^* \vee \gamma_+ = 1$, $\varphi_- \leq \overline{\gamma}_-$ holds. \square

In fact, the proof that $m^r p^r \leq \text{id}$ and $p^r m^r \leq \text{id}$ in Proposition 6.1 is a direct translation of the proof that $\overline{\gamma} \sqsubseteq \gamma$ in [10]. The only, but very important, difference is that $\overline{\gamma}$ was originally defined only for the consistent predicates whereas (p^r, m^r) is defined for the whole d-frame. On top of that, the previous theorem provides the proof-theoretic negation with a bitopological interpretation.

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A Appendix: Proofs omitted from the main text

A.1 Proofs related to Remark 4.3

Proposition A.1 *Satisfying any one of the following axioms is equivalent to requiring that $p \circ m = \text{id}$:*

$$\begin{aligned} (\neg\neg L) \quad & \neg\neg\varphi \subset \varphi \\ (\neg\supset L) \quad & \neg(\varphi \supset \psi) \subset \varphi \wedge \neg\psi \end{aligned}$$

and satisfying any one of the following axioms is equivalent to requiring that p preserves finite suprema:

$$\begin{aligned} (\neg\wedge R) \quad & \neg(\varphi \wedge \psi) \supset \neg\varphi \vee \neg\psi \\ (\neg\sqcup R) \quad & \neg(\varphi \sqcup \psi) \supset \neg\varphi \sqcup \neg\psi \end{aligned}$$

Proof. $(\neg\neg L)$ is equivalent to $p \circ m = \text{id}$ as, by Lemma 4.1 (L2), $\mathcal{L} \models \varphi \supset \neg\neg\varphi$ is equivalent to $v(\varphi)_+ \leq v(\neg\neg\varphi)_+ = p(m(v(\varphi)_+))$ for all valuations v . To show the same for $(\neg\supset L)$, again by Lemma 4.1 (L2), $\mathcal{L} \models \varphi \wedge \neg\psi \supset \neg(\varphi \supset \psi)$ is equivalent to $v(\varphi)_+ \wedge p(v(\psi)_-) \leq p(m(v(\varphi)_+)) \wedge p(v(\psi)_-)$ for all valuations v . This has to hold for all φ and ψ . If ψ is such that $\psi_- = 1$ we get $v(\varphi)_+ \wedge p(1) \leq p(m(v(\varphi)_+)) \wedge p(1)$ and, since both p and m preserve 1, we get $v(\varphi)_+ \leq p(m(v(\varphi)_+))$ as we wanted.

For the second part we use Lemma 4.1 (L2) once again to show that $(\neg\wedge R)$ is equivalent to $p(\varphi_- \vee \psi_-) = p(\varphi_-) \vee p(\psi_-)$ for all φ and ψ . Notice that, $p(x \vee y) \geq p(x) \vee p(y)$ holds always from monotonicity of p , and $\mathcal{L} \models \neg(\varphi \wedge \psi) \supset \neg\varphi \vee \neg\psi$ is equivalent to $p(v(\varphi)_- \vee v(\psi)_-) \leq p(v(\varphi)_-) \vee p(v(\psi)_-)$ for all valuations v . The same argument applies for $(\neg\sqcup R)$. \square

We can also assume the following axiom

$$(\supset 3) \quad (\varphi \supset \psi) \supset \varphi \supset \varphi$$

which is equivalent to assuming that L_+ is a Boolean algebra as it forces $((v(\varphi)_+ \rightarrow v(\psi)_+) \rightarrow v(\varphi)_+) \rightarrow v(\varphi)_+ = 1$ to hold for all valuations v .

A.2 Proofs of the main theorems in Section 5

Lemma A.2 *Let \mathcal{L} be an nd -frame. Then, $(\Phi_+(x))^{\circ\tau-} \supseteq \Phi_-(m(x))$ and $(\Phi_-(y))^{\circ\tau+} \supseteq \Phi_+(p(y))$ in $\Sigma(\mathcal{L})$, for all $x \in L_+$ and $y \in L_-$.*

Proof. The second statement is true because, from (dp_p) , $\Phi_+(p(y)) \subseteq \Phi_-(y)$. \square

Proof of Theorem 5.3 (following the proof of Theorem 4.1 in [10]). Clearly, (iii) implies (ii) which implies (i). As in the original paper, η_X is bicontinuous, biopen onto the image and natural in X we see that (iii) and (iv) are equivalent. To show that (i) implies (iv) assume that $\iota: X \cong \Sigma(\mathcal{L})$. If we prove that $\eta_{\Sigma(\mathcal{L})}$ is a bijection, we get that also η_X is because the following square commutes (from naturality of η):

$$\begin{array}{ccc}
 X & \xrightarrow{\iota} & \Sigma(\mathcal{L}) \\
 \downarrow \eta_X & & \downarrow \eta_{\Sigma(\mathcal{L})} \\
 \Sigma\Omega(X) & \xrightarrow{\Sigma\Omega(\iota)} & \Sigma\Omega\Sigma(L)
 \end{array}$$

Observe that $\eta_{\Sigma(\mathcal{L})}$ is injective for all \mathcal{L} . For the surjectivity of $\eta_{\Sigma(\mathcal{L})}$, take a $(\mathcal{F}_+, \mathcal{F}_-) \in \Sigma\Omega\Sigma(L)$ and define

$$F_+ = \{x \in L_+ \mid \Phi_+(x) \in \mathcal{F}_+\} \quad F_- = \{y \in L_- \mid \Phi_-(y) \in \mathcal{F}_-\}.$$

We will show that (F_+, F_-) is an nd-point of L . As in [10], the pair (F_+, F_-) is a d-point. To show that it is actually an nd-point, we need to show that it satisfies (dp_p) and (dp_m) . For the former, let $p(x) \in F_+$. This is equivalent to $\Phi_+(p(x)) \in \mathcal{F}_+$. From Lemma A.2 we also know that $\Phi_+(p(x)) \subseteq (\Phi_-(x))^{\circ\tau_+} \in \mathcal{F}_+$. And, since $(\mathcal{F}_+, \mathcal{F}_-)$ is an nd-point, we know that $\Phi_-(x) \in \mathcal{F}_-$. Finally, from the definition of F_- we get that $x \in F_-$ as we wanted. The proof of (dp_m) is the same but dual.

The argument that $\eta_{\Sigma(\mathcal{L})}(F_+, F_-) = (\mathcal{F}_+, \mathcal{F}_-)$ is exactly the same as in [10]. \square

Lemma A.3 $\epsilon_{\mathcal{L}}$ is an onto nd-frame homomorphism. Moreover, ϵ is natural in \mathcal{L} .

Proof. Since ϵ is defined the same way as for d-frames, we see that $\epsilon_{\mathcal{L}}$ is a d-frame homomorphism and that ϵ is natural in \mathcal{L} . We need to show that it is indeed an nd-frame homomorphism. That is that it satisfies $\epsilon_{\mathcal{L}}(-\alpha) \sqsubseteq \neg\epsilon_{\mathcal{L}}(\alpha)$ for all $\alpha \in L_+ \times L_-$ but this is exactly the same statement as Lemma A.2. \square

Proof of Theorem 5.4 (following the proof of Theorem 5.1 in [10]). The implications (iii) \implies (ii) \implies (i) are immediate, and (iv) \implies (iii) follows from the fact that ϵ_L is onto (by Lemma A.3).

To show that (i) implies (v), it is enough to show that (s_p) and (s_m) hold for all images of Ω (the other conditions were already proved in [10] for d-frames). Clearly, for $U_+ \in \tau_+$ and $V_- \in \tau_-$, $U_+ \subseteq V_-^{\circ\tau_+}$ iff $U_+ \subseteq V_-$. Therefore, if $U_+ \not\subseteq V_-^{\circ\tau_+}$, then there exists an $x \in U_+ \setminus V_-$ such that $U_+ \in \mathcal{U}_+(x)$ and $V_- \notin \mathcal{U}_-(x)$. Moreover, $(\mathcal{U}_+(x), \mathcal{U}_-(x))$ is an nd-point.

Finally, (v) implies (iv). As already proved in [10], ϵ_L is injective and reflects con and tot. It remains to prove that $\epsilon_L^{-1}(\neg(\Phi_+(x), \Phi_-(y))) \sqsubseteq \neg\epsilon_L^{-1}(\Phi_+(x), \Phi_-(y))$. Let us focus on the plus coordinates, that is to prove that

$$(\epsilon_L)_+^{-1}(\Phi_-(y)^{\circ\tau_+}) \leq p(\epsilon_L)_-^{-1}(\Phi_-(y)).$$

Observe that $\Phi_-(y)^{\circ\tau_+} = \Phi_+(x)$ for some $x \in L_+$ because $\epsilon_{\mathcal{L}}$ is onto. Since ϵ_L is injective, $(\epsilon_L)_+^{-1}(\Phi_-(y)^{\circ\tau_+}) = (\epsilon_L)_+^{-1}(\Phi_+(x)) = x$ and also $p((\epsilon_L)_-^{-1}(\Phi_-(y))) = p(y)$. Now, assume for a contradiction that $x \not\leq p(y)$. Then, from (s_p) , there exists an nd-point (F_+, F_-) such that $x \in F_+$ and $y \notin F_-$ but this is impossible since $\Phi_+(x) = \Phi_-(y)^{\circ\tau_+} \subseteq \Phi_-(y)$. \square