A note on coherence of dcpos ☆

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Abstract

In this note, we prove that a well-filtered dcpo L is coherent in its Scott topology if and only if for every $x, y \in L$, $\uparrow x \cap \uparrow y$ is compact in the Scott topology. We use this result to prove that a well-filtered dcpo L is Lawson-compact if and only if it is patch-compact if and only if L is finitely generated and $\uparrow x \cap \uparrow y$ is compact in the Scott topology for every $x, y \in L$.

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1. Introduction

In this paper, we investigate the *coherence* with respect to the Scott topology on directed-complete partial ordered sets (*dcpo*'s for short). Coherence, which states that the intersection of any two compact saturated sets is again compact, is an important property in domain theory [1, 3]. For instance, coherence is equivalent to Lawson compactness on pointed continuous domains [5]. This equivalence enabled the second author to characterise the Lawson compactness of continuous domains by the so-called "property M", and use this element-level characterization to classify the category of continuous domains with respect to the cartesian closedness [5, 6].

In [9, 8], the equivalence between coherence and Lawson compactness was generalised to quasicontinuous domains. In Chapter 3 of [3], one even sees that on finitely generated quasicontinuous domains the compactness of $\uparrow x \cap \uparrow y$ for any $x, y \in L$, which seems much weaker than what coherence requires, already implies the Lawson compactness of L. In this note, we greatly generalize this result to *well-filtered* dcpos. Indeed, since every quasicontinuous domain is locally finitary compact and sober (see for example, [4]), our proof drops the locally finitary compact property and only uses well-filteredness, which is even strictly weaker than sobriety [7].

2. Preliminaries

We refer to [1, 3] for the standard definitions and notations of order theory and domain theory, and to [4] for topology.

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A topological space is called *well-filtered* if, whenever an open set U contains a filtered intersection $\bigcap_{i\in I} Q_i$ of compact saturated subsets, then U contains Q_i for some $i \in I$. Any sober space is well-filtered (see [3, Theorem II-1.21]). We take *coherence* of a topological space to mean that the intersection of any two compact saturated subsets is compact. A *stably compact* space is a topological space which is compact, locally compact, well-filtered and coherent. We call a dcpo L well-filtered (respectively, compact, sober, coherent, locally compact, stably compact) if L with its Scott topology $\sigma(L)$ is a well-filtered (respectively, compact, sober, coherent, locally compact, stably compact) space. Without further reference, we always equip L with the Scott topology $\sigma(L)$. Finally, a dcpo L is said to be *core-compact* if its Scott topology $\sigma(L)$ is a continuous lattice in the inclusion order.

For a topological space X, we denote the set of all compact saturated sets of X by $\mathcal{Q}(X)$. We consider the *upper Vietoris topology* v on $\mathcal{Q}(X)$, generated by the sets

$$\Box U = \{K \in \mathcal{Q}(X) \mid K \subseteq U\},$$

where U ranges over the open subsets of X. We use $\mathcal{Q}_v(X)$ to denote the resulting topological space. For a dcpo L, we use $\mathcal{Q}_v(L)$ to denote $\mathcal{Q}_v((L, \sigma(L)))$.

3. Main results

Lemma 3.1. *Let L be a well-filtered dcpo. Then L is coherent if and only if $\uparrow x \cap \uparrow y$ is compact for all $x, y \in L$.*

Proof. If L is coherent, it is obvious that $\uparrow x \cap \uparrow y$ is compact for all $x, y \in L$, since $\uparrow x, \uparrow y$ are compact saturated.

For the reverse, suppose $\uparrow x \cap \uparrow y$ is compact for all $x, y \in L$. We proceed to prove that for any compact saturated sets $A, B \subseteq L$, $A \cap B$ is compact in L. To this end, fix some element $a \in L$; we define a function f from L to $\mathcal{Q}_v(L)$ by sending an element x to the compact saturated set $\uparrow x \cap \uparrow a$. We claim that f is continuous. Indeed, for every Scott open subset $U \subseteq L$, $f^{-1}(\Box U) = \{x \mid \uparrow x \cap \uparrow a \subseteq U\}$ is obviously an upper set. Let $D \subseteq L$ be a directed subset with $\sup D \in f^{-1}(\Box U)$, then one has $\uparrow(\sup D) \cap \uparrow a \subseteq U$, that is, $\bigcap_{d\in D}(\uparrow d \cap \uparrow a) \subseteq U$. Note that L is well-filtered and $\{\uparrow d \cap \uparrow a \mid d \in D\}$ is a filtered family of compact saturated sets by assumption, so we have some $d \in D$ such that $\uparrow d \cap \uparrow a \subseteq U$, i. e., $d \in f^{-1}(\Box U)$. Hence f is continuous.

Since f is continuous, for the given compact saturated subset $A \subseteq L$, $f(A) = \{\uparrow x \cap \uparrow a \mid x \in A\}$ is a compact subset of $\mathcal{Q}_v(L)$. We now claim that the union of $f(A)$, which is just $A \cap \uparrow a$, is compact in L. Indeed, for any compact subset $\mathcal{C}$ of $\mathcal{Q}_v(L)$, let $\{U_\alpha\}$ be a directed family of open sets of L covering $\bigcup \mathcal{C}$. By compactness, every element K of $\mathcal{C}$ is already covered by one U_α; in other words, $K \in \Box U_\alpha$. It follows that $\{\Box U_\alpha\}$ is a directed family covering $\mathcal{C}$, and now the compactness of $\mathcal{C}$ tells us that $\mathcal{C} \subseteq \Box U_\alpha$ for some α. Hence $\bigcup \mathcal{C} \subseteq U_\alpha$ for this α. (This argument is similar to the one employed by Andrea Schalk in [10, Chapter 7] for showing that $\bigcup\colon \mathcal{Q}_v(\mathcal{Q}_v(X)) \to \mathcal{Q}_v(X)$ is well-defined.)

Now for such A the above argument enables us to define another function g from L to $\mathcal{Q}_v(L)$ as: $g(x) = \uparrow x \cap A$ for every $x \in L$. A similar deduction shows that g is continuous. So for the compact saturated subset B of L, $g(B)$ is compact in $\mathcal{Q}_v(L)$, and again the union of $g(B)$, which is $A \cap B$, is compact in L. So L is coherent. $\Box$

Corollary 3.2. *Every well-filtered complete lattice L is coherent.*

Proof. For every $x, y \in L$ the intersection of $\uparrow x$ and $\uparrow y$, which is $\uparrow(x \vee y)$, is always compact, so the statement follows from Lemma 3.1. □

The following fact about core-compact complete lattices is essentially due to G. Gierz and K.H. Hofmann [2]; we collect it here as a corollary to the previous result.

Corollary 3.3. *For a complete lattice L, the following statements are equivalent:*

1. *L is core-compact, i. e., $\sigma(L)$ is a continuous lattice;*
2. *$(L, \sigma(L))$ is stably compact.*

Proof. The only interesting part is that *1* implies *2*. Suppose L is a complete lattice and $\sigma(L)$ is continuous, then $(L, \sigma(L))$ is a locally compact sober space by [3, PropositionVII-4.1]. Since sober spaces are well-filtered, L is coherent by Corollary 3.2. Finally, L is obviously compact in its Scott topology since it has a least element. □

We now come to a characterization of the compactness of Lawson and patch topologies on L. Recall that the patch topology on L arises by taking all Scott closed sets together with all compact saturated sets as a subbasis for the closed sets; whereas the (coarser) Lawson topology is generated by the Scott closed subsets and principal upper sets $\uparrow x$. The following theorem is a generalization of [3, Theorem III-5.8] which is stated for quasicontinuous domains.

Theorem 3.4. *Let L be a well-filtered dcpo. Then the following statements are equivalent:*

1. *L is patch-compact, i. e., L is compact in the patch topology;*
2. *L is Lawson-compact;*
3. *L is compact and $\uparrow x \cap \uparrow y$ is compact for every $x, y \in L$;*
4. *L is finitely generated and $\uparrow x \cap \uparrow y$ is compact for every $x, y \in L$;*
5. *L is finitely generated and coherent.*

Proof. (*1*$\Rightarrow$*2*): That *1* implies *2* is true for all dcpos since the patch topology is finer than the Lawson topology.
(*2*$\Rightarrow$*3*): It is obvious that L is compact since the Lawson topology is finer than the Scott topology. For every $x, y \in L$, $\uparrow x \cap \uparrow y$ is Lawson closed; therefore it is Lawson-compact, thus Scott compact.
(*3*$\Rightarrow$*4*): Since L is compact, by the Hausdorff Maximality Principle, every element is above some minimal element of L. Denote the set of all minimal elements of L by M; M must be finite. Otherwise, the family $\{M \backslash F \mid F \subseteq_{\text{fin}} M\}$ is a filtered set of non-empty Scott closed sets with an empty intersection, which contradicts compactness.
(*4*$\Rightarrow$*5*): This is from Lemma 3.1.
(*5*$\Rightarrow$*1*): This is a straightforward consequence of [3, Lemma VI-6.5][1]. □

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[1]This lemma works for sober dcpos in [3]. However, one can find that its proof only uses well-filteredness.

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