The Classification of Continuous Domains*

Extended Abstract
Achim Jung†
Technische Hochschule Darmstadt
and
Imperial College of Science and Technology, London

Abstract

The long-standing problem of finding the maximal cartesian closed categories of continuous domains is solved. The solution requires the definition of a new class of continuous domains, called *FS-domains*, which contains all retracts of SFP-objects. The properties of FS-domains are discussed in some detail.

Keywords: continuous domains, SFP-objects, Lawson-topology, Smyth's Theorem, FS-domains, L-domains

1 Introduction

The first spaces suitable for the interpretation of programming language constructs were *continuous lattices* discovered by Dana Scott in the late sixties. Continuous lattices turned out to have numerous connections to other fields of mathematics such as algebra, topology, and convex analysis. An indication of this is the voluminous *Bibliography of Continuous Lattices* contained in [4].

In Computer Science, however, it was soon recognized that the subclass of *algebraic lattices* is fully sufficiant for the purposes of semantics. Indeed, the basic concept of finite pieces of information corresponds nicely to the idea of compact elements in these structures. Generality was sought in a different direction, namely, in the way the least upper bound of pieces of information was to be formed. This led to a variety of different classes of *domains*: Lattices, meet-semilattices (= Scott-domains), SFP-objects, to name a few.

It is our belief that continuous domains do have a similar importance for computer science in areas largely still to be developed. One application is the analysis of probabilistic algorithms. Here the central domain is clearly the unit interval, a non-algebraic but continuous lattice. Some work in this direction has been carried out in [3, 5].

Looking at all those different definitions of domains the novice in the field will naturally ask for some orientation. And indeed, it is possible to give a rather complete overview once the basic assumption is shared that a collection of domains should form a cartesian closed category. Michael Smyth [9] showed 1983 that there is a largest cartesian closed full subcategory in the class of all countably based algebraic dcpo's with least element. In his doctoral thesis [6] the present author completely described all categories of algebraic domains with respect to that criterion of cartesian closedness. It is the purpose of the present note to do the same for continuous domains.

It is an easy exercise to show that any Scott-continuous retract of an algebraic dcpo is a continuous dcpo and it is equally simple to see that the class of all such retracts is cartesian closed if one starts with a cartesian closed category. This immediately gives us a class of continuous domains for each class of algebraic domains. It is then an obvious question whether this will give

*Part of the research was carried out while the author was employed in a project funded by the English SERC.

†Department of Computing, Imperial College, 180 Queen's Gate, London, SW7 2BZ, England.

us the whole variety on the continuous side. As Smyth notes at the end of his paper [9] "(this result) does not come out by manipulating retractions". It turned out to be a very hard problem, indeed. The solution, which was partly provided in [6] and is completed here, involves the definition of two new classes of domains: *L-domains* and *FS-domains*. We will show below that each cartesian closed category of continuous domains with least element consists of continuous L-domains or of FS-domains. The special question, whether the retracts of SFP-objects form a maximal class we leave unanswered. They are FS-domains but we do not know whether this containment is proper.

FS-domains do have a distinctive advantage over SFP-retracts: They are easy to discover. This is illustrated below by showing that the collection of all closed discs in the plane together with the plane itself (the ordering being reversed inclusion) forms a countably based FS-domain. Even for this well-structured concrete example it appears to be extremely hard to decide whether it is an SFP-retract.

2 Background

Our notation will be fairly standard. We call directed-complete partial orders *dcpo's* and do not generally assume that they have a least element. If a dcpo does have a bottom element then we call it *pointed*. A dcpo is *continuous* if every element is the directed supremum of elements way-below it, where an element x is *way-below* an element y ($x \ll y$) if whenever the sup of a directed set is above y then some element of the directed set is above x. A subset B is a *basis* if every element x is the directed sup of base elements way-below x. A dcpo is *countably based* or *ω-continuous* if it has a countable basis.

Our functions are *Scott-continuous* that is, they preserve directed sups. Dcpo's together with Scott-continuous maps form a cartesian closed category **DCPO**. The full subcategories **CONT** and $\mathbf{CONT}_\perp$ of continuous dcpo's (with bottom) are not cartesian closed. It is the purpose of this note to describe all maximal cartesian closed full subcategories of $\mathbf{CONT}_\perp$.

The basic properties of the way-below relation are sumarized in the following lemma.

Lemma 1 *If D is a continuous dcpo then the following holds for all $x, x', y, y' \in D$:*

(i) $x \ll y \Longrightarrow x \le y$.

(ii) $x' \le x \ll y \le y' \Longrightarrow x' \ll y'$.

(iii) $x \ll y \Longrightarrow \exists z. x \ll z \ll y$. ▮

Given a pointed dcpo E and a dcpo D and given elements $e \in E$ and $d \in D$ we can define the *step function* $(d \searrow e)$ as follows:

$$(d \searrow e)(x) = \begin{cases} e, & \text{if } d \ll x; \\ \perp, & \text{otherwise.} \end{cases}$$

A step function is always Scott-continuous. If x' is way-below x in D then the step function $(x \searrow x')$ is way below the identity function id_D on D. In a continuous dcpo we can interpolate between x' and x with elements y' and y: $x' \ll y' \ll y \ll x$. It is then easy to check that the step function $(x \searrow x')$ is way-below the step function $(y \searrow y')$ in the dcpo $[D \longrightarrow D]$.

3 Continuous L-domains

Definition. A dcpo D is an *L-domain* if it is pointed and if every principal ideal in D is a complete lattice. The category of continuous L-domains is denoted by **cL**.

L-domains were discovered by the present author and by T.Coquand [1, 2] independently. A thorough treatment of their main properties can be found in [6], where it was already shown that they form a maximal cartesian closed full subcategory of $\mathbf{CONT}_\perp$ ('Theorem 4.25').

Continuous L-domains occur in 'nature': Given a compact connected and locally connected space X the collection of all closed connected nonempty subsets of X ordered by reversed inclusion forms a continuous L-domain. This example is due to Klaus Keimel and Jimmie Lawson.

4 FS-domains

It was generally conjectured (see for example [9, 7]) that the retracts of SFP-objects (or rather: bifinite domains, see [6, 10]) form another maximal cartesian closed full subcategory of $\mathbf{CONT}_\perp$ and that there are no other. In what follows we shall characterize the second maximal class and show that every cartesian closed full subcategory of $\mathbf{CONT}_\perp$ is contained in one of the two. This second class will consist of FS-domains, which are introduced here for the first time. They contain the retracts of SFP-objects, but it is open whether this inclusion is strict. However, we hope to convince the reader that FS-domains are preferable to SFP-retracts anyway.

Definition. Let $f, g: D \to E$ be functions from a set D to a dcpo E. We say that f is *finitely separated* from g if there exists a finite subset M of E such that for every $x \in D$ there is some $m \in M$ such that $f(x) \leq m \leq g(x)$ holds. The function f is *strongly finitely separated* from g if there exists a finite set M of pairs $(m', m) \in E \times E$ with $m' \ll m$ such that for every $x \in D$ there is some pair from M between $f(x)$ and $g(x)$. We will mostly need functions $f: D \to D$ separated from the identity id_D.

Lemma 2 *Let $f: D \to D$ be a Scott-continuous function on a dcpo D finitely separated from id_D. Then $f(x) \ll x$ holds for every $x \in D$, $f \circ f$ is strongly finitely separated from id_D and way-below id_D in $[D \longrightarrow D]$.* ▮

Definition. A pointed dcpo D is called an *FS-domain* if there exists a directed family $(f_i)_{i \in I}$ of Scott-continuous functions, each finitely separated from id_D, with supremum id_D. The category of FS-domains with Scott-continuous functions as arrows is denoted by **FS**.

By the preceding lemma it is obvious that FS-domains are continuous. Considering the characterization ('Theorem 4.1') in [6], it is also clear that **FS** contains all retracts of bifinite domains. In fact, **FS** has all closure properties one usually expects from a category of domains:

Theorem 3 *Any product of FS-domains is an FS-domain and the inverse limit of FS-domains is an FS-domain. Also,* **FS** *is a cartesian closed subcategory of* $\mathbf{CONT}_\perp$.

Proof. We show that the function space $[D \longrightarrow E]$ for FS-domains D and E is again an FS-domain. Let $f: D \to D$ be finitely separated from id_D and $g: E \to E$ be finitely separated from id_E. We show that the function $F: [D \longrightarrow E] \to [D \longrightarrow E]$, defined by $\lambda h. g \circ g \circ h \circ f \circ f$, is finitely separated from $\mathsf{id}_{[D \to E]}$. Let M_f, M_g be finite separating sets for f and g, respectively. Define an equivalence relation on $[D \longrightarrow E]$ by

$$h_1 \sim h_2 \iff \forall m \in M_f.\ \uparrow g \circ h_1(m) \cap M_g = \uparrow g \circ h_2(m) \cap M_g.$$

Obviously there are only finitely many equivalence classes on $[D \longrightarrow E]$. Let $\widetilde{M_F}$ be a set of representatives from each class. We show that $M_F = g \circ \widetilde{M_F} \circ f$ is a separating set for F. Given $h \in [D \longrightarrow E]$ let $\bar{h}$ be the corresponding representative in $\widetilde{M_F}$. We calculate for an $x \in D$:

$$\begin{array}{rcll} h(x) & \geq & h(m_f) & \text{;for some } m_f \in M_f \\ & & & \ \text{with } f(x) \leq m_f \leq x \\ & \geq & m_g & \text{;for some } m_g \in M_g \text{ with} \\ & & & \ g(h(m_f)) \leq m_g \leq h(m_f) \\ & \geq & g(\bar{h}(m_f)) & \text{;because } g(h(m_f)) \leq m_g \\ & & & \ \text{and } h \sim \bar{h} \\ & \geq & g(\bar{h}(f(x))) & \text{;because } f(x) \leq m_f. \end{array}$$

By symmetry we also have $\bar{h}(x) \geq g(h(f(x)))$ and hence $g \circ \bar{h} \circ f \geq g \circ g \circ h \circ f \circ f$. So indeed: $h \geq g \circ \bar{h} \circ f \geq F(h)$. ▮

Definition. For a continuous dcpo D the *Lawson-topology* λ_D is generated by the subbasic open sets $\Uparrow x, x \in D$ and $D \setminus \uparrow x, x \in D$.

On continuous dcpo's the Lawson-topology will always be Hausdorff. With the results in [6] or in [8] it is easy to see that FS-domains are Lawson-compact. In fact, the Lawson-topology is closely connected to the function space:

Theorem 4 *Let D be an FS-domain.*

(i) *For each* $f \ll \mathsf{id}_D$ *we define an entourage* $\mathcal{U}_f = \{(x,y) \in D \times D \mid f(x) \leq y \wedge f(y) \leq x\}$ *resulting in a basis* $(\mathcal{U}_f)_{f \ll \mathsf{id}_D}$ *for a uniformity on* D*. The corresponding topology equals* λ_D*.*

(ii) *A function* $f: D \to D$ *is way-below* id_D *if and only if it is strongly finitely separated from* id_D*.* ▌

We finish this section with the discussion of a concrete example of an FS-domain. (It was suggested to me by Jimmie Lawson.) Let Disc be the collection of all closed discs in the plane plus the plane itself, ordered by reversed inclusion. One checks that the filtered intersection of discs is again a disc, so Disc is a dcpo. A disc d_1 is way-below a disc d_2 if and only if d_1 is a neighborhood of d_2. This proves that Disc is continuous. For every $\epsilon > 0$ we define a map f_ϵ on Disc as follows. All discs inside the open disc with radius $\frac{1}{\epsilon}$ are mapped to their closed ϵ-neighborhood, all other discs are mapped to the plane which is the bottom element of Disc. Because the closed discs contained in some compact set form a compact space under the Hausdorff subspace topology, these functions are finitely separated from the identity map. This proves that Disc is a countably based FS-domain. We do not know whether this domain is a retract of an SFP-object.

5 The classification

The following lemma, which we cite from [6] is the starting point for our classification:

Lemma 5 *Let* D *and* E *be continuous pointed dcpo's with property m. If* $[D \longrightarrow E]$ *is continuous then* E *is an L-domain or* D *is Lawson-compact.* ▌

Lawson-compact dcpo's do not form a cartesian closed category. Indeed, we are now going to show that FS-domains are the largest cartesian closed full subcategory of $\mathbf{CONT}_\perp$ which consists of Lawson-compact domains only.

Definition. For any dcpo D and any $d \in D$ the retraction $r_d: D \to D$ is defined by $r_d(x) = x$ if $x \leq d$ and $r_d(x) = d$ otherwise.

Lemma 6 *If a function* $f: D \to D$ *is below* r_x *and* r_y *then* $f(x) \leq x, f(x) \leq y, f(y) \leq x$*, and* $f(y) \leq y$*.* ▌

The following Lemma appears also in [8]

Lemma 7 *If* D *is a dcpo with continuous and Lawson-compact function space* $[D \longrightarrow D]$ *and if* $f \ll \mathsf{id}_D$ *holds then there exist pairs* $x_1' \ll x_1, \ldots, x_n' \ll x_n$ *such that every upper bound of the step functions* $(x_i \searrow x_i')$*,* $i = 1, \ldots, n$*, is above* f*.*

Proof. $\Uparrow f$ is a Lawson-neighborhood of $\uparrow \mathsf{id}_D$. Since $[D \longrightarrow D]$ is Lawson-compact, each of the sets $\uparrow(x_1 \searrow x_1') \cap \ldots \cap \uparrow(x_m \searrow x_m')$, for any finite set of pairs $x_1' \ll x_1, \ldots, x_m' \ll x_m$, is Lawson-compact. The intersection of all these sets is filtered and equals $\uparrow \mathsf{id}_D$. Therefore one of them is already contained in $\Uparrow f$. ▌

Theorem 8 *If* D *and* $[D \longrightarrow D]$ *are continuous and Lawson-compact and if* $f \ll \mathsf{id}_D$ *then* f *is finitely separated from* id_D*.*

Proof. Let $g \ll \mathsf{id}_D$ be such that $f \leq g \circ g$ and let $X_1 = (x_1 \searrow x_1'), \ldots, X_n = (x_n \searrow x_n')$ be step functions such that any upper bound of them is above g according to Lemma 7. For each $i \in I = \{1, \ldots, n\}$ interpolate between x_i' and x_i to get y_i', y_i such that $x_i' \ll y_i' \ll y_i \ll x_i$ and let Y_i be the step function $(y_i \searrow y_i')$. We noted above that $X_i \ll Y_i$ holds in $[D \longrightarrow D]$. Also note that for each $x \in D \setminus O$ — where $O = \Uparrow y_1 \cup \ldots \cup \Uparrow y_n$ — $g(x) = \perp$ holds. That is because the function which maps each element of O onto itself and everything else onto bottom is above all Y_i and hence above g.

For each $x \in O$ consider the retraction r_x. The element x is way-above some of the y_i but not necessarily above all of them. Call the subset of I for which $y_i \ll x, I_x$. Then r_x is above all Y_i with $i \in I_x$, because $r_x(e) = x \geq y_i \geq y_i' \geq$

$Y_i(e)$ for $e \not\leq x$ and $r_x(e) = e = \mathsf{id}_D(e) \geq Y_i(e)$ otherwise.

Claim: *If $h: D \to D$ is below r_x and above all X_i with $i \in I_x$, then $h|_{\downarrow x} \geq g|_{\downarrow x}$.*

Define $h': D \to D$ by $h'(e) = e$ if $e \not\leq x$ and $h'(e) = h(e)$ otherwise. This is continuous because $h'|_{\downarrow x} = h|_{\downarrow x} \leq r_x|_{\downarrow x} = \mathsf{id}_D|_{\downarrow x}$ and $h'\big|_{D\setminus\downarrow x} = \mathsf{id}_D\big|_{D\setminus\downarrow x}$. The map h' is above all step functions X_i:

Case 1: $e \not\leq x$: $h'(e) = e = \mathsf{id}_D(e) \geq X_i(e), i \in I$.

Case 2a: $e \leq x, i \in I_x$: $h'(e) = h(e) \geq X_i(e)$ by assumption.

Case 2b: $e \leq x, i \in I \setminus I_x$: $e \notin \Uparrow x_i \subseteq \Uparrow y_i$ by the definition of I_x, so $h'(e) = h(e) \geq \bot = X_i(e)$.

So h' is above g and hence $h|_{\downarrow x} = h'|_{\downarrow x} \geq g|_{\downarrow x}$. This proves our claim.

Now let J be some subset of I. Since $[D \longrightarrow D]$ is Lawson-compact there exists a finite set M_J contained in $\bigcap\{\uparrow X_i \mid i \in J\}$ such that every upper bound of $\{Y_i \mid i \in J\}$ is above some $h \in M_J$, that is

$$\bigcap_{i\in J} \uparrow Y_i \subseteq \bigcup_{h\in M_J} \uparrow h.$$

In particular, for a given $x \in D$, there is $h \in M_{I_x}$ with $h \leq r_x$. We now take all h from each M_J that we need, that is:

$$\begin{aligned} FM = \{h \in \textstyle\bigcup_{J\subseteq I} M_J \mid \quad & \exists x \in D. I_x = J \wedge \\ & \wedge h \leq r_x \wedge h \in M_J\}. \end{aligned}$$

A function in FM will in general be below many r_x with $J = I_x$. We select just one x_h for each $h \in FM$ and define

$$M = \{h(x_h) \mid h \in FM\}.$$

It remains to show that M separates f from the identity on D. To this end, let x be some arbitrary but fixed element in D and let $h \in M_{I_x}$ be such that $r_x \geq h$. x is not necessarily equal to x_h but we have $h \leq r_x, r_{x_h}$ and we can apply Lemma 6: $h(x_h) \leq x$ and $h(x) \leq x_h$, also, $h \geq X_i$ for all $i \in I_x$ by construction. Hence by the 'Claim' above, $h|_{\downarrow x} \geq g|_{\downarrow x}$ and $h|_{\downarrow x_h} \geq g|_{\downarrow x_h}$. So we can calculate:

$$\begin{array}{rcll} x & \geq & h(x_h) & \text{Lemma 6} \\ & \geq & g(x_h) & \text{`Claim'} \\ & \geq & g(h(x)) & \text{Lemma 6} \\ & \geq & g(g(x)) & \text{`Claim'} \\ & \geq & f(x) & \text{by construction.} \end{array}$$

Thus with $m = h(x_h)$ we have found a separating element in M between x and $f(x)$. ▌

Corollary 9 *If D and $[D \longrightarrow D]$ are Lawson-compact and continuous then D is an FS-domain.* ▌

Corollary 10 *Every cartesian closed full subcategory of* $\mathbf{CONT}_\bot$ *is contained in* $\mathbf{cL}$ *or in* $\mathbf{FS}$. ▌

If we restrict our attention to continuous domains with a countable basis, then we must have Lawson-compactness. This was shown in [6]. So we also have the following continuous analogue to Smyth's Theorem for continuous domains:

Theorem 11 *The class ω-$\mathbf{CONT}_\bot$ of pointed continuous countably based dcpo's contains a largest cartesian closed full subcategory, the class of all countably based FS-domains.* ▌

It is an interesting observation that the techniques of our present paper yield a proof of Smyth's Theorem which does not use the Axiom of Choice. Also, it is possible to extend the results of this paper to dcpo's without bottom element. Most of the work for this was done in [6] already. One gets four maximal cartesian closed full subcategories of **CONT**.

Acknowledgement

I would like to thank our baby Elena for keeping me awake one night long enough to find the crucial step in the proof of Theorem 8.

References

[1] T. Coquand. Categories of embeddings. In *Logic in Computer Science*, pages 256–263. IEEE Computer Society Press, 1988.

[2] T. Coquand. Categories of embeddings. *Theoretical Computer Science*, 68:221–238, 1989.

[3] S. Graham. Closure properties of a probabilistic powerdomain construction. In M. Main, A. Melton, M. Mislove, and D. Schmidt, editors, *Mathematical Foundations of Programming Language Semantics*, volume 298 of *Lecture Notes in Computer Science*, pages 213–233. Springer Verlag, 1988.

[4] R.-E. Hoffmann and K. H. Hofmann, editors. *Continuous Lattices and their Applications*, volume 101 of *Lecture Notes in Pure and Applied Mathematics*. Marcel Dekker Inc., 1987.

[5] C. Jones and G. Plotkin. A probabilistic powerdomain of evaluations. In *Logic in Computer Science*, pages 186–195. IEEE Computer Society Press, 1989.

[6] A. Jung. *Cartesian Closed Categories of Domains*, volume 66 of *CWI Tracts*. Centrum voor Wiskunde en Informatica, Amsterdam, 1989.

[7] T. Kamimura and A. Tang. Finitely continuous posets. Technical Report TR-84-1, University of Kansas, 1984.

[8] J. D. Lawson. The versatile continuous order. In M. Main, A. Melton, M. Mislove, and D. Schmidt, editors, *Mathematical Foundations of Programming Language Semantics*, volume 298 of *Lecture Notes in Computer Science*, pages 134–160. Springer Verlag, 1988.

[9] M. B. Smyth. The largest cartesian closed category of domains. *Theoretical Computer Science*, 27:109–119, 1983.

[10] P. Taylor. *Recursive Domains, Indexed category Theory and Polymorphism.* PhD thesis, Cambridge University, 1987.