Stably Compact Spaces and Closed Relations

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Abstract

Stably compact spaces are a natural generalization of compact Hausdorff spaces in the T_0 setting. They have been studied intensively by a number of researchers and from a variety of standpoints.

In this paper we let the morphisms between stably compact spaces be certain "closed relations" and study the resulting categorical properties. Apart from extending ordinary continuous maps, these morphisms have a number of pleasing properties, the most prominent, perhaps, being that they correspond to preframe homomorphisms on the localic side. We exploit this Stone-type duality to establish that the category of stably compact spaces and closed relations has bilimits.

1 Introduction

The research reported in this paper derives its motivation from two sources. For some time, we have tried to extend Samson Abramsky's *Domain Theory*

in Logical Form to continuous domains, [1,15,14,17]. This has led to a number of insights, the most important perhaps being that in order to perform domain constructions *strictly logically*, one can invoke a version of Gentzen's cut elimination theorem. This, however, requires that we consider a purer logic than Abramsky did. Semantically, it then turns out that the notion of morphisms so captured consists of certain *relations*, rather than functions, [14, Proposition 6.5]. This is quite in line with developments in denotational semantics, where the need for (or the advantages of) relations has been noticed for some time, [5,3].

Our second motivation stems from the desire to circumvent some of the difficulties connected to classical domain theory. As is well known, in order to get a cartesian closed category of continuous domains, one has to restrict to a subcategory of FS-domains, [13,1]. Unlike general continuous domains, a straightforward characterisation of FS-domains via their Stone dual, for example, is not known. Perhaps as a result of the relative weakness of our tools for FS-domains, certain basic questions about them remain unresolved. We still do not know whether they coincide with retracts of bifinite domains or whether the probabilistic powerdomain can be restricted to this category, [16].

The semantic spaces which we put forward in this paper, in contrast to FS-domains, are very well behaved and understood. They are the so-called *stably-compact spaces*. Many equivalent characterisations are known and many properties have been discovered for them. Also, they do encompass most categories of continuous domains which have played a role in denotational semantics. As is clear from what we have said at the beginning, we are interested in the category SCS^* of stably compact spaces with closed *relations* as morphisms. Although a similar set-up has been considered some time ago, [26, Prop. 11.2.5], the explicit relational presentation appears to be new.

The purpose of this paper is to examine the suitability of SCS^* as a semantic universe. To this end we look at finitary closure properties and the bilimit construction. The latter, to our great satisfaction, behaves in a very natural and intuitive way. Specifically, we show that the bilimit coincides with a classical topological limit although it is constructed *order-theoretically*.

2 The category of of stably compact spaces and closed relations

2.1 The spaces

We assume standard domain theoretic notation as it is used in [8,1], for example. Slightly less well known, perhaps, are the following notions and results. If X is a topological space and A an arbitrary subset of X then the *saturation* of A is defined as the intersection of all neighborhoods of A. For any T_0-topological space X, the *specialization order* of X is the relation $\sqsubseteq_X$ given by

$x \sqsubseteq_X y$ if every neighborhood of x is also a neighborhood of y. The saturation of a subset A can then also be described as the upward closure with respect to $\sqsubseteq_X$. Open set are always upper, that is, saturated. An important fact is that the saturation of a compact set is again compact, for a set A has exactly the same open covers as its saturation.

For any topological space X the set of open subsets forms a complete lattice $\Omega(X)$ with respect to subset inclusion. Vice versa, for every complete lattice L the set of completely prime filters, denoted $\mathsf{pt}(L)$, carries the topology $\{O_a \mid a \in L\}$ where $F \in O_a$ if $a \in F$. A space is T_0 if the assignment, which associates with a point $x \in X$ the open neighborhood filter $\mathcal{N}(x)$, is injective. A space is called *sober* if the assignment is bijective. See [1, Section 7] for a detailed introduction to this topic. We are now ready to define the objects of interest in this paper:

Definition 2.1 A topological space is called *stably compact* if it is sober, compact, locally compact and finite intersections of compact saturated subsets are again compact.

Stably compact spaces have been studied intensively (and under many different names), [8,10,9,24,19,15] but, unfortunately, apart from [17] there is no single comprehensive reference for their many properties. We therefore state the main facts needed in the sequel. Our principal technical tool is the Hofmann-Mislove Theorem, [11,18]:

Theorem 2.2 *Let X be a sober space. There is an order-reversing bijection between the set $\mathcal{K}(X)$ of compact saturated subsets of X (ordered by reversed inclusion) and Scott-open filters in $\Omega(X)$ (ordered by inclusion). It assigns to a compact saturated set the filter of open neighborhoods and to a Scott-open filter of open sets their intersection.*

One consequence of this which we will need later is that every Scott-open filter in $\Omega(X)$ is equal to the intersection of all completely prime filters containing it. Another is the fact that the set $\mathcal{K}(X)$ is a dcpo when equipped with reversed inclusion. For stably compact spaces even more is true:

Proposition 2.3 *Let X be a stably compact space.*

(i) *$\mathcal{K}(X)$ is a complete lattice in which suprema are calculated as intersections and finite infima as unions.*

(ii) *$\Omega(X)$ and $\mathcal{K}(X)$ are stably continuous frames.*

(iii) *In $\Omega(X)$ we have $O \ll O'$ if and only if there is $K \in \mathcal{K}(X)$ with $O \subseteq K \subseteq O'$.*

(iv) *In $\mathcal{K}(X)$ we have $K \ll K'$ if and only if there is $O \in \Omega(X)$ with $K' \subseteq O \subseteq K$.*

As in [15] we use *stably continuous frame* to denote continuous distributive

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lattices in which the way-below relation is *multiplicative*, that is, in which $x \ll y, z$ implies $x \ll y \wedge z$ and in which $1 \ll 1$. They are precisely the Stone duals of stably compact spaces, see [10, Theorem 1.5]. Note that the proposition tells us that the complements of compact saturated sets form another topology on X, called the *co-compact topology* for X and denoted by X_κ. Original and co-compact topology are closely related:

Proposition 2.4 *Let X be a stably compact space.*

(i) *The open sets of X_κ are the complements of compact saturated sets in X.*

(ii) *The open sets of X are the complements of compact saturated sets in X_κ.*

(iii) *X_κ is stably compact and $(X_\kappa)_\kappa$ is identical to X.*

(iv) *The specialization order of X is the inverse of the specialization order of X_κ.*

For a stably compact space X, the *patch topology* of X is the common refinement of the original topology and the co-compact topology. It is denoted by X_π. It is the key to making the connection to much earlier work by Leopoldo Nachbin, [21]: A *partially ordered space* or *pospace* is a topological space X with a partial order relation $\sqsubseteq_X$ such that the graph of $\sqsubseteq_X$ is a closed subset of $X \times X$. Such a space must be Hausdorff because the diagonal relation, i.e., the intersection of $\sqsubseteq_X$ and the opposite partial order $\sqsupseteq_X$, is closed.

Theorem 2.5 *For a stably compact space X the specialization order together with the patch topology makes X_π into a compact ordered space. Conversely, for a compact ordered space $(X, \sqsubseteq)$ the open upper sets $\uparrow U = U \in \Omega(X)$ form the topology for a stably compact space $X^\uparrow$, and the two operations are mutually inverse.*

Moreover, for a stably compact space X the upper closed sets of X_π are precisely the compact saturated sets of X.

Notice that for a compact Hausdorff space X, the diagonal relation Δ_X is a closed (trivial) partial order. By applying Theorem 2.5 to the pospace (X, Δ_X), we see that the upper opens and lower opens are just the opens of the original topology. So $X = X_\kappa = X_\pi$. The converse also holds.

Corollary 2.6 *A space X is compact Hausdorff if and only if it is a stably compact space for which $X = X_\kappa$.*

Proof. The patch topology for any stably compact space is Hausdorff. In the case of a stably compact space for which $X = X_\kappa$, the patch topology is simply the original. □

We can thus think of stably compact spaces as the T_0 generalization of compact Hausdorff spaces. The fact that $X \neq X_\kappa$ in general forces us to tread carefully in Section 2.2 as we generalize from closed relations between compact Hausdorff spaces to closed relations between stably compact spaces.

The importance of stably compact spaces for domain theory is that almost all categories used in semantics are particular categories of stably compact spaces.

Proposition 2.7 *FS domains, and hence in particular Scott domains and continuous lattices, equipped with their Scott topologies, are stably compact spaces.*

2.2 The morphisms

The obvious category of stably compact spaces is that of continuous functions, i.e. the full subcategory SCS of the category of topological spaces Top. The category that we are really interested in, however, is one that generalizes KHaus*, the category of compact Hausdorff spaces and closed relations. We quote the basic definitions and results from [14].

The specialization order of a stably compact space X is generally not closed in $X \times X$. Indeed, were it closed, X would be a pospace, hence would be Hausdorff. Thus, specialization would be trivial. Specialization, on the other hand, is reversed by taking the co-compact topology (again, in the Hausdorff case $X = X_\kappa$ so the "reversal" is trivial). Thus:

Proposition 2.8 *The specialization order of a stably compact space X is closed in $X \times X_\kappa$.*

Proof. Suppose that $x \not\sqsubseteq_X y$. Then there is an open set U containing x and not y. By local compactness, we can assume that U is contained in a compact saturated neighborhood K of x that also does not contain y. U is an upper set containing x. The complement of K is a lower set containing y. Thus $U \times (X \setminus K)$ is a neighborhood of $\langle x, y \rangle$ in $X \times X_\kappa$ that does not meet $\sqsubseteq_X$. □

For stably compact spaces X and Y, we call a closed subset $R \subseteq X \times Y_\kappa$ a *closed relation from X to Y* and we write it as $R \colon X \longrightarrow Y$. If we spell out this condition then it means that for $x \in X$ and $y \in Y$ such that $x \not\!\! R\, y$ we find an open neighborhood U of x and a compact saturated set $K \subseteq Y$ that doesn't contain y such that $U \times (Y \setminus K) \cap R = \emptyset$. [cf. the proof Proposition 2.8.] Note that every closed relation R satisfies the rule $x' \sqsubseteq_X x\, R\, y \sqsubseteq_Y y' \implies x'\, R\, y'$.

The composition of closed relations is the usual relation product, $R\,;S = \big\{ \langle x, z \rangle \mid (\exists y)\, x\, R\, y \text{ and } y\, S\, z \big\}$. Note that, following usual practice, we write the composition of relations from left to right, whereas for functions it is from right to left. To avoid ambiguity we use ";" to indicate left-to-right composition. Notice that the specialization order of any stably compact space X acts as identity under taking the relation product with closed relations from or to X and also that the composition of two closed relations is again closed. We call the category of stably compact spaces with closed relations SCS*.

The Hausdorff case is worth considering separately as it helps to illuminate the definition of closed relations. As we have noted, a stably compact space is Hausdorff if and only if its topology agrees with its co-compact topology. Thus our closed relations from X to Y are simply closed subsets of $X \times Y = X \times Y_\kappa$ whenever Y is Hausdorff. Thus SCS^* correctly generalizes KHaus^*, in which we could take the morphisms simply as closed subsets of $X \times Y$. The fact that we could get away with this apparently simpler notion of morphism in the Hausdorff setting is due essentially to the fact that in compact Hausdorff spaces the co-compact topology is "hidden from view." In particular, KHaus^* is a full subcategory of SCS^* (as well as being a subcategory of Rel).

Note that the obvious forgetful "functor" from SCS^* to Rel, the category of sets with relations, preserves composition but *not* identities. The only stably compact spaces for which identity is preserved are those with trivial specialization orders, i.e., the compact Hausdorff spaces.

Relations between sets can be understood as multi-functions. As the following proposition shows this carries over to our topological setting in an interesting way.

Proposition 2.9 *Let X and Y be stably compact spaces and $R \colon X \longrightarrow Y$ a closed relation then*

$$f_R(x) := \{y \in Y \mid x \, R \, y\}$$

defines a continuous function from X to $\mathcal{K}(Y)$, where the latter is equipped with the Scott topology. Conversely, if $f \colon X \to \mathcal{K}(Y)$ is continuous then

$$\{\langle x, y \rangle \in X \times Y \mid y \in f(x)\}$$

is a closed relation from X to Y. Moreover, these two translations are mutually inverse.

To extend this correspondence to the composition of relations and multi-functions, respectively, we first have to define a law of composition on the latter. To this end recall that $\mathcal{K}(X)$ with its Scott topology is again a stably compact space by Propositions 2.3 and 2.7. Hence we can make $\mathcal{K}$ into an endofunctor on SCS by mapping a continuous function $f \colon X \to Y$ to the function $\mathcal{K}(f) \colon \mathcal{K}(X) \to \mathcal{K}(Y)$ that takes a compact saturated subset $K \subseteq X$ to $\uparrow f[K]$. This endofunctor is part of a monad whose unit takes the saturation of points and whose multiplication is simply union [22]. Consequently, the canonical composition of multi-functions is Kleisli composition which turns out to be the analogue of ordinary relation product.

Proposition 2.10 *The category of closed relations SCS^* is isomorphic to the Kleisli category $\mathsf{SCS}_\mathcal{K}$.*

It is generally the case that a category C with a monad T is embedded in the Kleisli category C_T simply by post-composing with the unit of the monad.

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Moreover, if the units of the monad are monic, then the embedding is faithful. Hence, SCS is a subcategory of $\mathsf{SCS}_{\mathcal{K}}$ and thus also of SCS^*. Concretely, this embedding works by taking the *hypergraph* of a function. The following proposition characterizes those relations that are really embedded functions:

Proposition 2.11 *If* $f\colon X \to Y$ *is a continuous function then the hypergraph*

$$\big\{ \langle x, y \rangle \in X \times Y \mid f(x) \sqsubseteq y \big\}$$

is a closed relation from X to Y. Conversely, if $R\colon X \mathrel{\longrightarrow\!\!\!\!\!\!+\,\,} Y$ is a closed relation such that for all $x \in X$ the set $f_R(x)$ has a least element $r(x)$ then $r\colon X \to Y$ is a continuous function, and this operation is the inverse of the previous.

Again, the Hausdorff case may help to illuminate this. If $f\colon X \to Y$ is a continuous function with Y a compact Hausdorff space, then the hypergraph is simply the graph of f. This is a closed relation just as classical topology tells us it should be. Conversely, suppose that a closed relation from X to Y is the graph of a function g. Then clearly $f_R(x)$ has a least element $g(x)$ for each x. Thus g is a continuous function.

2.3 The category

The left adjoint from SCS to the Kleisli category $\mathsf{SCS}_{\mathcal{K}} \cong \mathsf{SCS}^*$ preserves coproducts. Hence, they are given in SCS^* simply as topological coproducts, i.e., as disjoint unions.

In the category Rel of sets and relations for every relation $R\colon X \mathrel{\longrightarrow\!\!\!\!\!\!+\,\,} Y$ there is the reciprocal relation R_κ that is given by $y \, R_\kappa \, x \iff x \, R \, y$. This is the main ingredient that makes Rel into an allegory [7]. Our category SCS^* fails to be an allegory exactly because, as we shall see, it lacks a true reciprocation operation. On the other hand, if $R\colon X \mathrel{\longrightarrow\!\!\!\!\!\!+\,\,} Y$ is a closed relation between stably compact spaces then $R_\kappa\colon Y_\kappa \mathrel{\longrightarrow\!\!\!\!\!\!+\,\,} X_\kappa$ is a closed relation between the co-compact topologies, and $(\cdot)_\kappa$ is an involution on SCS^*. The problem is that it doesn't fix objects. We can think of X_κ as an upside-down version of X since the specialization order $\sqsubseteq_{X_\kappa}$ for the co-compact topology is simply $\sqsupseteq_X$, i.e. the dual of the one for the original space.

Nonetheless, the maps $X \mapsto X_\kappa$ and $R \mapsto R_\kappa$ comprise a contravariant functor, showing that SCS^* is a self-dual category. Consequently, categorical products (denoted here by $X \times^* Y$ to avoid conflict with topological products $X \times Y$) are also given by disjoint union:

$$X \times^* Y \cong (X_\kappa + Y_\kappa)_\kappa = (X_\kappa \,\dot\cup\, Y_\kappa)_\kappa = (X_\kappa)_\kappa \,\dot\cup\, (Y_\kappa)_\kappa = X \,\dot\cup\, Y = X + Y.$$

If a self-dual category is cartesian closed then all objects are isomorphic and hence the category is equivalent to the category with only one (identity) morphism. This shows that SCS^* cannot be cartesian closed.

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Since categorical products in SCS^* are the same as co-products, let us look at cartesian products. In SCS they are the categorical product and we can lift them to SCS^* to make SCS^* into a symmetric monoidal category. The tensor product takes the cartesian product of the spaces with the product topology and we also embed the morphisms needed for the symmetric monoidal structure from SCS as described in Proposition 2.11. The definition of the tensor product of two closed relations R and S is pointwise, i,e, $\langle x, y \rangle \ R \otimes S \ \langle x', y' \rangle : \iff \ x \ R \ y$ and $x' \ R \ y'$. This defines a closed relation and extends to products of continuous functions; for the details see [17, Section 3.2.4].

With respect to $\otimes$, the category SCS^* is closed: Because of $(X \times Y)_\kappa = X_\kappa \times Y_\kappa$ we see that closed subsets of $(X \times Y) \times Z_\kappa$ are the same thing as closed subsets of $X \times (Y_\kappa \times Z)_\kappa$ which proves $\mathsf{SCS}^*(X \otimes Y, Z) \cong \mathsf{SCS}^*(X, Y_\kappa \otimes Z)$. This internal homset $Y_\kappa \otimes Z$, however, does not correspond to the "real" homset $\mathsf{SCS}^*(Y, Z)$.

The homset $\mathsf{SCS}^*(Y, Z)$ consists of the closed subsets of $Y \times Z_\kappa$ which by Theorem 2.5 are precisely the compact saturated subsets of the dual $(Y \times Z_\kappa)_\kappa$. Hence, we can write the *relation space* as $[Y \Rightarrow Z] := \mathcal{K}(Y_\kappa \times Z)$. With this definition and Proposition 2.10 we get

$$\mathsf{SCS}^*(X \otimes Y, Z) \cong \mathsf{SCS}^*(X, Y_\kappa \otimes Z) \cong \mathsf{SCS}\big(X, \mathcal{K}(Y_\kappa \otimes Z)\big) = \mathsf{SCS}\big(X, [Y \Rightarrow Z]\big).$$

So, we see that $(- \otimes Y)$ and $[Y \Rightarrow -]$ are almost adjoint. The problem is that the induced morphism $X \longmapsto [Y \Rightarrow Z]$ is not uniquely determined.

The canonical evaluation morphism is a *functional* closed relation and for the induced morphism we can always choose a functional one, and as such it is unique, i.e. these morphisms come from SCS rather than SCS^*. In [23] such a situation is called a *Kleisli exponential*. There is an alternative description of the relation space by observing $\mathsf{SCS}^*(Y, Z) \cong \mathsf{SCS}\big(Y, \mathcal{K}(Z)\big)$. Thus the normal function space $[Y \to \mathcal{K}(Z)]$ with the compact-open topology, which is simply the Scott topology, yields a space that is homeomorphic to $[Y \Rightarrow Z]$. This construction was first studied in [25], although it seems that some of subtleties concerning the fact that this is only a Kleisli exponential were overlooked.

3 Stone Duality

Next we develop the Stone duality of closed relations. The morphisms between open set lattices corresponding to closed relations turn out to be preframe homomorphisms, [2], preserving finite meets and directed suprema. They have been studied in a similar framework before, see [26, Prop. 11.2.5], but the duality with relations seems to be new.

3.1 Relational preimage

If $R\colon X \longrightarrow Y$ is a relation and $A \subseteq X$ a subset, then we write

$$[A]R := \big\{ y \in Y \ \big| \ (\exists x \in A) \ x \ R \ y \big\}$$

for the usual forward image. The definition of the preimage of a subset $B \subseteq Y$ under the relation R is a bit more tricky as there are several candidates. Here, we are only interested in the *universal preimage* given by

$$(\forall R)[B] := \big\{ x \in X \ \big| \ (\forall y \in Y) \ x \ R \ y \Longrightarrow y \in B \big\}.$$

This definition is useful because $\forall R$ turns out to be the right adjoint to $[\cdot]R$:

Lemma 3.1 *If $R \subseteq X \times Y$ is a relation and A and B are subsets of X and Y, respectively, then we have*

$$[A]R \subseteq B \iff A \subseteq (\forall R)[B].$$

In the usual functional setting the situation is analogous; preimage is right adjoint to direct image. The connection between relational and functional preimage is the following.

Lemma 3.2 *If $f\colon X \to Y$ is a continuous function between stably compact spaces and $F\colon X \longrightarrow Y$ the corresponding closed relation given by the hypergraph, then for all upper sets $A = {\uparrow}A \subseteq Y$ we have*

$$f^{-1}[A] = (\forall F)[A].$$

We now describe the translation from topological spaces to frames in the relational setting.

Proposition 3.3 *If $R\colon X \longrightarrow Y$ is a closed relation then $\forall R$ is a continuous semilattice homomorphism from $\Omega(Y)$ to $\Omega(X)$, i.e. it preserves finite infima and directed suprema.*

Proof. First, we have to check that for any open $V \subseteq Y$ the preimage $(\forall R)[V]$ is open. So let $x \in (\forall R)[V]$, or equivalently $f_R(x) = [x]R \subseteq V$. We know from Proposition 2.9 that f_R is continuous and thus Proposition 2.3 gives us an open neighborhood U of x such that $f_R(x') \subseteq V$ for all $x' \in U$. We conclude $x \in U \subseteq (\forall R)[V]$, thus showing that for a closed relation the universal preimage of an open set is open.

As we have seen in Lemma 3.1, $\forall R$ as a function between the full powersets is a right adjoint. As such it preserves all intersections and thus the finite meets in $\Omega(Y)$.

Thus, it is a monotone map and, consequently, to show that it also preserves directed suprema we only have to verify $(\forall R)\big[\bigcup^{\uparrow} V_i\big] \subseteq \bigcup^{\uparrow} (\forall R)[V_i]$. So,

we consider an $x \in (\forall R)\big[\bigcup^{\uparrow} V_i\big]$ which means $f_R(x) \subseteq \bigcup^{\uparrow} V_i$. But as $f_R(x)$ is compact we can find an index i such that $f_R(x) \subseteq V_i$ and, equivalently, such that $x \in (\forall R)[V_i]$. $\qquad\square$

We call $\Omega^* R$ the restriction and co-restriction of $\forall R$ to the open subsets of X and Y to simplify notation. Going from a relation to the forward image function is well-known to be functorial, and so is taking adjoints. By Lemma 3.1 this implies that universal preimage is also functorial. Clearly, $\Omega^* \sqsubseteq_X$ is the identity on $\Omega^*(X) = \Omega(X)$ as all open sets are upper sets. Thus Ω^* is a contravariant functor from SCS^* to the category of stably continuous frames and Scott continuous semilattice homomorphisms which we denote by SCF^*.

Just like Ω we also have to adjust the functor pt to the relational setting. Consider a homomorphism $\phi \colon L \to M$. We define the relation $\mathsf{pt}^*(\phi) \colon \mathsf{pt}^*(M) \relbar\joinrel\mapsto \mathsf{pt}^*(L)$ by

$$ Q \; \mathsf{pt}^*(\phi) \; P \; :\Longleftrightarrow \; \phi^{-1}[Q] \subseteq P $$

where pt^* on objects behaves just like the usual pt, i.e., P and Q are completely prime filters in L and M, respectively. Alternatively, we can identify completely prime filters with their characteristic functions which are frame morphisms to $\mathbf{2}$, the two-element lattice. For two such *points* $p \colon L \to \mathbf{2}$ and $q \colon M \to \mathbf{2}$ the above definition becomes

$$ q \; \mathsf{pt}^*(\phi) \; p \; :\Longleftrightarrow \; q \circ \phi \sqsubseteq p. $$

Proposition 3.4 *If $\phi \colon L \to M$ is a continuous semilattice homomorphism, then $\mathsf{pt}^*(\phi) \colon \mathsf{pt}^*(M) \relbar\joinrel\mapsto \mathsf{pt}^*(L)$ is a closed relation.*

Proof. Suppose $Q \subseteq M$ and $P \subseteq L$ are completely prime filters such that $\phi^{-1}[Q] \nsubseteq P$. As ϕ is Scott continuous and Q completely prime and thus, in particular, Scott open, the set $\phi^{-1}[Q]$ is also Scott open. Because it is also not contained in P and L is a continuous lattice we can find an $x \in \phi^{-1}[Q] \setminus P$ such that $\Uparrow x \nsubseteq P$. On the other hand Q, as an upper set, is the union of principal filters $\uparrow y$ for $y \in Q$ and hence we get $\phi^{-1}[Q] = \phi^{-1}\big[\bigcup\{\uparrow y \mid y \in Q\}\big] = \bigcup\big\{\phi^{-1}[\uparrow y] \;\big|\; y \in Q\big\} \ni x$. This means that we can find a $y \in Q$ such that $x \in \phi^{-1}[\uparrow y]$.

As L is stably continuous, the set $\Uparrow x$ is a Scott open filter which corresponds to the compact saturated subset $\big\{P \in \mathsf{pt}^*(L) \;\big|\; \Uparrow x \subseteq P\big\}$ of $\mathsf{pt}^*(L)$ by the Hofmann-Mislove theorem. Now, we consider the open subset of $\mathsf{pt}^*(M) \times \mathsf{pt}^*(L)_{\kappa}$ which is given as the product of the open set corresponding to y and to the complement of the compact saturated set corresponding to $\Uparrow x$, and we claim that this is a neighborhood of $\langle Q, P \rangle$ that doesn't meet R_{ϕ}. Clearly, $\langle Q, P \rangle$ is in this set, and if $Q' \in \mathsf{pt}^*(M)$ and $P' \in \mathsf{pt}^*(L)$ are such that $y \in Q'$ and $\Uparrow x \nsubseteq P'$ we get $\phi^{-1}[Q'] \supseteq \phi^{-1}[\uparrow y] \ni x$ and thus $\phi^{-1}[Q'] \supseteq \Uparrow x$ which implies $\phi^{-1}[Q'] \nsubseteq P'$. $\qquad\square$

Now we have all the ingredients for a duality between SCS^* and SCF^*. It remains to check that the categorical conditions are indeed met.

Theorem 3.5 *The contravariant functors Ω^* and pt^* are part of a dual equivalence between the categories SCF^* and SCS^*.*

Proof. We begin by showing that pt^* is indeed a functor. Clearly, $\mathsf{pt}^*(\mathsf{id}_L) = \sqsubseteq_{\mathsf{pt}^*(L)}$, the identity closed relation on $\mathsf{pt}^*(L)$. The interesting direction for functoriality is to show that $\mathsf{pt}^*(\psi \circ \phi) \subseteq \mathsf{pt}^*(\psi); \mathsf{pt}^*(\phi)$, where $\phi \colon L \to M$ and $\psi \colon M \to N$ are continuous semilattice morphisms. Let $P \in \mathsf{pt}^*(N)$ and $P' \in \mathsf{pt}^*(L)$ be such that P $(\mathsf{pt}^*(\psi \circ \phi))$ P', or equivalently that $\phi^{-1}\big[\psi^{-1}[P]\big] \subseteq P'$. We need to find a completely prime filter $Q \subseteq M$ that satisfies $\psi^{-1}[P] \subseteq Q$ and $\phi^{-1}[Q] \subseteq P'$. Unfortunately, $\psi^{-1}[P]$ in general is only a Scott open filter, not a point in M.

However, by the Hofmann-Mislove Theorem, 2.2, we have $\psi^{-1}[P] = \bigcap\{Q \in \mathsf{pt}^*(M) \mid \psi^{-1}[P] \subseteq Q\}$. So for the sake of contradiction, assume there exists $x_Q \in \phi^{-1}[Q] \setminus P'$ for all $Q \supseteq \psi^{-1}[P]$. Then the supremum $\bigvee x_Q$ of all these elements does not belong to P' because P' is completely prime; on the other hand, $\phi(\bigvee x_Q)$ belongs to all $Q \supseteq \psi^{-1}[P]$ by monotonicity of ϕ, hence to $\psi^{-1}[P]$. This contradicts the assumption $\phi^{-1}[\psi^{-1}[P]] \subseteq P'$.

To show that Ω^* and pt^* give rise to a duality between SCF^* and SCS^* we have to check that their actions on morphisms are mutually inverse. So, suppose $R \colon X \longrightarrow\!\!\!\!+ Y$ is a closed relation and $\mathcal{N}(x)$ and $\mathcal{N}(y)$ are the open neighborhood filters of two points $x \in X$ and $y \in Y$. We get

$$
\begin{aligned}
\mathcal{N}(x) \, (\mathsf{pt}^*(\forall R)) \, \mathcal{N}(y) &\iff (\forall R)^{-1}\big[\mathcal{N}(x)\big] \subseteq \mathcal{N}(y) \\
&\iff \big(\forall V \in \Omega^*(Y)\big) \, V \in (\forall R)^{-1}\big[\mathcal{N}(x)\big] \implies V \in \mathcal{N}(y) \\
&\iff \big(\forall V \in \Omega^*(Y)\big) \, x \in (\forall R)[V] \implies y \in V \\
&\iff \big(\forall V \in \Omega^*(Y)\big) \, [x]R \subseteq V \implies y \in V
\end{aligned}
$$

Clearly, $x \mathrel{R} y$ implies this last condition and the converse follows from the fact that $[x]R$ is saturated.

Finally, we take a continuous semilattice morphism $\phi \colon L \to M$ and show that $\big(\Omega^*(\mathsf{pt}^*(\phi))\big)\big(\{P \in \mathsf{pt}^*(L) \mid x \in P\}\big) = \big\{Q \in \mathsf{pt}^*(M) \mid \phi(x) \in Q\big\}$ for any $x \in L$:

$$
\begin{aligned}
&\big(\forall \mathsf{pt}^*(\phi)\big)\Big(\{P \in \mathsf{pt}^*(L) \mid x \in P\}\Big) \\
&= \Big\{Q \in \mathsf{pt}^*(M) \mid \big(\forall P \in \mathsf{pt}^*(L)\big) \, Q \, (\mathsf{pt}^*(\phi)) \, P \implies x \in P\Big\} \\
&= \Big\{Q \in \mathsf{pt}^*(M) \mid \big(\forall P \in \mathsf{pt}^*(L)\big) \, \phi^{-1}[Q] \subseteq P \implies x \in P\Big\}
\end{aligned}
$$

As before we use the fact that $\phi^{-1}[Q]$ is a Scott-open filter and hence by the Hofmann-Mislove Theorem equal to the intersection of all completely prime filters containing it. The expression then re-writes to $\{Q \in \mathsf{pt}^*(M) \mid x \in \phi^{-1}[Q]\}$ which is equal to $\{Q \in \mathsf{pt}^*(M) \mid \phi(x) \in Q\}$ as desired. $\qquad\square$

It is interesting to consider the Stone dual of the involution on SCS^* that we

discussed in Section 2.3. The co-compact topology on a stably compact space has precisely the compact saturated subsets of the original space as closed sets which implies $\Omega^*(X_\kappa) = \Omega(X_\kappa) \cong \mathcal{K}(X)$. From the Hofmann-Mislove Theorem we know that $\mathcal{K}(X)$ is in one-to-one correspondence to the Scott open filters in $\Omega(X)$. The latter can also be understood via their characteristic functions which are precisely the continuous semilattice homomorphisms to $\mathbf{2}$, the two-element lattice. Putting it all together we get $\Omega(X_\kappa) \cong \mathcal{K}(X) \cong$ $\mathsf{SCF}^*\big(\Omega(X), \mathbf{2}\big)$ and we see that this self-duality in localic terms is exactly the Lawson duality of stably continuous semilattices [20].

3.2 Functions revisited

We know from Proposition 2.11 that SCS embeds faithfully in SCS^* and also how to recognize the morphisms that arise from this embedding as hypergraphs of functions. We refer to a closed relation as *functional* if it is the hypergraph of a continuous function. Similarly the category SCF^* contains a subcategory of functional arrows.

Proposition 3.6 *If $R\colon X \longmapsto Y$ is a functional closed relation then $\Omega^*(R)$ preserves finite (and consequently all) suprema. Conversely, if $\phi\colon L \to M$ is a frame homomorphism then $\mathsf{pt}^*(L)$ is functional.*

Proof. If ϕ is a frame homomorphism then for any completely prime filter $Q \subseteq M$ the preimage $\phi^{-1}[Q]$ is completely prime. Hence, this is the least completely prime filter $P \subseteq L$ such that $\phi^{-1}[Q] \subseteq P$.

For the converse observe that the forward image $[x]R$ of any point x has a least element and hence will be contained in either U or V iff it is contained in $U \cup V$. This shows that $\forall R$ preserves finite suprema. $\qquad\square$

This result, of course, is very similar to the classical Stone duality between SCS, the category of stably compact spaces with continuous functions, and $\mathsf{SCF}^*_\vee$, stably continuous lattices with frame homomorphisms. There the functors Ω and pt act on morphisms as follows: $\Omega(f)$ is simply the preimage function $f^{-1}[\cdot]$ and similarly $\mathsf{pt}(\phi)$ takes a completely prime filter P to the completely prime filter $\phi^{-1}[P]$. As a corollary of the previous proposition we get that pt^* and Ω^* commute with the embeddings of the functional subcategories.

Corollary 3.7 *The diagram of functors*

$$
\begin{array}{ccc}
\mathsf{SCS} & \underset{\mathsf{pt}}{\overset{\Omega}{\rightrightarrows}} & \mathsf{Frm} \\
\Big\uparrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle j} \\
\mathsf{SCS}^* & \underset{\mathsf{pt}^*}{\overset{\Omega^*}{\rightleftarrows}} & \mathsf{SCF}^*
\end{array}
$$

commutes in the sense that $j \circ \Omega = \Omega^* \circ i$ *and* $i \circ \mathsf{pt}^* = \mathsf{pt} \circ j$.

Proof. The first equality was proved in Lemma 3.2. For the second, take a frame morphism $\phi \colon L \to M$. It is mapped by $i \circ \mathsf{pt}$ to the hypergraph of the preimage function, i.e. the closed relation that relates $Q \in \mathsf{pt}(M) = \mathsf{pt}^*(M)$ to $P \in \mathsf{pt}(L) = \mathsf{pt}^*(L)$ if and only if $\phi^{-1}[Q] \subseteq P$ which is precisely $\mathsf{pt}^*(j(\phi))$. $\square$

As a consequence of this corollary the operation which extracts from a functional relation the underlying continuous function (which exists by Proposition 2.11) is just the composition $\mathsf{pt} \circ \Omega^*$. It follows that this is functorial. We denote it by U.

There is a more categorical way to identify the functional morphisms in the two dual categories. As we have seen in Section 2.3, the products on the functional subcategory give rise to a symmetric monoidal structure on the larger relational category. In addition, the diagonals $\Delta_A \colon A \to A \times A$ and morphisms $!_A$ to the terminal object induce a *diagonal structure*. The functional morphisms are then characterized as the *total* and *deterministic* morphisms, i.e. the ones for which $!$ and Δ, respectively, are natural transformations. For more details see [17, Section 3.3].

4 Subspaces

There are a number of different concepts of "good subspace" in Topology as often simply carrying the induced topology is too weak. One very useful one that is well-known in domain theory is that of an *embedding-projection pair*. It combines the categorical notion of section retraction pair with the order theoretic notion of adjunction. It is then an immediate corollary that the space that is the codomain of the section carries the subspace topology. In the following we will generalize this to the relational setting.

4.1 Perfect relations

We start by defining a special class of relations that will be important when we characterize relations that have adjoints.

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Definition 4.1 We say that a closed relation $R\colon X \longmapsto Y$ is *perfect* if for all compact saturated sets $K \subseteq Y$ the preimage $(\forall R)[K]$ is compact.

Perfect relations can alternatively be characterized in terms of their Stone duals.

Proposition 4.2 *A closed relation $R\colon X \longmapsto Y$ is perfect if and only if $\Omega^*(R)$ preserves the way-below relation.*

Proof. Let us assume that R is perfect and $U \ll V$ are open subsets of Y. Then there is a compact saturated set $K \subseteq Y$ such that $U \subseteq K \subseteq V$ and we get $\Omega^*(R)(U) = (\forall R)[U] \subseteq (\forall R)[K] \subseteq (\forall R)[V] = \Omega^*(R)(V)$. By assumption $(\forall R)[K]$ is compact and hence we conclude $\Omega^*(R)(U) \ll \Omega^*(R)(V)$.

Conversely, suppose $\Omega^*(R)$ preserves way-below and $K \subseteq Y$ is compact saturated. As a saturated set, K it is the intersection of all the open sets that contain it and we compute

$$(\forall R)[K] = (\forall R)\Big[\bigcap\nolimits_{\downarrow}\{U \in \Omega^*(Y) \mid K \subseteq U\}\Big] = \bigcap\nolimits_{\downarrow}\big\{(\forall R)[U] \mid K \subseteq U\big\}$$

where the last equality follows because, by Lemma 3.1, $\forall R$ is a right adjoint and hence preserves arbitrary intersections in $\mathfrak{P}(Y)$. Now we claim that this last intersection is taken over a filterbase for a Scott open filter in $\Omega^*(X) = \Omega(X)$. The set $\big\{(\forall R)[U] \mid K \subseteq U\big\}$ is clearly filtered. To see that it is generates a Scott open filter take $U \in \Omega(Y)$ that contains K. Since Y is locally compact, the neighborhood filter of the compact set K has a basis of compact saturated sets. This means that there is an open set V and a compact set K' such that $K \subseteq V \subseteq K' \subseteq U$. This implies $V \ll U$ and hence by assumption $(\forall R)[V] \ll (\forall R)[U]$.

By the Hofmann-Mislove Theorem the intersection over a Scott open filter of open sets, and hence also of a filterbase for such a filter, is compact saturated. This shows that $(\forall R)[K]$ is compact and finishes the proof. $\square$

This extends the classical situation of functions between stably compact spaces (or, more generally, locally compact sober spaces), [10, Remark 1.3]. Since the Stone dual of a function has an upper adjoint, perfectness in that situation can be further characterized by the adjoint being Scott-continuous (loc. cit.). Because of Corollary 3.7 we have that a continuous function between stably compact spaces is perfect in the classical sense if and only if the corresponding relation given by the hypergraph is perfect in our sense.

It may be worthwhile to add a few words about terminology here. As we quoted, perfect maps have (at least) three different characterizations and furthermore many useful properties. Depending on what is considered essential in a given situation, additional assumptions are made in order to preserve certain key properties in the absence of local compactness, sobriety or both. This has led to an abundance of different concepts for which it now appears impossible to establish a coherent terminology. Either of "proper" [4,10] or

"perfect" [12,9,6] is usually used but it is not clear where the boundary between the two ought to be drawn. Our choice of "perfect" follows the more recent custom of reserving "proper" for slightly stronger requirements even in the case of locally compact sober spaces.

We also note that perfect *functions* between stably compact spaces are exactly those which are continuous with respect to both original and co-compact topology. This implies that they are exactly those maps which are monotone and patch continuous. To summarize:

Proposition 4.3 *Let $f\colon X \to Y$ be a function between stably compact spaces and $R\colon X \longmapsto Y$ the corresponding hypergraph. Then the following are equivalent:*

(i) *R is perfect;*

(ii) *f is perfect with respect to the original topologies;*

(iii) *f is perfect with respect to the co-compact topologies;*

(iv) *f is monotone and patch continuous.*

There is yet another approach to perfectness via uniform continuity: For every stably compact space there is a unique quasi-uniformity $\mathcal{U}$ such that $\mathcal{U}$ induces the topology and $\mathcal{U}^{-1}$ induces the co-compact topology. A continuous function $f\colon X \to Y$ between stably compact spaces is perfect if and only if it is uniformly continuous with respect to these unique quasi-uniformities on X and Y. For details see [25, Theorem 3].

In a way, perfect continuous functions seem to be a better notion of morphisms for the category SCS than just continuous ones, as open and compact saturated sets play similarly important roles. Moreover, with these morphisms we can explain in which way the patch topology is a "natural" construction: Every continuous function between compact Hausdorff spaces is perfect, and hence this category embeds fully and faithfully into SCS with perfect maps. Now, taking the patch topology is simply the right adjoint, i.e. the co-reflector, for this inclusion functor, [6].

Returning to closed relations again, perfectness is linked to openness. We say that a closed relation $R\colon X \longmapsto Y$ is *open* if for all open sets $U \subseteq X$ the forward image $[U]R$ is open.

For the next proposition we need the following observation which relates forward image, universal preimage, complementation and reciprocation:

Lemma 4.4 *If $R\colon X \longmapsto Y$ is a relation in Rel and $M \subseteq X$ is an arbitrary subset then $[X \setminus M]R = Y \setminus (\forall R_\kappa)[M]$.*

Proof. For $y \in Y$ we have

$$y \in [X \setminus M]R \iff (\exists x \in X \setminus M)\, x\, R\, y$$
$$\iff y \notin (\forall R_\kappa)[M]$$
$$\iff y \in Y \setminus (\forall R_\kappa)[M].$$

$\square$

Proposition 4.5 *A closed relation $R: X \longrightarrow Y$ is open if and only if the reciprocal relation $R_\kappa: Y_\kappa \longrightarrow X_\kappa$ is perfect.*

Proof. Let us assume that R is open. We take a compact saturated set $K \in \mathcal{K}(X_\kappa)$ and have to show that $(\forall R_\kappa)[K]$ is compact in Y_κ. By Theorem 2.5 the condition $K \in \mathcal{K}(X_\kappa)$ is equivalent to $X \setminus K \in \Omega(X)$ and the openness of R means that $[X \setminus K]R$ is open. By the previous lemma we have $[X \setminus K]R = Y \setminus (\forall R_\kappa)[K] \in \Omega(Y_\kappa)$ which, again by Theorem 2.5, implies that $(\forall R_\kappa)[K]$ is a compact saturated subset of Y_κ.

Conversely, if R_κ is perfect and $U \in \Omega(X)$ then $X \setminus U$ is compact saturated in X_κ. From the previous lemma we get $(\forall R_\kappa)[X \setminus U] = Y \setminus Y \setminus (\forall R_\kappa)[X \setminus U] = Y \setminus [X \setminus (X \setminus U)]R = Y \setminus [U]R$ which is a compact saturated subset of Y_κ because of the perfectness of R_κ. Consequently, its complement $[U]R$ is an open subset of Y. $\square$

4.2 Adjunctions

As usual in an order-enriched category, we say that for two closed relations $R: X \longrightarrow Y$ is the *left* or *lower adjoint* of $S: Y \longrightarrow X$ if $S\,;R: X \longrightarrow X$ is below the identity and if $R\,;S: Y \longrightarrow Y$ is above the identity on Y. Likewise, S is called the *right* or *upper adjoint* of R. The question is what is the right order on the homsets $\mathsf{SCS}^*(X, Y)$. One choice is subset inclusion but it turns out to be better to use the one induced from the corresponding homsets $\mathsf{SCS}\big(X, \mathcal{K}(Y)\big)$, in keeping with Proposition 2.10. Since $\mathcal{K}(Y)$ is ordered by reverse inclusion this means that the relations in the homsets for SCS^* are also ordered by reverse inclusion of their graphs. Note that adjoints determine each other uniquely as is the case in any order-enriched category.

Lemma 4.6 *The functors Ω^* and pt^* preserve the order on the homsets, thus making SCS^* and SCF^* dually equivalent as order-enriched categories. Consequently, we have $R \dashv S$ for closed relations if and only if $\Omega^*(S) \dashv \Omega^*(R)$.*

Proof. The first claim can easily be verified from the definition of the two functors. Then the second is an immediate consequence. Note, however, that because of contravariance the role of lower and upper adjoint are reversed. $\square$

Upper adjoints have a very concise characterization:

Theorem 4.7 *A closed relation $R: X \longrightarrow Y$ has a lower adjoint if and only if it is perfect and functional.*

Proof. From the previous lemma we know that R has a lower adjoint if and only if $\Omega^*(R)$ has an upper adjoint. As we know, $\Omega^*(R)$ is a continuous semilattice homomorphism and as a monotone function between the complete lattices $\Omega^*(Y) = \Omega(Y)$ and $\Omega^*(X) = \Omega(X)$ it is a lower adjoint if and only if it preserves all suprema. By Proposition 3.6 this is the case precisely when R is functional.

In this case we have an upper adjoint $u \colon \Omega^*(X) \to \Omega^*(Y)$, but it need not be a continuous semilattice homomorphism. As an upper adjoint it preserves all infima, but it is Scott continuous if and only if its adjoint $\Omega^*(R)$ preserves the way-below relation (see [1, Proposition 3.1.14]). From Proposition 4.2 we know that this is equivalent to R being perfect. □

Using Proposition 4.3 above we can rephrase this as follows.

Corollary 4.8 *A closed relation has a lower adjoint if and only if it is functional and the corresponding function is patch continuous, i.e. continuous with respect to the patch topologies.*

In the case of Hausdorff spaces the last condition is trivially true since the patch topology is simply the original topology. Hence, we get the following result.

Corollary 4.9 *A closed relation between compact Hausdorff spaces is a continuous function if and only if it has a lower adjoint in* SCS^*.
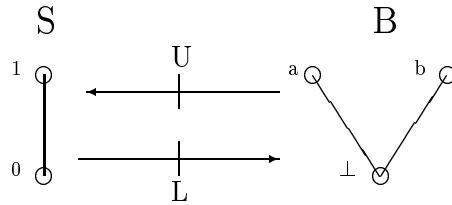

Fig. 1. A non-functional embedding retraction pair.

Consider the two posets given in Figure 1. We define two closed relations $L := \{0\} \times B \cup \{1\} \times \{a,b\}$ and $U := \{\bot\} \times S \cup \{a,b\} \times \{1\}$ which is the hypergraph of the function that maps $\bot$ to 0 and identifies a and b by mapping them to 1. We have $L \,;\, U = id_S$ and also $U \,;\, L \sqsubseteq id_B$ which shows that they form a embedding-projection pair in the sense that L is a lower adjoint section and U the corresponding upper adjoint retraction. This example shows that embeddings need not be functional.

We can, however, say explicitly what this lower adjoint does. Essentially it is just taking preimages under the function corresponding to its adjoint:

Proposition 4.10 *Let* $u \colon X \to Y$ *be a perfect continuous function between stably compact spaces,* $U \colon X \longrightarrow Y$ *its hypergraph and* L *the lower adjoint.*

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Then we have

$$y \, L \, x \iff x \in (\forall U)[\uparrow y] \iff y \leq u(x)$$

and the corresponding multi-function $f_L \colon Y \to \mathcal{K}(X)$ *satisfies*

$$f_L(y) = u^{-1}[\uparrow y].$$

Proof. Note that we have $x \in (\forall U)[\uparrow y] \iff x \in u^{-1}[\uparrow y]$ by Lemma 3.2, and hence the descriptions of the adjoint given in the proposition agree.

We begin by showing that L is a closed relation. The easiest proof is to show that f_L is continuous: It factorizes as $Y \xrightarrow{\uparrow} \mathcal{K}(Y) \xrightarrow{u^{-1}[\cdot]} \mathcal{K}(X)$ where the first function is already known to be continuous. The spaces $\mathcal{K}(Y)$ and $\mathcal{K}(X)$ carry the Scott topology and directed suprema are given by filtered intersections which are preserved by the preimage function $u^{-1}[\cdot]$. So, f_L is a composition of continuous functions.

To show $L \dashv U$ we have to check $\sqsubseteq_X = \mathsf{id}_X \subseteq U; L$ and $L; U \subseteq \mathsf{id}_Y = \sqsubseteq_Y$ since the order on the homsets is reversed inclusion. So, for $x \sqsubseteq x'$ we have $x \, U \, u(x) \, L \, x'$ since $u(x) \sqsubseteq u(x')$. For the second inclusion, $y \, L \, x \, U \, y'$ implies $y \sqsubseteq u(x) \sqsubseteq y'$.

$\square$

5 Bilimits

As our final topic we consider bilimits in SCS^*. In domain theory such bilimits are usually taken over directed diagrams of embedding-projection pairs. As pointed out in [1] the construction doesn't depend on the fact that the morphisms are sections and retractions but exclusively on the properties of the adjunctions. Hence, we discuss the construction of bilimits using this setup.

Both SCS^* and SCF^* are order enriched categories and support the notion of an adjoint pair. We denote the subcategories of lower adjoints by SCS^*_l and SCF^*_l, respectively. The dual categories of upper adjoints are denoted by SCS^*_u and SCF^*_u.

In the following we discuss bilimits of directed diagrams of adjoint closed relations between stably compact spaces, or to be more precise, colimits for functors from a directed poset I to the subcategory of lower adjoint closed relations SCS^*_l.

Theorem 5.1 *Every directed diagram in* SCS^*_l *has a bilimit.*

This means that it has a colimit which is also a colimit for the whole category SCS^*. *Moreover, the corresponding upper adjoints for the colimiting cocone make it into limit for the upper adjoints of the diagram and this is also a limit in the ambient category* SCS^*.

Proof. We prove this via the Stone dual. So let I be a directed set and

$D\colon I \to \mathsf{SCS}_l^*$ a directed diagram. We consider the composition $\Omega^* \circ D \to \mathsf{SCF}_u^*$ where we denote the objects as $L_i := \Omega^*(D(i))$ and the morphisms as ϕ_{ji} and their upper adjoints as ψ_{ij}. Such a diagram can be considered to consist of dcpo's and Scott-continuous maps. Hence the general domain theoretic machinery can be brought to bear, cf. [1, Section 3.3] and [8, Section IV-3]. From this we know that the (domain-theoretic) bilimit is given by

$$\Big\{ (x_i)_{i\in I} \in \prod_{i\in I} L_i \ \big| \ (\forall i < j)\, \psi_{ij}(x_j) = x_i \Big\}$$

and that the (Scott-continuous) maps $\psi_j \colon L \to L_j$, $\psi_j((x_i)_{i\in I}) = x_j$ form a limiting cone over the diagram $((L_i)_{i\in I}, (\psi_{ij})_{i\leq j})$ in the category DCPO. Furthermore, the (Scott-continuous) maps $\phi_i \colon L_i \to L$, $\phi_i(x) = \big(\bigsqcup^{\uparrow}_{k\geq i,j} \psi_{jk}(\phi_{ki}(x))\big)_{j\in I}$ form a colimiting cocone of the diagram $((L_i)_{i\in I}, (\phi_{ji})_{i\leq j})$ in DCPO. The following relationships hold:

(i) For all $i \in I$, ϕ_i is a lower adjoint of ψ_i.

(ii) $\mathsf{id}_L = \bigsqcup^{\uparrow}_{i\in I} \phi_i \circ \psi_i$.

(iii) $(\forall i, j \in I)\, \psi_j \circ \phi_i = \bigsqcup^{\uparrow}_{k\geq i,j} \psi_{jk} \circ \phi_{ki}$.

(iv) For any cone $(M, (\mu_i)_{i\in I})$ (of Scott-continuous maps) over the diagram $((L_i)_{i\in I}, (\psi_{ij})_{i\leq j})$ the mediating morphism $\mu\colon M \to L$ is given by $\mu = \bigsqcup^{\uparrow}_{i\in I} \phi_i \circ \mu_i$.

(v) For any cocone $(M, (\mu_i)_{i\in I})$ (of Scott-continuous maps) over the diagram $((L_i)_{i\in I}, (\phi_{ji})_{i\leq j})$ the mediating morphism $\mu\colon L \to M$ is given by $\mu = \bigsqcup^{\uparrow}_{i\in I} \mu_i \circ \psi_i$.

The objects and morphisms of the category SCF^* have additional structure, so we need to show the following:

(a) L is a complete lattice.

(b) L is continuous.

(c) L is distributive.

(d) The way-below relation on L is multiplicative and $\underline{1} \ll \underline{1}$.

(e) For all $i \in I$, ϕ_i and ψ_i preserve finite infima.

(f) Assuming that the cone (resp. cocone) maps preserve finite infima, so do the mediating morphisms.

For the sake of brevity, we will from now on write $\underline{x}$ for a sequence $(x_i)_{i\in I}$ wherever possible.

(a) The ψ_{ij}, as upper adjoints, preserve all infima. Hence these are calculated pointwise in L.

(b) Continuity follows for dcpo's already, see Theorem 3.3.11 in [1]. However, it will be necessary for the remaining claims to have a characterization of the way-below relation on L at hand. For this observe that the ϕ_i preserve way-below, [1, Proposition 3.1.14(2)]; we can therefore employ property 2

above to get $\underline{x} \ll \underline{y}$ iff there exists an index $j \in I$ and elements $x \ll y$ in L_i such that $\underline{x} \leq \phi_j(\overline{x)} \ll \phi_j(y) \leq \underline{y}$.

We need to do (e) next: The ψ_i preserve infima because they are upper adjoints. For the lower adjoints we exploit the fact that finite meets commute with directed joints in continuous lattices, [8, Corollary I-2.2]. The claim then follows directly from the formula for the ϕ_i.

(c) We need to invoke the continuity of L for this: Assume $\underline{a} \ll \underline{x} \wedge (\underline{y} \vee \underline{z})$. Using the continuity of supremum and infimum we know that there are additional sequences $\underline{a}'$, $\underline{b}$ and $\underline{c}$ such that $\underline{a} \leq \underline{a}' \wedge (\underline{b} \vee \underline{c})$ and $\underline{a}' \ll \underline{x}$, $\underline{b} \ll \underline{y}$ and $\underline{c} \ll \underline{z}$. By our characterization of way-below on L it follows that we can find elements x, y, z in some approximating lattice L_j such that $\underline{a}' \leq \phi_j(x) \leq \underline{x}$, etc. Now we can calculate $\underline{a} \leq \underline{a}' \wedge (\underline{b} \vee \underline{c}) \leq \phi_j(x) \wedge (\phi_j(y) \vee \phi_j(z)) = \phi_j(x \wedge (y \vee z)) = \phi_j((x \wedge y) \vee (x \wedge z)) = (\phi_j(x) \wedge \phi_j(y)) \vee (\phi_j(x) \wedge \phi_j(z)) \leq (\underline{x} \wedge \underline{y}) \vee (\underline{x} \wedge \underline{z})$.

(d) This is similar to the previous item: For $\underline{x} \ll \underline{y}, \underline{z}$ find $x \ll y$, $x' \ll z$ in some L_j such that $\underline{x} \leq \phi_j(x) \ll \phi_j(y) \leq \underline{y}$ and $\underline{x} \leq \phi_j(x') \ll \phi_j(z) \leq \underline{z}$. The claim then follows from multiplicativity of $\ll$ in L_j: $\underline{x} \leq \phi_j(x) \wedge \phi_j(x') = \phi_j(x \wedge x') \ll \phi_j(y \wedge z) = \phi_j(y) \wedge \phi_j(z) \leq \underline{y} \wedge \underline{z}$.

For $\underline{1} \ll \underline{1}$ just observe that $1 \ll 1$ holds in each L_i and the lower adjoints are SCF^* maps, that is, they preserve the empty meet.

(f) Like (e), this follows from the defining formulas for mediating morphisms and the fact that finite meets commute with directed suprema. $\qquad\square$

The limit-colimit coincidence for SCF^* which we established in the preceding proof says (among other things) that directed colimits in SCF^*_l are also colimits in the original category of semilattice homomorphisms. Both the diagram maps ϕ_{ji} and the cocone maps ϕ_i are in fact lower adjoints and consequently sup-preserving, which means that they are frame maps. Frame maps between continuous semilattices, however, are not necessarily lower adjoints. Nonetheless, directed colimits in SCF^*_l are also colimits of frames, as our next lemma shows.

Lemma 5.2 *The embedding of SCF^*_l into the category Frm of frames and frame homomorphisms preserves directed colimits.*

Proof. The colimit L of a directed diagram $((L_i)_{i \in I}, (\phi_{ji})_{i \leq j})$ in SCF^*_l as constructed in the proof of the previous theorem yields a distributive continuous lattice, hence a (spatial) frame, [8, Theorem 5.5]. The colimiting maps ϕ_i are lower adjoints in addition to being SCF^* morphisms, so they are frame homomorphisms. What needs to be shown is that the mediating morphism μ for a cocone $(\mu_i)_{i \in I}$ of frame homomorphisms is again a frame homomorphism. Since we already know that μ will be a continuous semilattice homomorphisms all that remains to be shown is preservation of (finite) suprema. The proof of this property is a beautiful interplay between formulas 2 and 3 from the preceding theorem. Let X be a set of elements of the colimit L. We calculate

for the non-trivial inequality:

$$\mu(\bigsqcup X) = \bigsqcup_{j \in I}^{\uparrow} \mu_j \circ \psi_j(\bigsqcup X) \qquad \text{definition of } \mu$$

$$= \bigsqcup_{j \in I}^{\uparrow} \mu_j \circ \psi_j(\bigsqcup_{\underline{x} \in X} \bigsqcup_{i \in I}^{\uparrow} \phi_i \circ \psi_i(\underline{x})) \qquad \text{formula 2}$$

$$= \bigsqcup_{j \in I}^{\uparrow} \bigsqcup_{i \in I}^{\uparrow} \mu_j \circ \psi_j(\bigsqcup_{\underline{x} \in X} \phi_i \circ \psi_i(\underline{x})) \qquad \text{associativity}$$

$$= \bigsqcup_{j \in I}^{\uparrow} \bigsqcup_{i \in I}^{\uparrow} \mu_j \circ \psi_j \circ \phi_i(\bigsqcup_{\underline{x} \in X} \psi_i(\underline{x})) \qquad \phi_i\text{'s are lower adjoints}$$

$$= \bigsqcup_{j \in I}^{\uparrow} \bigsqcup_{i \in I}^{\uparrow} \mu_j(\bigsqcup_{k \geq i,j}^{\uparrow} \psi_{jk} \circ \phi_{ki}(\bigsqcup_{\underline{x} \in X} \psi_i(\underline{x}))) \qquad \text{formula 3}$$

$$= \bigsqcup_{j \in I}^{\uparrow} \bigsqcup_{i \in I}^{\uparrow} \bigsqcup_{k \geq i,j}^{\uparrow} \mu_j \circ \psi_{jk}(\bigsqcup_{\underline{x} \in X} \phi_{ki} \circ \psi_i(\underline{x})) \qquad \begin{array}{l}\mu_j\text{'s are continuous \&} \\ \phi_{ki}\text{'s are lower adjoints}\end{array}$$

$$= \bigsqcup_{j \in I}^{\uparrow} \bigsqcup_{i \in I}^{\uparrow} \bigsqcup_{k \geq i,j}^{\uparrow} \mu_k \circ \phi_{kj} \circ \psi_{jk}(\bigsqcup_{\underline{x} \in X} \phi_{ki} \circ \psi_{ik} \circ \psi_k(\underline{x})) \qquad \text{(co)cone condition}$$

$$\leq \bigsqcup_{j \in I}^{\uparrow} \bigsqcup_{i \in I}^{\uparrow} \bigsqcup_{k \geq i,j}^{\uparrow} \mu_k(\bigsqcup_{\underline{x} \in X} \psi_k(\underline{x})) \qquad \text{adjointness of } \phi \text{ and } \psi$$

$$= \bigsqcup_{k \in I}^{\uparrow} \mu_k(\bigsqcup_{\underline{x} \in X} \psi_k(\underline{x})) \qquad \text{redundant indices}$$

$$= \bigsqcup_{k \in I}^{\uparrow} \bigsqcup_{\underline{x} \in X} \mu_k \circ \psi_k(\underline{x}) \qquad \mu_k\text{'s are frame maps}$$

$$= \bigsqcup_{\underline{x} \in X} \bigsqcup_{k \in I}^{\uparrow} \mu_k \circ \psi_k(\underline{x}) \qquad \text{associativity}$$

$$= \bigsqcup_{\underline{x} \in X} \mu(\underline{x}) \qquad \text{definition of } \mu$$

$\square$

Theorem 5.3 *The functor U from* SCS_u^* *to* SCS *preserves inverse limits.*

Proof. The dual equivalence between SCS_u^* and $\mathsf{SCF^*}_l$ transforms inverse limits into direct colimits. The latter are preserved by the inclusion of $\mathsf{SCF^*}_l$ into Frm according to the preceding lemma. Stone duality translates them into inverse limits in Top. $\square$

The reader may still feel a bit numb from all these calculations and not immediately recognize the force of this theorem. Let us therefore elaborate on its content a little bit. Top is a complete category and limits are calculated in the usual way: If $D \colon \mathsf{I} \to \mathsf{Top}$ is a functor (for any diagram D) then the points of $\lim D$ are given by *threads*:

$$\lim D = \big\{ (x_i)_{i \in \mathrm{obj}(\mathsf{I})} \in \prod_{i \in \mathrm{obj}(\mathsf{I})} D(i) \;\big|\; (\forall (f \colon i \to j) \in \mathrm{mor}(\mathsf{I})) \, D(f)(x_i) = x_j \big\}$$

The topology is inherited from the product space $\prod_{i \in \mathrm{obj}(\mathsf{I})} D(i)$. Upper adjoint relations between stably compact spaces are functional and the functor U associates with every such relation the generating (perfect) function. Theorem 5.3 then states that a bilimit in N^* is calculated topologically as the limit of the corresponding inverse diagram of perfect maps. One can turn this around and say that the content of the theorem is to recognize inverse limits of perfect maps as bilimits in an order-enriched setting, yielding a limit-colimit coincidence with respect to closed relations. This appears to be an important first step in making stably compact spaces a suitable universe for semantic interpretations.

Acknowledgements

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