On the bitopological nature of Stone duality

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Abstract

Based on the theory of frames we introduce a Stone duality for bitopological spaces. The central concept is that of a *d-frame*, which axiomatises the two open set lattices.

Exploring the resulting concept of d-sobriety we find this to be a much more inclusive concept than usual sobriety. Spatial d-frames suggest additional axioms that lead us to define *reasonable d-frames*; these have an alternative presentation as *partial frames*.

We explore natural notions of regularity and compactness for bitopological spaces, and their manifestation in d-frames. This yields the machinery to locate precisely within this general landscape a number of classical Stone-type dualities, namely, those of Stone for Boolean algebras and bounded distributive lattices, those of the present authors for strong proximity lattices (with negation), and the duality of classical frames.

The general duality can be given a logical reading by viewing the open sets of one topology as *positive* extents of formulas, and those of the other topology as *negative* extents. This point of view emphasises the fact that formulas may be undecidable in certain states and may be self-contradictory in others. We also obtain two natural orders on the set of formulas, one related to Scott's information order and the other being the usual logical implication. The interplay between the two can be said to be the main organising principle of this study.

Contents

1 Introduction and overview

In his landmark paper — published in two parts as [Sto36] and [Sto37a] — Marshall Stone showed that every Boolean algebra is isomorphic to a subalgebra of a powerset. The powerset is that of the set of prime filters of the given algebra, which in this paper we will call the *spectrum*. Stone also showed that when the spectra are suitably topologised, homomorphisms between algebras correspond exactly to continuous functions between the corresponding spectra (but the direction is reversed). Furthermore, the spaces that arise as spectra are exactly the compact totally disconnected spaces, now called *Stone spaces*. In modern terminology, he thus established a dual equivalence between the category of Boolean algebras and Boolean algebra homomorphisms, and the category of Stone spaces and continuous functions. It was the first time that topology was used to resolve a problem in algebra and Stone's work can rightly be called a milestone in the history of mathematics.

Observing that the representation of Boolean algebras by their spectra does not require negation, Stone generalised his work to bounded distributive lattices in [Sto37b], but the resulting topological spaces are no longer Hausdorff and satisfy a list of axioms that must have seemed rather esoteric at the time. The morphisms, too, are not just the continuous ones but it must be further required that inverse images of compact open sets are compact. It is perhaps for these reasons that this work did not receive much attention until Hilary Priestley showed in [Pri70, Pri72] that the topology on the spectrum can be enriched in a natural way so that a compact Hausdorff space is obtained. Furthermore, the spectra carry an order relation and indeed the maps between spectra that are dual to lattice homomorphisms are the continuous *order-preserving* ones.

To a topologist, this work offered the tantalising prospect of replacing the second-order structure of topological spaces by the first-order structure of lattices, except that the translation is only available for the very special spaces that appear in the theorems of Stone and Priestley. However, the spirit of the representation theorem can be maintained for a very large class of spaces if one axiomatises the set of *all* open sets (and not just the clopen or compact-open ones). According to

[Joh82, Notes on Chapter II] this step appears to have been first taken by Charles Ehresmann and his student Jean Bénabou [Bén59], and it led to the development of frame and locale theory, sometimes referred to as *pointfree topology*.

Although frame theory does succeed in replacing topology by algebra, it can do so only by considering the infinitary operation of "arbitrary join of open sets" and is therefore not first-order. Motivated by this fact and by applications in computer science, Michael Smyth in [Smy92a] tried to extend the original Stone duality without sacrificing the finitary nature of the algebras. He proposed to use *proximity lattices*, where the algebraic structure of bounded distributive lattices is extended with a "proximity relation" $\prec$. In the paper [JS96], the first author in collaboration with Philipp Sünderhauf showed that when the definition of proximity lattices is augmented with one further axiom, one obtains a perfectly self-dual algebraic structure for which the definition of the spectrum is furthermore much simplified and is seen to be a direct generalisation of Stone's approach. Nonetheless, the class of spaces that have a representation via *strong* proximity lattices is the same as that studied by Smyth, namely, the class of *stably compact spaces*. These cover all compact Hausdorff spaces plus most of the T_0 spaces that appear in Dana Scott's domain theory for denotational semantics, [Sco72, AJ94, GHK$^+$03].

The representation of stably compact spaces via strong proximity lattices was successfully used in subsequent work, [JKM99, JKM01, MJ02] to provide a "continuous" version of Samson Abramsky's *Domain Theory in Logical Form*, [Abr91], but at the fundamental level some important questions were left unanswered. Foremost of these is the "correct" definition of a homomorphism between strong proximity lattices, which should be structure-preserving in a natural way and correspond to a topological morphism of some kind between spectra. At least three possibilities appear in the papers cited above, and each leads to a duality with certain topological morphisms, but in the absence of a clear understanding of the meaning of the proximity relation $\prec$ it is hard to choose between them. One might also hope to be able to express the duality by considering the maps into a dualising object (which is the set of truth values in the Boolean algebra case, and the two-element lattice in the case of frames), but none of the three existing definitions works in this respect. Finally, one may wonder whether the duality in [JS96] can be modified so that *locally* stably compact spaces are covered, but despite several attempts no satisfying solution has so far emerged.

The principal purpose of the present paper is demonstrate that the dualities of Stone, Ehresmann-Bénabou, and Jung-Sünderhauf are all special cases of a very general and very straightforward duality between bitopological spaces and what we will define below, *d-frames*. Priestley's duality as such is not covered but her work is one of the sources of inspiration for bringing a second topology into the picture.

Bitopology may at first sound rather abstract and perhaps technical but in fact there are at least three examples where two natural topologies coexist. The most important of these is the real line where we know that the usual topology is the join of the upper and the lower topologies (with open sets all intervals (r, ∞) and $(-\infty, r)$, respectively). The second example is given by the Vietoris topologies for hyperspaces $H \subseteq \mathcal{P}X$ of a topological space $(X; \tau)$; one is generated by the sets $\Box O = \{A \in H \mid A \subseteq O\}$, $O \in \tau$, and the other by $\Diamond O = \{A \in H \mid A \cap O \neq \emptyset\}$. Finally, in Scott's domain theory one has the Scott topology and the weak lower topology, the join of which is known as the Lawson topology.

An alternative and illuminating route to the constructions and results of this paper is given by the logical reading of Stone duality. For this recall that any Boolean algebra can be viewed as the set of formulas of a propositional theory factored by interprovability. The elements of the spectrum are then nothing other than the *models* of the theory and Stone's representation theorem is equivalent to the completeness of the proof calculus. The basic open sets $\Phi_+(\gamma)$ of the topology consist of those models in which the formula γ holds, but the impact of this is less clear, unless there is an independent interpretation of this structure. The examples from computer science come in handy at this point, as domains bring a natural notion of convergence with them, based on the approximation of computable elements (e.g., functions) by partial ones. The fact that the sets $\Phi_+(\gamma)$ are open can then be rephrased as saying that the property γ holds for a computable element if and only if it holds for some finite approximating element already.

Smyth emphasised this reading by pointing out that a Scott-open set in a domain corresponds to a *finitely observable property* of states of a computational system, [Smy83, Smy92b]. This is closely related to interpretations of open sets as semi-decidable properties in recursion theory, or as "states of knowledge" in intuitionistic logic. When motivating our definitions and constructions we will stick to "(finitely) observable" throughout; note that some qualification is necessary, as set-theoretically the extent of a predicate can be any subset of the state space.

In the duality of propositional logic (Boolean algebras), the extents of formulas are clopen sets, in other words, both the extent and its complement are open. This is appropriate because the complement is the extent of the negated formula, but in general one would not expect the state space so neatly to separate into those states where a formula observably holds and those where it observably fails. To give an example from recursion theory, the set of halting Turing machines is finitely observable but its complement is not. There *are* Turing machines for which we can observe in finite time that they will run forever, namely those which return to a previous state of computation, but by Turing's famous theorem no such finite observational criterion can exhaust all non-terminating machines.

We hope that at this stage it is natural to consider separate topologies, one for the subsets of a state space where a property observably holds, and one whose open sets are those where the property observably fails. For every proposition γ we thus have the pair $(\Phi_+(\gamma), \Phi_-(\gamma))$. In general, there are states where γ can neither be established to hold, nor to fail. As a simple example, consider the proposition "$x > 0$" for real numbers; if numbers are presented as streams of decimal digits, then we will be able to affirm the property whenever x is indeed greater than zero, and refute it for negative numbers, but for any initial segment consisting of zeros we cannot affirm or refute.

It was phenomena such as these which led Stephen Kleene to consider three-valued logic, where true and false are augmented with the value $\perp$, meaning that the proposition is (perhaps currently) undecided. The phenomenon is also well-known in knowledge representation, where an agent may not have enough information to decide a given proposition. In this latter setting it is equally conceivable that an agent is provided with conflicting information, that is, that its state of knowledge is contained in both $\Phi_+(\gamma)$ and $\Phi_-(\gamma)$. Thus we are led to Nuel Belnap's four-valued logic, [Bel77], where there can be not only too little information (denoted by $\perp$), but also too much (denoted by $\top$). As it will turn out, Belnap's four-element lattice of truth values is the dualising object in our duality. So from the logical perspective, we are again led to consider bitopologies. In addition, the logic suggests that the following two special cases are of importance: some propositions γ do not self-contradict, that is, the extents $\Phi_+(\gamma)$ and $\Phi_-(\gamma)$ on the state space are disjoint; we call such propositions *consistent*. Dually, some propositions δ are observable at every state, that is $\Phi_+(\delta)$ and $\Phi_-(\delta)$ cover the state space; these will be called *total*. The algebraic abstraction of the situation consists of two frames L_+ and L_- plus two relations con and tot between them; we call this a *d-frame*.

At this stage we hope we have given enough information to be able to outline the contents of the paper. In Section 2 we review the classical dualities mentioned above and present the mathematical machinery that underlies them. In Section 3 we begin our study of d-frames and the duality with bitopological spaces. The definitions and constructions will turn out to be straightforward adaptations of the duality of frames. We show that d-frames can be defined as algebraic structures on a *single* carrier set (to be thought of as $L_+ \times L_-$). The most important outcome of this (mathematically simple) exercise is that d-frames carry an information order $\sqsubseteq$ *and* a logical order $\leq$. Semantically, the former can be thought of as increasing positive and negative extents of a formula, the latter as increasing the positive and decreasing the negative. The two orders are connected via simple algebraic laws.

In Section 4 we look at those bitopological spaces that are faithfully represented by d-frames, and find that this class is considerably wider than the class of sober spaces. The dual concept, spatial d-frames, is examined in Section 5. These satisfy a number of additional axioms (not necessary for the underlying duality) and we select a subset which appears to be both natural and powerful. We show that for the resulting concept of a *reasonable d-frame*, free structures still exist. Some attention is given to the relationship between (reasonable) d-frames and the "biframes" of Banaschewski, Brümmer, and Hardie, [BBH83]. These are an alternative to d-frames but formalise the two topologies together with their join. Obviously, there is a forgetful functor from biframes to d-frames, and we show that it has a left adjoint. An interesting open problem is whether the d-frames that arise from biframes can be characterised independently.

Section 6 examines two special cases of d-frames which are of particular importance to Stone duality, namely, the regular and compact regular ones. The latter can be shown to be spatial (using the Axiom of Choice) and are the duals of stably compact spaces. Many consequences of spatiality, however, can be shown directly and without reference to the Axiom of Choice.

In Section 7 we demonstrate that the structure of a (reasonable) d-frame $\mathcal{L}$ is completely captured by the subset $\mathcal{P}_{\mathcal{L}}$ of consistent propositions. The axiomatisation makes crucial use of *both* the information and the logical structure. In addition, the relation $\prec$ of strong proximity lattices is employed to capture the totality predicate. We call these structures *partial frames*. The equivalence with d-frames could now be summarised in the slogan that there is no difference between three-valued and four-valued logic. An important result from the transition to partial frames is a logical interpretation of $\gamma \prec \delta$: It means that every (information order) refinement γ' of γ implies (in the logical order) every refinement δ' of δ.

Sections 8 and 9 consider special cases of partial frames and their duality with bitopological spaces to locate precisely the classical dualities. To start with, the category of compact regular partial frames is equivalent to the category of strong proximity lattices, where morphisms of the latter category are adjoint pairs of consequence relations, first considered in [JKM01]. The necessity for these emerges most naturally from the requirement to match partial frame homomorphisms. If one assumes on top of compact regularity that the set of *reflexive* propositions (that is, those γ for which $\gamma \prec \gamma$) is dense in the partial frame, then one obtains a category that is equivalent to bounded distributive lattices and lattice homomorphisms.

If the language of (partial) propositions includes a negation operation (as, for example, considered by Belnap), then there is no distinction between positive and negative extents and one expects the two topologies on the spectrum to coincide. At the level of d-frames (or, equivalently, partial frames) the negation operation must be specified explicitly, as the mere existence of an isomorphism between L_+ and L_- is of little consequence. The theory then collapses to that of frames in the usual sense as we are really only dealing with one topology. Once regularity is assumed, there is at most one choice of negation on a d-frame, so here negation is a structural property rather than additional structure. Compact regular d-frames with negation then correspond to strong proximity lattices with negation, and these were shown by the second author to be the Stone duals of compact Hausdorff spaces, [Mos04]. If there is a sufficient supply of reflexive elements, finally, one obtains a Boolean algebra and Stone's original duality with Stone spaces. This presentation of Stone dualities clarifies a number of issues, as promised at the start of this introduction, and we look at this more carefully in the concluding section.

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2 Review of some Stone-type dualities

For motivation and ease of reference we briefly review the main components of some classical Stone-type dualities. This will allow us to point out how they fit into the general picture that we are about to develop. We present them in a formulation that is most convenient for this purpose, not one that is most accurate historically.

2.1 The dualities of Stone and Priestley

Stone's original papers, [Sto36, Sto37a], were concerned with the representation of Boolean algebras as sub-algebras of powersets. Tarski and Lindenbaum had shown that a Boolean algebra is isomorphic to a powerset if and only if it is complete and atomic. In this case, the isomorphism is with the powerset of the set of atoms. Atoms occupy a special position in the Boolean algebra (just above the element 0) but it is their property of being *prime* which is used in the proof: An element x is *prime* if $x \leq y \vee z$ implies $x \leq y$ or $x \leq z$. Seeing an analogy with Kummer's *ideals* in the theory of rings, Stone realised that it was the properties of the set $X \setminus \uparrow x := \{y \mid x \not\leq y\}$ that were important, not those of x itself, and, crucially, that there were potentially many more of the former than of the latter.

The definitions don't require the negation operation of Boolean algebras, so we first state Stone's representation theorem of 1937 for distributive lattices[1] (published as [Sto37b]).

Definition 2.1 *A subset F of a lattice L is called a* filter *if it is an upper set that is closed under finite meets (including $\mathit{tt} = \bigwedge \emptyset$). A filter is called* prime *if it is inaccessible by finite joins: $\bigvee \emptyset = \mathit{ff} \notin F$, and if $x \vee y \in F$ then either $x \in F$ or $y \in F$.*

The notions ideal *and* prime ideal *are defined dually.*

[1] In this paper, "lattice" will mean bounded lattice: $(L; \wedge, \vee, \mathit{tt}, \mathit{ff})$ and "lattice morphisms" will preserve this structure. Although Lemma 2.3 and Theorem 2.4 can be formulated without boundedness, nothing in this paper is gained by the greater generality. We use tt and ff rather than 1 and 0 because throughout the paper we are specifically interested in the connection with logic.

We denote the set of all prime filters of L with $\mathsf{pFilt}(L)$ and the set of prime ideals with $\mathsf{pIdl}(L)$.

Proposition 2.2 *In any lattice, the complement of a prime filter is a prime ideal and vice versa.*

The following is traditionally stated in terms of prime ideals and referred to as the *Prime Ideal Theorem* or "PIT". However, because of the previous proposition there is no real difference between using ideals or filters.

Lemma 2.3 *In any distributive lattice, if $x \not\leq y$ then there is a prime filter containing x but not y.*

Theorem 2.4 (Stone 1937) *Every distributive lattice is isomorphic to a sublattice of a powerset.*

Proof. Consider the powerset of the set of prime filters, and map an element x of the lattice to the set $\Phi_+(x)$ of prime filters containing it.[2] It is easily checked that $\Phi_+(x \wedge y) = \Phi_+(x) \cap \Phi_+(y)$ and $\Phi_+(x \vee y) = \Phi_+(x) \cup \Phi_+(y)$. By the definition of prime filters, it is clear that $\Phi_+(\mathit{ff}) = \emptyset$ and $\Phi_+(\mathit{tt}) = \mathsf{pFilt}(L)$. ∎

This covers Boolean algebras also, as a prime filter on a Boolean algebra is the same as an *ultrafilter*: it contains exactly one of x and $\neg x$. From this it follows that the subset of prime filters associated with x is exactly the complement of those that are associated with $\neg x$, and so the negation operation is faithfully modelled by complement.

Stone did not stop here but realised that it was just as important to represent the *homomorphisms* between the algebras. In order to single out the right set-theoretic maps between collections of prime filters, he employed topology. The basic opens of the *Stone topology* on $\mathsf{pFilt}(L)$ are exactly the sets $\Phi_+(x) := \{F \in \mathsf{pFilt}(L) \mid x \in F\}$ which were used in the proof of the representation theorem. The topological space so obtained is called the *spectrum* of L (borrowing terminology from Ring Theory).

[2]The choice of notation $\Phi_+(x)$ will become clearer later.

Theorem 2.5 (Stone 1936, 1937)

1. *The spectra of Boolean algebras are precisely the totally disconnected compact Hausdorff spaces.*

2. *The spectra of distributive lattices are precisely those topological spaces that*

   (i) *satisfy the T_0 separation axiom;*

   (ii) *are compact;*

   (iii) *have a basis consisting of compact open sets;*

   (iv) *are coherent, that is, intersections of compact open sets are compact;*[3]

   (v) *are well-filtered, that is, if the intersection of a filter base $(C_i)_{i \in I}$ of compact open sets is contained in an open set, then so is some C_i already.*

The spaces that appear in part (1) above are called *Stone spaces* and those in part (2) *spectral spaces*. The word "precisely" in the theorem refers to the translation from these spaces back to algebraic structures. In the case of Stone spaces one considers the collection of subsets that are both closed and open ("clopen"), for spectral spaces those that are both compact and open. Since a subset of a compact Hausdorff space is compact if and only if it is closed, (1) is a special case of (2).

The characterisation of homomorphisms on the side of the spectrum works as follows. For Boolean algebras one considers continuous functions between the spectra in the opposite direction; for distributive lattices one uses *perfect maps*.[4] These are continuous functions for which the inverse image of a compact open is compact. Again, any continuous function between compact Hausdorff spaces is perfect, so the Boolean algebra case is a special case.

In the language of Category Theory we have the following:

[3]This is also called *stably compact*.

[4]There is some variability regarding terminology here with some authors insisting that the right adjective for these functions should be "*proper*".

Bool, the category of Boolean algebras and Boolean algebra homomorphisms;
dLat, the category of bounded distributive lattices and $(\wedge, \vee, tt, ff)$-homomorphisms;
Stone, the category of Stone spaces and continuous functions;
Spec, the category of spectral spaces and perfect maps.

Theorem 2.6

1. *The categories* **Bool** *and* **Stone** *are dually equivalent.*

2. *The categories* **dLat** *and* **Spec** *are dually equivalent.*

As explained in the introduction, spectral spaces feature prominently in Scott's theory of semantic domains, but from a mathematical point of view they carry rather a lot of conditions. Hilary Priestley recognised that there was an alternate description as certain ordered spaces.

Definition 2.7 *A* Priestley space *is a topological space with a (topologically closed) partial order relation* $\leq_s$ *which is*

(i) *compact Hausdorff;*

(ii) *totally order disconnected, that is, for* $x \not\leq_s y$ *there is a clopen upper set* U *such that* $x \in U$ *and* $y \notin U$.

The category **Pries** *has Priestley spaces as objects and continuous order-preserving maps as morphisms.*

Theorem 2.8 (Priestley 1970) *The categories* **dLat** *and* **Pries** *are dually equivalent.*

Proof. Stone's topology on the spectrum of prime filters is enriched with open sets

$$\Phi_-(x) := \{F \in \mathsf{pFilt}(L) \mid x \notin F\}$$

which clearly results in a Hausdorff topology. The order is inclusion between prime filters.

In the reverse direction one considers the collection of clopen upper sets. ∎

The sets $\Phi_+(x)$ constitute a topological basis, and so do the sets $\Phi_-(x)$ separately. So Theorem 2.8 clearly suggests a bitopological view. Indeed, compactness of the Priestley topology is established in two steps. Following Stone's idea, one first shows that each set $\Phi_\pm(x)$ is compact with respect to the opposite ($\mp$) topology. Then one invokes the Alexander Sub-base Lemma to see that the join of the two topologies is compact. So the "hard" part of the proof is located at the transition from a bitopological setting to a topological one.

Instead of "prime filter of L" we could have said "homomorphism from L to $\mathbb{B} := \{\mathit{ff} < \mathit{tt}\}$" because prime filters are exactly the inverse images of tt under such homomorphisms. Likewise, instead of "clopen subset of the Stone space X" we could have said "continuous function from X to $\mathbb{D} := (\{n, y\}, \mathcal{D})$, where $\mathcal{D}$ is the discrete topology. For spectral spaces one uses Sierpinski space: $\mathbb{S} := (\{n, y\}, \mathcal{S})$ where $\{y\}$ is the only nontrivial open of $\mathcal{S}$, and for Priestley spaces one uses the discrete topology together with the order $n < y$.

The fact that $\mathbb{B}$, $\mathbb{D}$ and $\mathbb{S}$ have the same number of elements is not a coincidence. It follows from the fact that all five categories introduced so far are *concrete* over **Set**, with the forgetful functor having a left adjoint. In such a situation, the underlying set of a dual object is always given by the set of morphisms into a *dualising object*, and the dualising objects of the two categories involved have "the same" underlying set. See [Joh82, SectionVI.4.1] for a concise discussion of this phenomenon.

Let us also briefly review the logical reading of Stone duality. Every Boolean algebra is the *Lindenbaum algebra* of a propositional theory. A prime filter on the algebra corresponds to a *model* of the theory. The Prime Ideal Theorem (our Lemma 2.3) then states the completeness of propositional logic. The duality of bounded distributive lattices likewise corresponds to propositional logic without negation, or *positive logic*.

To every proposition x one can associate the binary partition of the spectrum into those models in which the proposition is true (its *positive extent* $\Phi_+(x)$) and those where it is false (the *negative extent* $\Phi_-(x)$). Stone duality equips the spectrum with a topology in which both extents are compact and open. With the Hausdorff assumption, this yields clopen extents for the Boolean algebra case. Priestley duality renders the positive extent a compact open (hence clopen) upper set, the negative extent a compact open (hence clopen) lower set.

2.2 Proximity lattices and (stably) compact spaces

Priestley duality associates a compact ordered space with a bounded distributive lattice, whereas Stone duality provides a certain T_0-space. The two are closely linked as the underlying set in both cases is provided by the collection of prime filters. The connection between a Priestley space and the corresponding spectral space is, in fact, purely topological and holds more generally. We recall the main points of this.

Definition 2.9 *A compact ordered space is a set with a compact topology τ and an order relation $\leq_s$ that is closed in the product topology.*

Compact ordered spaces were studied by Leopoldo Nachbin in a systematic fashion in [Nac65]. He established the following central property for these spaces.

Lemma 2.10 (Separation Lemma) *For $x \not\leq_s y$ in a compact ordered space there is an open upper set U and an open lower set O such that $x \in U$, $y \in O$, and $U \cap O = \emptyset$.*

One defines the *upper topology* τ_+ as the collection of all open upper sets and likewise the *lower topology* τ_-.

Proposition 2.11 *The topology of a compact ordered space is the join of the upper and lower topologies: $\tau = \tau_+ \vee \tau_-$.*

Proof. For $O \in \tau$, the complement $C := X \setminus O$ is compact. Consider $x \in O$; for any $y \in C$, either $x \not\leq_s y$ or $x \not\geq_s y$, so the Separation Lemma can be employed. The rest is standard. ∎

From the Separation Lemma it also follows that $\leq_s$ is the specialisation order of the upper topology (and the reverse of the specialisation order of the lower topology). A set that is closed in the lower topology is an upper set and therefore *saturated* in the upper topology, that is, it is the intersection of upper open sets. Furthermore, it is compact in the upper topology by the overall compactness of τ. The converse is again an easy consequence of separation. Thus:

Proposition 2.12 *The sets that are upper and closed in a compact ordered space are precisely the compact saturated sets with respect to the upper topology.*

In preparation for our general development later, we note that the bitopological space $(X;\tau_+,\tau_-)$ is compact and regular. Here is the definition:

Definition 2.13 *A bitopological space* $(X;\tau_+,\tau_-)$ *is called* compact *if every cover of* X *with elements from* $\tau_+ \cup \tau_-$ *has a finite sub-cover.*

Let U, U' *be two elements of* τ_+. *We say that* U' *is* well-inside U *(and write* $U' \lhd U$*) if there is* $O \in \tau_-$ *such that* $U' \cap O = \emptyset$ *and* $U \cup O = X$.

A bitopological space is called regular *if every open of* τ_+ *is the union of those* τ_+*-opens well-inside it, and the analogous statement holds for the opens of* τ_-.

By Alexander's Sub-base Lemma, a bitopological space is compact if and only if it is compact in the join of the two topologies. However, since most of our results do not depend on the Axiom of Choice, it makes sense to have a separate notion that does not require the join topology.

Proposition 2.14 *If* $(X;\tau;\leq_s)$ *is a compact ordered space then* $(X;\tau_+,\tau_-)$ *is compact and regular.*

We will maintain that the bitopological point of view is the best one to understand Stone duality, but to complete our brief presentation of the link with Priestley duality we observe that Proposition 2.12 suggests that the second topology is actually redundant. Indeed, the topologies that arise as upper topologies of compact ordered spaces can be characterised independently.

Definition 2.15 *A topological space is called* stably compact[5] *if it is*

(i) T_0*;*

(ii) compact;

(iii) locally compact;

(iv) coherent, that is, intersections of compact saturated sets are compact;

(v) well-filtered, that is, if the intersection of a filter base $(C_i)_{i \in I}$ *of compact saturated sets is contained in an open set, then so is some* C_i *already.*

Clearly, every spectral space, as characterised in Theorem 2.5, is stably compact, but the latter class is much bigger since zero-dimensionality is not required; for example, all compact Hausdorff spaces are included.

The following is an immediate consequence of the definitions.

Theorem 2.16 *A topological space* $(X; \tau)$ *is stably compact if and only if the bitopological space* $(X; \tau, \tau_{cc})$ *is compact and regular, where* τ_{cc} *is the topology whose closed sets are (generated by) the compact saturated sets with respect to* τ*.*

Theorem 2.17 *A topological space is stably compact if and only if it is the upper topology of a compact ordered space.*

Proof. We have done most of the work for the "if" direction explicitly. For the converse one considers the co-compact topology whose closed sets are defined as the compact saturated sets of a given topology. It is easy to see that an ordered Hausdorff space is obtained this way, but to establish compactness one needs to call upon the Axiom of Choice in the form of Alexander's Sub-base Lemma. ∎

For more detail we refer to [AMJK04] and [GHK$^+$03, Section VI-6]. For our present purposes we can say that the theorem establishes the topological link between the dualities of Stone and Priestley for bounded distributive lattices. The

[5] These spaces were called *coherent* in [JS96].

dualities themselves can be generalised to the setting of the theorem by considering a refinement of bounded distributive lattices. Following [JS96], we outline the main results.

Definition 2.18 *A* strong proximity lattice[6] *is a structure $\mathcal{X}$ consisting of a distributive lattice $(X; \wedge, \vee, tt, ff)$ together with a transitive binary relation $\prec$ such that*

$$
\begin{array}{lllll}
(\prec\!-tt) & x \prec tt & & & \\
(ff-\!\prec) & ff \prec x & & & \\
(\prec\!-\wedge) & x \prec y,\ x \prec y' & \iff & x \prec y \wedge y' & \\
(\vee\!-\!\prec) & x \prec y,\ x' \prec y & \iff & x \vee x' \prec y & \\
(\wedge\!-\!\prec) & a \wedge x \prec y & \Longrightarrow & \exists a' \in X.\ a \prec a' \ \textit{and}\ a' \wedge x \prec y & \\
(\prec\!-\vee) & x \prec y \vee a & \Longrightarrow & \exists a' \in X.\ a' \prec a \ \textit{and}\ x \prec y \vee a' &
\end{array}
$$

By letting $x = tt$ in $(\wedge\!-\!\prec)$ one sees that the relation $\prec$ is *interpolative*, that is, $x \prec y$ implies that there is z such that $x \prec z \prec y$. Strong proximity lattices generalise distributive lattices in that the lattice order $\leq$ on a distributive lattice satisfies the definition. We will consider morphisms on strong proximity lattices in Section 8.1.

Definition 2.19 *An upper subset A of a strong proximity lattice is called* round *if for every $x \in A$ there is $x' \in A$ with $x' \prec x$. A round lower set is defined dually.*

The carrier of the spectrum of a strong proximity lattice consists of all round prime filters, the collection of which we denote by $\mathsf{rpFilt}(\mathcal{X})$. The following are the analogues to Proposition 2.2 and Lemma 2.3:

Proposition 2.20 *If F is a round prime filter on strong proximity lattice $\mathcal{X}$, then $\downarrow\!(X \setminus F) := \{x \in X \mid \exists x' \in X \setminus F.\ x \prec x'\}$ is a round prime ideal, and vice versa. Furthermore, the translations are inverses of each other.*

[6] The qualifier "strong" distinguishes the concept from its precursor in [Smy92a], where $(\wedge\!-\!\prec)$ was not a requirement.

Lemma 2.21 *In a strong proximity lattice, if F is a round filter not intersecting an ideal I then F can be extended to a round prime filter still disjoint from I.*

We topologise the spectrum as in Stone duality, that is, taking the sets $\Phi_+(x) = \{F \in \mathsf{rpFilt}(\mathcal{X}) \mid x \in F\}$ as the basic opens.

Theorem 2.22 *The spectrum of a strong proximity lattice is a stably compact space, and every stably compact space arises in this way.*

Proof. The details of the first half are in [JS96]. For the second statement one associates with a stably compact space all pairs $U \subseteq K$ where U is open and K is compact saturated. The lattice operations on these pairs are defined componentwise and the proximity is given by $(U, K) \prec (U', K')$ iff $K \subseteq U'$. ∎

While this result establishes an interesting link between a wide class of topological spaces and certain algebraic structures, it is not clear what the representation problem is that is solved by it. Sure enough, it is still true that the map $x \mapsto \Phi_+(x)$ preserves the bounded lattice structure, but how is $\prec$ modelled by it? And do we have a prior idea what it should be modelled by? The paper [JS96] did not address the second question, as the emphasis was on an algebraic (or logical) description of certain given spaces.

The problem is also apparent when we look at morphisms. In [JS96] a duality is established between continuous functions on the side of stably compact spaces and certain *approximable relations* between strong proximity lattices. This choice was motivated by applications in Domain Theory. In later work, [JKM99], we introduced *continuous consequence relations* which correspond to certain continuous relations between the spaces; this was motivated by the logical reading of the duality. However, neither choice generalises Stone duality for bounded distributive lattices.

Another problem with Theorem 2.22 is that the functors are not given by the set of morphisms into a dualising object. Indeed, the two types of morphism for strong proximity lattices mentioned above are not even functions so there is no forgetful functor into **Set**. Following the general methodology laid out in [Joh82, SectionVI.4.1], we should find a dualising object by constructing the free stably compact space over the one-point set (which is the one-point topological space) and dualise this. The construction outlined in the proof of Theorem 2.22 yields a three-element proximity lattice $\mathit{ff} < \perp < \mathit{tt}$ with $\mathit{ff} \prec x \prec \mathit{tt}$ for any x but $\perp \not\prec \perp$. If we were dealing with a concrete duality, then the dualising object among stably compact spaces would also have three elements but there is no such space that yields the right proximity lattice.

Our answer to the riddles above is to suggest that the duality of stably compact spaces and strong proximity lattices is a special case of a general duality between certain algebraic structures, which we will introduce below, and bitopological spaces. Given the information above, the reader can perhaps guess the general outline of how this might go. For example, in Stone duality one can look at the bitopological space where one topology has the collection of $\Phi_+(x)$ as its basis, and the other is generated by the $\Phi_-(x)$. Stone's perfect maps between spectral spaces are then nothing else but bicontinuous maps.

The duality of Theorem 2.22 is also bitopological in nature; instead of thinking of the tokens (U, K) as consisting of an open and a compact saturated set, we should view them as consisting of an open and a co-compact open $X \setminus K$. The correct morphisms on the spatial side should again be bicontinuous maps and whatever we choose as morphism between strong proximity lattices should mirror these.

Finally, recall the logical reading of Stone duality as briefly outlined at the end of Section 2.1. For strong proximity lattices we see that the positive extent $\Phi_+(x)$ of a lattice element x is disjoint from the negative extent $\Phi_-(x)$ but it is not necessarily the set-theoretic complement. Lattice elements can thus be seen as *partial predicates* which are true in some models, false in others, and whose status is unknown (or undecidable in finite time) in the remaining cases. A chief aim of the present paper is to convince the reader that this is indeed a fruitful point of view.

2.3 Frames

The Stone dualities reviewed so far lead to rather special topological spaces. If one is interested in a duality that applies to all spaces then frame theory is the answer. Although there is an excellent text available on the subject, [Joh82], we review its main components as we will make constant use of frame-theoretic ideas throughout the paper.

Definition 2.23 *A* frame *is a complete lattice in which finite meets distribute over arbitrary joins. We denote with* $\sqsubseteq$, $\sqcap$, $\bigsqcup$, 0, *and* 1 *the order, finite meets, arbitrary joins, least and largest element, respectively.*[7]

A frame homomorphism preserves finite meets and arbitrary joins; thus we have the category **Frm**.

For $(X;\tau)$ a topological space, $(\tau;\subseteq)$ is a frame; for $f\colon (X;\tau) \to (X';\tau')$ a continuous function, $f^{-1}\colon \tau' \to \tau$ is a frame homomorphism. These are the constituents of the contravariant functor $\Omega\colon$ **Top** $\to$ **Frm**. It is represented by **Top**$(-,\mathbb{S})$ where $\mathbb{S}$ is *Sierpinski space* – the same space that arose in Theorem 2.5(2). In this representation, an "open" is identified with a continuous map from X to $\mathbb{S}$. The frame operations on such maps are defined point-wise.

The collection $\mathcal{N}(a)$ of open neighbourhoods of a point a in a topological space $(X;\tau)$ forms a *completely prime filter* in the frame ΩX, that is, it is an upper set, closed under finite intersections, and whenever $\bigcup \mathcal{O} \in \mathcal{N}(a)$ then $\mathcal{O} \cap \mathcal{N}(a) \neq \emptyset$. This leads one to consider the set of *points* (sometimes called "abstract points" for emphasis) of a frame L to be the collection $\mathsf{spec}\, L$ of completely prime filters. Abstract points are exactly the pre-images of $\{1\}$ under homomorphisms from L to $2 = \{0 < 1\}$. So an alternative representation takes a point to be a homomorphism from L to 2.

A frame L induces a topology on $\mathsf{spec}\, L$ whose opens are of the form $\Phi(x) = \{F \in \mathsf{spec}\, L \mid x \in F\}$ with $x \in L$. In the alternative representation, $\mathsf{spec}\, L$ takes the weakest topology such that for each $x \in L$, the evaluation map $p \mapsto p(x)$ is continuous as a map from $\mathsf{spec}\, L$ to $\mathbb{S}$. A frame homomorphism $h\colon L \to L'$ induces a continuous function $\mathsf{spec}\, h\colon \mathsf{spec}\, L' \to \mathsf{spec}\, L$ by letting $\mathsf{spec}\, h(F) := h^{-1}(F)$ for $F \in \mathsf{spec}\, L'$. These are the components of the contravariant functor spec from **Frm** to **Top**, represented by **Frm**$(-,2)$.

[7] This lattice notation is different from that chosen for distributive lattices. The reason for this will become clear as our theory unfolds.

Theorem 2.24 *The functors* Ω *and* spec *constitute a dual adjunction between* **Top** *and* **Frm**.

The unit and co-unit of this adjunction are simply $\mathcal{N}$ and Φ. That is, for any space $(X;\tau)$ the map $\eta_X\colon X \to \mathsf{spec}\,\Omega\,X$, given by $a \mapsto \mathcal{N}(a)$, is continuous; it is also open onto its image. Likewise, for any frame L the $\epsilon_L\colon L \to \Omega\,\mathsf{spec}\,L$, given by $x \mapsto \Phi(x)$ is a frame homomorphism; it is also surjective.

A brief comparison of the dual adjunction between frames and spaces and Stone's original theorems is instructive. In Stone's duality for distributive lattices and spectral spaces, the co-unit, $x \mapsto \Phi_+(x)$, is clearly a surjective distributive lattice homomorphism onto the compact opens of the spectrum. The prime ideal theorem is used to show that if $x \not\leq y$ holds in a distributive lattice, then there is a point in the spectrum (that is, a prime filter) showing that $\Phi_+(x)$ is not a subset of $\Phi_+(y)$. This establishes that the co-unit is injective. Armed with this, one then sees that the co-unit is in fact an isomorphism. The axioms for spectral spaces are engineered to ensure that (a) compact opens form a distributive lattice (a minimal requirement) (b) there are enough compact opens to distinguish points, so the unit is also injective and (c) there are enough points so that the unit is also surjective.

For frames, the prime ideal theorem can not help us establish that the co-unit is an isomorphism. The abstract points of a frame are completely prime filters, not merely prime filters. But the complement of a completely prime filter is a principle prime ideal, so the prime ideal theorem is powerless to find an abstract point that would separate $\Phi(x)$ from $\Phi(y)$ for $x \not\sqsubseteq y$. So in general, the co-unit of the $\Omega \dashv \mathsf{spec}$ adjunction may not be an isomorphism. Similarly, we have not assumed any separation on spaces that would ensure that the unit is injective (although this is easily remedied as injectivity is precisely the T_0 axiom) nor that spaces have enough points to ensure that the unit is surjective.

We can ask when a frame L is *spatial* in the sense that it is isomorphic to $\Omega\,X$ for some space X. The adjunction transfers isomorphisms: $L \cong \Omega\,X$ if and only if $X \cong \mathsf{spec}\,L$. So L is spatial if and only if ϵ_L is a frame isomorphism. Because ϵ_L is already a surjective frame homomorphism, this holds if and only if ϵ_L is injective. Examples of non-spatial frames are found in [Joh82].

Similarly, we can ask when a space X is *sober* in the sense that it is homeomorphic to $\mathsf{spec}\, L$ for some frame L. By the same reasoning as in frames, this holds if and only if η_X is a homeomorphism. Because η_X is already continuous and open onto its image, it suffices for η_X to be a bijection. As mentioned above, injectivity is precisely the T_0 axiom. Surjectivity says that every completely prime filter of opens is the neighbourhood filter of a point.

Theorem 2.25 *The functors* Ω *and* spec *restrict to a dual equivalence between sober spaces and spatial frames.*

A key property of **Frm** is the fact that free frames exist; the construction for the free frame over a set of generators is described in [Joh82, Section II.1.2], and for a presentation with generators and relations in [Joh82, Section II.2.11]. Existence of free frames, but not the details of construction, will be the basis for various free constructions for d-frames in what follows.

A frame can alternatively be regarded as a "hybrid" structure: a distributive lattice and directed complete partial order in which (a) the lattice operations are Scott continuous (preserve directed suprema) and (b) the lattice order and the directed complete order coincide. Then a frame homomorphism is a Scott continuous lattice homomorphism. So frames are special objects in the category of dcpo distributive lattices. On this view, a frame "thinks about its data" in two ways: first, as propositions of a positive propositional logic where finitary meets and joins make sense; second, as data in an information order where accumulation of directed joins makes sense. The following two important concepts in frames highlight this distinction.

On any frame L, say that $x \in L$ is *well-inside* $y \in L$ provided there exists some $w \in L$ so that $x \sqcap w = 0$ and $y \sqcup w = 1$. We write this as $x \triangleleft y$. Notice that this relation is meaningful in any lattice; it says nothing about directed joins. On the other hand, with distributivity $x \triangleleft y$ implies $x \sqsubseteq y$. Also, for any y the set of all $x \triangleleft y$ is directed because $x \triangleleft y$ and $x' \triangleleft y$ implies $x \sqcup x' \triangleleft y$. So far, this depends only on the fact that L is a distributive lattice. On the other hand, in a frame the join of elements well-inside y exists due to directed completeness, and the join is always below or equal to y. A frame is called *regular* if the join of elements well-inside y is always equal to y. So one can take regularity to be a condition on the interaction of logic and information within a frame.

Say that x is *way below* y provided that for any directed set D, if $y \sqsubseteq \bigsqcup^{\uparrow} D$ then $x \sqsubseteq d$ for some $d \in D$. We write this as $x \ll y$. This relation is meaningful in any dcpo, as it says nothing about finite meets and finite joins. On the other hand, because a frame has finite joins, the set of all x such that $x \ll y$ is directed. A frame is called *continuous* if the join of elements way below y is always equal to y. Again, continuity is a condition on the interaction of logic and information.

Topologically, well-inside and way-below have very different intuitions. In ΩX, an open set is well-inside another provided that the closure of first is contained in the second. This splits the definition of *clopen* in the sense that U is clopen if and only if $U \triangleleft U$. On the other hand, the way-below relation splits the definition of compactness: $U \ll V$ provided every open covering of V contains a finite sub-covering of U. So U is compact in the usual sense if and only if $U \ll U$.

Because a frame is a certain kind of dcpo, the Scott topology on a frame plays an important role. A subset U of a frame is *Scott open* if and only if U is an upper set and is inaccessible by directed joins: if D is a directed subset of the frame, then $\bigsqcup^{\uparrow} D \in U$ implies $D \cap U \neq \emptyset$. For example, a completely prime filter is automatically Scott open.

One important application of the Scott topology on frames is the Hofmann-Mislove Theorem. Because we will encounter bitopological versions of the theorem in Sections 5 and 6, we state it here for reference, along with its frame-theoretic version.

Theorem 2.26 *[HM81, KP94] In a sober space (X, τ), there is a bijection between the set of compact saturated subsets of X and the set of Scott open filters in τ.*

Theorem 2.27 *[GHK+03, Corollary V-5.4] A Scott-open filter in a frame is equal to the intersection of the collection of completely prime filters containing it. Moreover, this collection is compact in the spectrum of the frame.*

In later sections, we will consider structures in which a bounded distributive lattice and a dcpo interact but do not coincide in their orders. Indeed, we shall see that the relationship between frames and the duality theorems of Sections 2.1 and 2.2 hinges on the interaction of logical structure and information structure, and that bitopology allows us to make the needed distinction between the two.

3 Bitopological spaces and d-frames

We now begin to lay out the general framework within which the dualities reviewed above can all be seen as special cases. As mentioned several times, the correct setting for this is *bitopological spaces*. So we first must establish a bitopological analogue of frames.

3.1 Stone duality for bitopological spaces

Our notation for the two topologies of a bitopological space is τ_+ and τ_-, which is meant to suggest that the opens of τ_+ are the positive extents of predicates, that is, those models where a certain proposition is (perhaps observably) true. Likewise, an open from τ_- is the negative extent of a predicate, that is those models where a certain proposition is false. With this understanding, pairs of opens $(U_+, U_-) \in \tau_+ \times \tau_-$ are the denotations of predicates; U_+ is the set of models in which the predicate is true, U_- the set in which the predicate is false.

Classically, a predicate takes a definite truth value in every model. Here we allow a predicate to be undefined in a particular model ($x \notin U_+ \cap U_-$), or to be over-defined ($x \in U_+ \cap U_-$).

If U_+ and U_- are disjoint, the predicate is *consistent*. That is, the predicate can not be both true and false in the same model. Similarly, if $U_+ \cup U_- = X$, then the predicate is *total*. It must be either true or false in any model, but of course it may not be consistently so.

Note that in general no relationship between τ_+ and τ_- is assumed, nor is any separation axiom for either topology.

A *bicontinuous map* between bitopological spaces X and X' is a function from X to X' which is continuous with respect to both topologies separately. Thus we obtain the category **biTop**.

For the "Stone duals" of a bitopological spaces, we start with *pairs of frames* (L_+, L_-), and pairs of frame maps. That is, we start in the category **Frm**$\times$**Frm**. This is not enough, however, as it makes no provision for linking the two frames; they are, after all, intended to describe the same set of points. For this reason we add two relations $\mathsf{con}, \mathsf{tot} \subseteq L_+ \times L_-$ with the intended meaning that $\langle x, y\rangle \in \mathsf{con}$ indicates that the "opens" x and y do not intersect. The intended meaning of $\langle x, y\rangle \in \mathsf{tot}$ is that x and y cover the whole space. We will also refer to these as the "disjointness" and "covering" relations.

As a preliminary (unofficial) definition, call a structure $(L_1, L_2; \mathsf{con}, \mathsf{tot})$ a *d-frame*. Certainly, "biframe" would have been more appropriate but unfortunately that terminology is already taken.[8] Morphisms between d-frames $\mathcal{L}$ and $\mathcal{L}'$ are pairs of frame homomorphisms $h_1\colon L_1 \to L_2'$ and $h_2\colon L_2 \to L_2'$ such that $\langle x, y\rangle \in \mathsf{con}$ implies $(h_1(x), h_2(y)) \in \mathsf{con}'$ and similarly for the tot relation.

As an aside, the biframes of Banaschewski, Brümmer, and Hardie, [BBH83] consist of *three* frames L_0, L_1, and L_2, where L_1 and L_2 are sub-frames of L_0 and together form a generating set for it. Clearly, this is a frame-theoretic version of the common refinement $\tau_0 := \tau_1 \vee \tau_2$ for a bitopological space $(X; \tau_1, \tau_2)$ together with the two given topologies. In biframes, the relation between the elements of L_1 and L_2 is made fully explicit by virtue of being included in the common frame L_0. D-frames, on the other hand, only encode when two "opens" are disjoint and/or covering (consistent and/or total). We will study the connection between these two approaches in greater detail later.

Because con and tot are subsets of the product $L_1 \times L_2$, we visualise a d-frame concretely as the product (see the right column of Figure 1 for examples). This is much more than a heuristic aid, though, as the structure of the product frame itself will play an important role in the theory. This leads us to our "official" definition.

[8] In fact, "d-" is the next best thing, as you can take it to abbreviate the *Proto-Indo-European* word "dvô," meaning "two."

If L_1 and L_2 are frames, then $(1,0)$ and $(0,1)$ are complements – $(1,0) \sqcap (0,1) = (0,0)$ and $(1,0) \sqcup (0,1) = (1,1)$ – in the product $L_1 \times L_2$, and the pairs of frame homomorphisms (f,g) from L_1 to L_1' and from L_2 to L_2' are bijective with the frame homomorphisms from $L_1 \times L_2$ to $L_1' \times L_2'$ that preserve $(1,0)$ and $(0,1)$.

Conversely, consider a frame $(L; \bigsqcup, \sqcap, \bot, \top)$ with designated constants tt and ff that are complementary: $tt \sqcap ff = \bot$ and $tt \sqcup ff = \top$. Then L is isomorphic to the product of the two intervals $[\bot, tt]$ and $[\bot, ff]$ (both of which are frames themselves). The isomorphism from L to $[\bot, tt] \times [\bot, ff]$ is given by $\alpha \mapsto (\alpha \sqcap tt, \alpha \sqcap ff)$. In the reverse direction, $\langle x, y \rangle \mapsto x \sqcup y$. It is also easily shown that if L' has designated complements tt' and ff', then the frame homomorphisms $h\colon L \to L'$ that preserve the constants are bijective with pairs of frame maps $f\colon [\bot, tt] \to [\bot, tt']$ and $g\colon [\bot, ff] \to [\bot, ff']$. All told, the category **Frm**$\times$**Frm** is equivalent to the category of frames equipped with complementary pairs.

Thus we take a *d-frame* officially to be a structure $(L; tt, ff; \mathsf{con}, \mathsf{tot})$ where L is a frame, tt and ff are complements, and $\mathsf{con}, \mathsf{tot} \subseteq L$. The foregoing discussion shows that this is equivalent (categorically) to the "unofficial" version, but has the advantage of constituting a concrete category. A d-frame homomorphism is a frame homomorphism that preserves tt, ff, con and tot. We denote the category of d-frames by **dFrm**.

Because a d-frame $(L; tt, ff; \mathsf{con}, \mathsf{tot})$ is determined up to isomorphism by the two frames $[\bot, tt]$ and $[\bot, ff]$ together with con and tot, and because we often find it easier to consider the two frames separately, we continue on occasion to use the "unofficial" notation $(L_1, L_2; \mathsf{con}, \mathsf{tot})$ to abbreviate the d-frame $(L_1 \times L_2; (1,0), (0,1); \mathsf{con}, \mathsf{tot})$. This has the advantage of being more obviously motivated by the two topologies on a bitopological space. Note that $\bot := (0,0)$ and $\top := (1,1)$ are the least and greatest elements of the product d-frame.

Since the underlying frame of a d-frame $\mathcal{L}$ as isomorphic to the concrete product of frames, $[\bot, tt] \times [\bot, ff]$, we will find that explicit notation for the isomorphism is needed. Specifically, let $L_+ := [\bot, tt]$ and $L_- := [\bot, ff]$. The projections from L to L_+ and L_- are given by $\alpha \mapsto \alpha \sqcap tt$ and $\alpha \mapsto \alpha \sqcap ff$. The isomorphism from $L_+ \times L_-$ to L is given by $\langle x, y\rangle \mapsto x \sqcup y$. We denote these operations as follows, for $\alpha \in L$, $x \in L_+$ and $y \in L_-$:

$$\begin{aligned} \alpha_+ &:= \alpha \sqcap tt \\ \alpha_- &:= \alpha \sqcap ff \\ \langle x, y\rangle &:= x \sqcup y \end{aligned}$$

Clearly, a d-frame homomorphism $h\colon \mathcal{L} \to \mathcal{L}'$ is determined by its operation on L_+ and L_-. These restrictions, denoted by h_+ and h_-, are separately frame homomorphisms into L'_+ and L'_-. Together they preserve con and tot. That is, if $\langle x, y\rangle \in \mathsf{con}$, then $\langle h_+(x), h_-(y)\rangle \in \mathsf{con}$, and similarly for tot.

The (contravariant) functor Ω from bitopological spaces to d-frames associates a space $(X; \tau_+, \tau_-)$ with the d-frame $(\tau_+, \tau_-; \mathsf{con}, \mathsf{tot})$ where $(U, V) \in \mathsf{con}$ if and only if $U \cap V = \emptyset$ and $(U, V) \in \mathsf{tot}$ if and only if $U \cup V = X$. The functor associates with a bicontinuous function the map determined by the two inverse image maps. A trivial bit of set theory will convince the reader that the disjointness and covering predicates are preserved. Figure 1 shows some small examples. The bitopological space $\mathbb{S}.\mathbb{S}$, which looks like a product of two copies of Sierpinski space[9], allows us to represent the functor Ω as $\mathbf{biTop}(-, \mathbb{S}.\mathbb{S})$. Note how the four elements of $\mathbb{S}.\mathbb{S}$ correspond to the four ways in which an element of the space can be related to an open from τ_+ and an open from τ_-: it can be in one of the two but not the other, it can be in both, or it can be in neither.

For a functor in the reverse direction, we continue to follow the theory of frames by considering d-frame morphisms from $\mathcal{L} = (L; tt, ff; \mathsf{con}, \mathsf{tot})$ to 2.2, depicted in the upper right corner of Figure 1. Such morphisms are determined by pairs of frame homomorphisms $p_+ \colon L_+ \to 2$ and $p_- \colon L_- \to 2$ that together preserve con and tot. So they correspond to pairs of completely prime filters $F_+ \subset L_+$, $F_- \subset L_-$ such that

$$\begin{aligned} &(\mathrm{dp}_{\mathsf{con}}) \qquad \alpha \in \mathsf{con} \implies \alpha_+ \notin F_+ \text{ or } \alpha_- \notin F_-; \\ &(\mathrm{dp}_{\mathsf{tot}}) \qquad \alpha \in \mathsf{tot} \implies \alpha_+ \in F_+ \text{ or } \alpha_- \in F_-. \end{aligned}$$

[9]We will make clear below in which sense $\mathbb{S}.\mathbb{S}$ can be seen as a product.

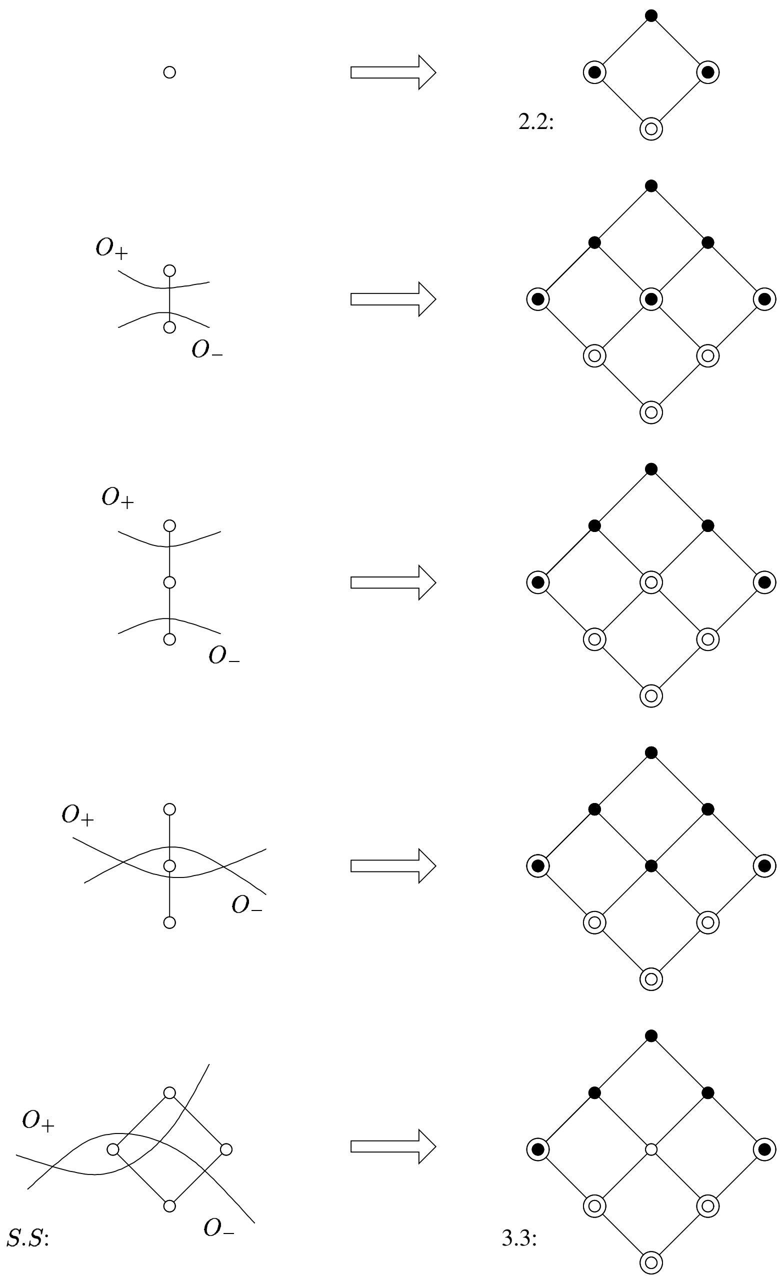


Figure 1: Some bitopological spaces and their concrete d-frames. (D-frame elements in the con-predicate are indicated by an additional circle, those in the tot-predicate are filled in.)

On L itself, a point manifests itself as a pair $(\hat{F}_+, \hat{F}_-)$ of completely prime filters on L that satisfy the analogue of (dp$_{\mathsf{con}}$) and (dp$_{\mathsf{tot}}$), plus

$$\begin{array}{ll} (\mathrm{dp}_+) & \mathit{tt} \in \hat{F}_+; \\ (\mathrm{dp}_-) & \mathit{ff} \in \hat{F}_-; \end{array}$$

Figure 2 illustrates the idea that $(\hat{F}_+, \hat{F}_-)$ determines four "quadrants" so that con does not intersect with the "upper quadrant" and tot does not intersect with the 'lower."

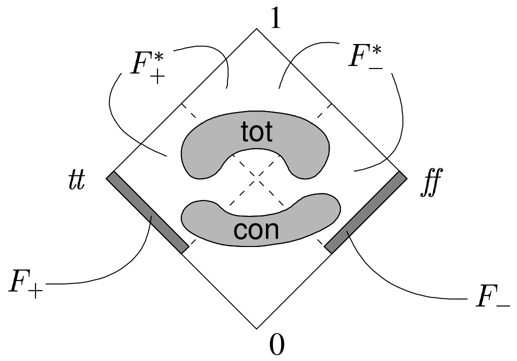


Figure 2: An abstract point in a d-frame.

In ordinary frame theory every point F is alternatively determined by the element $\bigsqcup L \setminus F$; the elements that arise in this fashion are exactly the $\sqcap$-prime ones. Translated to the setting of d-frames, we get elements $\rho \in L$ where $\rho^+ := \rho \sqcup \mathit{ff}$ and $\rho^- := \rho \sqcup \mathit{tt}$ are $\sqcap$-prime in L, no element β below ρ is in tot, and any element γ in con satisfies either $\gamma_+ \sqsubseteq \rho_+$ or $\gamma_- \sqsubseteq \rho_-$. However, this is not very helpful in actually finding points in a d-frame. A more "constructive" analysis of the situation is possible if we add further axioms to the definition of a d-frame; this will be presented in Section 5 below.

The set of d-points becomes a bitopological space by considering the collection of $\Phi_+(x) := \{(F_+, F_-) \mid x \in F_+\}$, $x \in L_+$, as the first topology $\mathcal{T}_+$, and the collection of $\Phi_-(y) := \{(F_+, F_-) \mid y \in F_-\}$, $y \in L_-$, as the second topology $\mathcal{T}_-$. Together, this is the *spectrum* of the d-frame $\mathcal{L}$, which we denote as $\mathsf{spec}\,\mathcal{L}$, following the usual notation for frames. The construction for objects is extended to a (contravariant) functor $\mathsf{spec}\colon \mathbf{dFrm} \to \mathbf{biTop}$ in the usual way, that is, by noting that the inverse image of a point under a d-frame morphism is again a point.

Theorem 3.1 *The functors* Ω *and* spec *establish a dual adjunction between* **biTop** *and* **dFrm**.

Our two categories are clearly concrete over **Set**. For bitopological spaces the left adjoint to the forgetful functor equips a set with two copies of the discrete topology. The free concrete d-frame over a given set is the product of the usual free frame with itself. The two predicates are chosen minimally, that is, as $\emptyset$. It follows that the dualising object in **dFrm** is given as $\Omega\, F_{\mathbf{biTop}}\, 1$ and that of **biTop** as $\mathsf{spec}\, F_{\mathbf{dFrm}}\, 1$. Indeed, we obtain 2.2 and $\mathbb{S}.\mathbb{S}$ this way.[10]

We note that the individual topologies on $\mathbb{S}.\mathbb{S}$ are not even T_0, only their join distinguishes all four elements. This explains why there is no schizophrenic object for the duality of strong proximity lattices and stably compact spaces, as $\mathbb{S}.\mathbb{S}$ is far from being stably compact. This also gives a clue as to why biframes are not suitable as the Stone duals of bitopological spaces. If a schizophrenic object were to exist, on the biframe side it would be the biframe derived from $F_{\mathbf{biTop}}\, 1$. But this is $(2, 2, 2)$. In other words, the schizophrenic object would have to be a two element bitopological space. None of the ten non-bihomeomorphic candidates yields the required representable functor.

3.2 Logical order on a d-frame

As mentioned above, in a d-frame $(L_1, L_2; \mathsf{con}, \mathsf{tot})$ given by two frames separately, the structure of the product $L_+ \times L_-$ plays an important role in our development. Recall that one can think of a frame as a logical structure (finite meets and joins) with an information structure (directed joins) where the two orders coincide. In a d-frame, a second distributive lattice that is "at 90 degrees" to the frame order also exists. This structure is a completely general phenomenon, known at least since [BK47]. Its proof is straightforward and can safely be left as an exercise.

[10] The free d-frame over the one-element set 1 looks like 3.3 in Figure 1, except that no elements should be marked as belonging to con and tot. The generator is the element in the middle.

Proposition 3.2 *Let $(L; \sqcap, \sqcup, 1, 0)$ be a bounded distributive lattice, and (t, f) a complemented pair in L, that is, $t \sqcap f = 0$ and $t \sqcup f = 1$. Then by defining*

$$\begin{array}{rcl} x \wedge y & := & (x \sqcap f) \sqcup (y \sqcap f) \sqcup (x \sqcap y) = (x \sqcup f) \sqcap (y \sqcup f) \sqcap (x \sqcup y) \\ x \vee y & := & (x \sqcup t) \sqcap (y \sqcup t) \sqcap (x \sqcup y) = (x \sqcap t) \sqcup (y \sqcap t) \sqcup (x \sqcap y) \end{array}$$

one obtains another bounded distributive lattice $(L; \wedge, \vee, t, f)$, in which $(1, 0)$ is a complemented pair. The original operations are recovered from it as

$$\begin{array}{rcl} x \sqcap y & = & (x \wedge 0) \vee (y \wedge 0) \vee (x \wedge y) = (x \vee 0) \wedge (y \vee 0) \wedge (x \vee y) \\ x \sqcup y & = & (x \vee 1) \wedge (y \vee 1) \wedge (x \vee y) = (x \wedge 1) \vee (y \wedge 1) \vee (x \wedge y) \end{array}$$

Furthermore, any two of the operations $\sqcap$, $\sqcup$, $\wedge$, and $\vee$ distribute over each other. If L is a frame, then $\wedge$ and $\vee$ are also Scott continuous.

This justifies our choice of symbols tt and ff in a d-frame, and suggests that we regard $(L; \wedge, \vee, tt, ff)$ as the *logical structure* of a d-frame. The logical structure makes a d-frame into a distributive "bilattice." See [Gin92, Fit91, MPS00] for introductions to bilattices. Bilattices are motivated by Belnap's four-valued logic [Bel77].

We can easily compute conjunction and disjunction in terms of elements of L_+ and L_-:

$$\begin{array}{rcl} \langle x, y\rangle \wedge \langle x', y'\rangle & := & \langle x \sqcap x', y \sqcup y'\rangle \\ \langle x, y\rangle \vee \langle x', y'\rangle & := & \langle x \sqcup x', y \sqcap y'\rangle \end{array}$$

Note the reversal of order in the second component. This makes sense, as we think of the second frame as providing negative answers.

The translation from the logical operations back to $\sqcap$ and $\sqcup$ means that we could have factored our definition of d-frames quite differently into a logical structure (a distributive lattice) and an information structure (a dcpo, not explicitly a frame) together with con and tot, and the needed axioms to make our definition recoverable. This approach would clearly emphasise our point that the information order is actually separate from logic, and that frames conflate the two. Separate treatment of the two orders is the primary motivation of investigations into bilattices. That research program, however, has not taken directed completeness of information into account.

4 Sobriety of bitopological spaces

Following the terminology of frames, we say that a bitopological space X is *(d-) sober* if it is bihomeomorphic to $\mathsf{spec}\,\mathcal{L}$ for some d-frame $\mathcal{L}$. As with frames and topological spaces, d-sobriety has an internal characterisation.

Theorem 4.1 *For a bitopological space X, the following are equivalent:*

1. *X is d-sober;*[11]
2. *X is bihomeomorphic to* $\mathsf{spec}\,\Omega X$.
3. *The unit map $\eta\colon X \to \mathsf{spec}\,\Omega X$ given by $x \mapsto (\mathcal{N}_+(x), \mathcal{N}_-(x))$ is a bihomeomorphism.*[12]
4. *The unit is a bijection.*

Proof. Clearly, (3) implies (2) and (2) implies (1). Furthermore, it is clear that for any bitopological space $(X; \tau_+, \tau_-)$ the map η is bicontinuous and bi-open onto the image. So it is a bihomeomorphism if and only if it is a bijection. Thus (4) and (3) are equivalent.

For (1) implies (4), assume X is bihomeomorphic to $\mathsf{spec}\,\mathcal{L}$. We prove that for $\mathsf{spec}\,\mathcal{L}$, the unit $\eta_{\mathsf{spec}\,\mathcal{L}}$ is a one-to-one correspondence between points on the given d-frame $\mathcal{L}$ and points on the second dual $\Omega\,\mathsf{spec}\,\mathcal{L}$. Then by naturality of η and the sobriety of X, η_X is also one-to-one.

Let (F_+, F_-) be a point of $\mathcal{L}$; we calculate its image under η according to the definitions:

$$\begin{aligned}
(\mathcal{F}_+, \mathcal{F}_-) &= \eta(F_+, F_-) \\
&= (\mathcal{N}_+(F_+, F_-), \mathcal{N}_-(F_+, F_-)) \\
&= (\{O_+ \in \mathcal{T}_+ \mid (F_+, F_-) \in O_+\}, \{O_- \in \mathcal{T}_- \mid (F_+, F_-) \in O_-\}) \\
&= (\{\Phi_+(x) \in \mathcal{T}_+ \mid x \in F_+\}, \{\Phi_-(x) \in \mathcal{T}_- \mid x \in F_-\})
\end{aligned}$$

[11]We will usually leave out the qualifier “d-” when it is clear that we are talking about a bitopological space.

[12]“$\mathcal{N}$” indicates the open neighbourhood filter.

Injectivity of η is clear as different points on $\mathcal{L}$ will give rise to different sets of open sets in at least one of the two canonical topologies on $\mathsf{spec}\,\mathcal{L}$. For surjectivity, we assume that $(\mathcal{F}_+, \mathcal{F}_-)$ is a point of $\Omega\,\mathsf{spec}\,\mathcal{L}$. We claim that

$$F_+ \;:=\; \{x \in L_+ \mid \Phi_+(x) \in \mathcal{F}_+\} \quad F_- \;:=\; \{y \in L_- \mid \Phi_-(y) \in \mathcal{F}_-\}$$

defines a point of $\mathcal{L}$ such that $\eta(F_+, F_-) = (\mathcal{F}_+, \mathcal{F}_-)$. Let's check the details:

Both F_+ and F_- are completely prime filters because the maps Φ_+, Φ_- are frame homomorphisms. Next assume that $\langle x, y\rangle \in \mathsf{con}$; in this case, no point (G_+, G_-) can have $x \in G_+$ and $y \in G_-$, and so (G_+, G_-) can not be both in $\Phi_+(x)$ and $\Phi_-(y)$. This means that $\Phi_+(x) \cap \Phi_-(y) = \emptyset$ and hence $\Phi_+(x) \notin \mathcal{F}_+$ or $\Phi_-(y) \notin \mathcal{F}_-$. This, finally, means that either $x \notin F_+$ or $y \notin F_-$. The argument for the tot-predicate is dual.

By the calculation at the beginning of the proof it is clear that (F_+, F_-) has no neighbourhoods other than those in $(\mathcal{F}_+, \mathcal{F}_-)$. ∎

Example 4.2 *All the bitopological spaces in Figure 1 are d-sober. For the one-point space this is clear, as the associated d-frame admits only one point. For the other four spaces one argues as follows: The underlying frame is the same in each case and it admits four completely prime filters:*

$$\begin{array}{lllllll} F_+^1 & := & \uparrow\mathit{tt} & & F_-^1 & := & \uparrow\mathit{ff} \\ F_+^2 & := & \uparrow(O_+, \emptyset) & & F_-^2 & := & \uparrow(\emptyset, O_-) \end{array}$$

The notation already indicates which of these can be used as the first, respectively second, component of a point. From this we get four possible combinations, and these are indeed all available in the last example. In the other three examples, the con/tot *labelling of the element (O_+, O_-) in the centre of the d-frame excludes certain combinations: if it belongs to* con, *then F_+^2 cannot be paired with F_-^2, and if it belongs to* tot *then F_+^1 cannot be paired with F_-^1.*

4.1 Bitopological analogues of topological concepts

As the foregoing examples show, d-sobriety is a subtle constraint on the interaction between τ_+ and τ_-. The remainder of this section explores this interaction, emphasising bitopological analogues of classical topological ideas.

Lemma 4.3 *Let τ_+ and τ_- be two T_0 topologies on a set X, and assume that the bitopological space $(X; \tau_+, \tau_-)$ is sober. Then the intersection of the two specialisation preorders equals identity: $\leq_+ \cap \leq_- =$ '$=$'.*

Proof. Assume $x \leq_+ y$, $x \leq_- y$ for two elements of X. Then $(\mathcal{N}_+(x), \mathcal{N}_-(y))$ is a point of $\Omega\, X$. Indeed, if $O_+ \cap O_- = \emptyset$ then either $y \notin O_+$ or $y \notin O_-$. In the first case, $x \notin O_+$ follows because $x \leq_+ y$.

Similarly, if $O_+ \cup O_- = X$ then either $x \in O_+$ or $x \in O_-$. In the second case, $y \in O_-$ follows because $x \leq_- y$.

Sobriety implies that $(\mathcal{N}_+(x), \mathcal{N}_-(y))$ must be a canonical point associated with a *single* element a of X. However, the T_0 separation axiom says that different points have different neighbourhood filters, so it must be the case that $x = a = y$. ∎

Corollary 4.4 *Let τ be a topology on a set X such that $(X; \tau, \tau)$ is d-sober. Then τ satisfies the T_1 separation axiom already.*

Proof. Any abstract point (F_+, F_-) of a d-sober space is of the form $(\mathcal{N}_+(x), \mathcal{N}_-(x))$ for some unique point x. So when the two topologies are the same, it must be the case that $F_+ = F_-$. Then if x and y are two distinct points of X it must be the case that their neighbourhood filters $\mathcal{N}_+(x) = \mathcal{N}_-(x)$ and $\mathcal{N}_+(y) = \mathcal{N}_-(y)$ differ, in other words, the topology τ must be T_0. By the preceding lemma the specialisation order must be equality so the topology is even T_1. ∎

We take these results to indicate "bi-T_1" ought to mean that $\leq_+ \cap \leq_-$ is equality. We refer to a bitopological space (X, τ, τ) as *symmetric*. Obviously, symmetry makes the category of topological spaces equivalent to a full sub-category of bitopological spaces. We will return to symmetric spaces in Section 9.

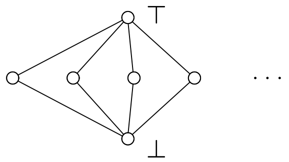
Counterexample 4.5 *If $(X; \tau)$ is a sober space in the usual sense, then $(X; \tau, \tau)$ is not necessarily d-sober. Indeed, by the previous statement this can only happen if $(X; \tau)$ is a T_1 space, but sobriety does not imply this, the Sierpinski space being* *the smallest example of a sober non-T_1 space. Another way of expressing this is to say that finite sets equipped with two T_0 topologies are not necessarily d-sober.*

Lemma 4.6 *If $(X; \tau_+, \tau_-)$ is a d-sober space then $(X; \tau_+ \vee \tau_-)$ is sober.*

Proof. For F a completely prime filter in $\tau_+ \vee \tau_-$, consider $(F_+, F_-) := (F \cap \tau_+, F \cap \tau_-)$. We show that this is a d-point. Indeed, if $O_+ \cap O_- = \emptyset$ then this intersection can not be an element of F. Therefore we can not have $O_+ \in F_+ \subseteq F$ and $O_- \in F_- \subseteq F$ at the same time. If $O_+ \cup O_- = X$ then because $X \in F$, we must have $O_+ \in F$ or $O_- \in F$ by primality. Hence $O_+ \in F_+$ or $O_- \in F_-$. ∎

Of course, the sobriety of $(X; \tau_+ \vee \tau_-)$ does not tell us anything about the d-sobriety of $(X; \tau_+, \tau_-)$; a counterexample is again provided by τ_+ and τ_- both being equal to the Sierpinski topology on the two-element set.[13] A more intricate example is required to show that Lemma 4.3 can not be reversed:

Counterexample 4.7 *Consider the set $X := \mathbb{N} \cup \{\bot, \top\}$ that can usefully be visualised as follows:*



As the positive topology on X we take all subsets of $\mathbb{N}$ plus all co-finite subsets of X. This is T_1 and sober, but not T_2. For the negative topology we take the weak upper topology, *whose closed sets are all of the form ${\downarrow}M$ with M a finite subset of X. This, too, is a sober topology. The specialisation order with respect to τ_+ is equality, and that with respect to τ_- is given in the diagram. Their intersection is, of course, equality.*

We claim that the bitopological space $(X; \tau_+, \tau_-)$ is not sober. Consider the pair $(\mathcal{N}_+(\bot), \mathcal{N}_-(\top))$. It satisfies the condition for total predicates because every non-empty open set of τ_- contains $\top$. The condition for consistent predicates is

[13] This differs from the theory of biframes where sobriety is taken to mean sobriety of $\tau_+ \vee \tau_-$.

also satisfied because every positive neighbourhood of $\perp$ intersects with every negative neighbourhood of $\top$ in co-finitely many natural numbers.

On the positive side we have:

Proposition 4.8 *Let $(X;\tau)$ be a sober space. The following are all d-sober bitopological spaces:*

1. *$(X;\tau,\mathcal{I})$ and $(X;\mathcal{I},\tau)$, where $\mathcal{I}$ is the indiscriminate topology;*
2. *$(X;\tau,\mathcal{D})$ and $(X;\mathcal{D},\tau)$, where $\mathcal{D}$ is the discrete topology;*

Proof. For the first claim, remember that the set $\{X\}$ is the only completely prime filter in the indiscriminate topology, and when paired with a completely prime filter of τ always gives rise to a d-point: As $\mathcal{I}$ has only two open sets, $O_+ \cap O_- = \emptyset$ can only happen if $O_+ = \emptyset$ or $O_- = \emptyset$. Likewise, $O_+ \cup O_- = X$ can only happen if $O_+ = X$ or $O_- = X$.

For the second claim, we note that the discrete topology is Hausdorff and hence guaranteed to be sober. Any d-point, therefore, has the form $(\mathcal{N}_+(x), \mathcal{N}_-(y))$. The rest of the proof depends on τ playing the role of τ_+ (the other case requiring a dual argument): If we had $x \not\leq_+ y$ then there would exist $O \in \tau$ with $x \in O$, $y \notin O$. The complement of O is an open set in $\mathcal{D}$ and so belongs to $\mathcal{N}_-(y)$, contradicting ($\mathsf{dp_{con}}$). Thus $x \leq_+ y$ and likewise $y \leq_+ x$. ∎

We interpret these results about d-sober spaces as telling us that it is more appropriate to consider $\leq_+ \cap \geq_-$ as the *specialisation (pre-) order* of a bitopological space, rather than $\leq_+ \cap \leq_-$. So "bi-T_0" should mean that $\leq_+ \cap \geq_-$ is a partial order. For this order the open sets of τ_+ become *upper* sets, and those of τ_-, *lower* sets. The bitopological spaces in Figure 1 have been drawn in this way. This view jibes with the situation in stably compact spaces where the specialisation order is opposite to that of the associated co-compact topology.

Let us pause to say a few more words about the bitopological space $\mathbb{S}.\mathbb{S}$, the dualising object in **biTop**. There is a forgetful functor from bitopological spaces to $\mathbf{Top^2}$, the category of pairs of topological spaces, which maps $(X;\tau_+,\tau_-)$ to $((X;\tau_+),(X;\tau_-))$. It has a right adjoint which maps a pair $((Y;\tau),(Y';\tau'))$ to

$(Y{\times}Y';\hat{\tau}_+,\hat{\tau}_-)$, where $\hat{\tau}_+ := \{U{\times}Y' \mid U \in \tau_+\}$ and $\hat{\tau}_- := \{Y{\times}U' \mid U \in \tau_-\}$. We denote the resulting bitopological space with $Y.Y'$. Notice that the usual Tychonoff product topology on $Y{\times}Y'$ is precisely the join of $\hat{\tau}_+$ and $\hat{\tau}_-$. The natural isomorphism between hom-sets $\mathbf{Top^2}(((X;\tau_+),(X;\tau_-)),((Y;\tau),(Y';\tau')))$ and $\mathbf{biTop}((X;\tau_+,\tau_-),Y.Y')$ is obvious. The dualising bitopological space $\mathbb{S}.\mathbb{S}$ is obtained in this way from two copies of Sierpinski space.

Definition 4.9 *A bitopological space $(X;\tau_+,\tau_-)$ is called* order-separated *if $\leq\, = \,\leq_+ \cap \geq_-$ is a partial order and $x \not\leq y$ implies that there are disjoint open sets $O_+ \in \tau_+$ and $O_- \in \tau_-$ such that $x \in O_+$ and $y \in O_-$.*

Lemma 4.10 *In an order-separated bitopological space the following are true:*

1. *$\leq_+ \,=\, \geq_-$;*
2. *$\leq_+ \cap \leq_- \,=\,$ '$=$'.*

Proof. For the first claim assume $x \not\leq_+ y$. This implies $x \not\leq y$ and we get a separating partial predicate (O_+,O_-). Since $y \in O_-$ but $x \notin O_-$ we conclude $x \not\geq_- y$. So $\not\leq_+ \,=\, \not\geq_-$ and this is equivalent to the first claim.

The second claim follows immediately from (1) and anti-symmetry of $\leq$. ∎

D-sobriety is a surprisingly inclusive concept. We illustrate this with two examples.

Example 4.11 *$\mathbb{R}$ with the usual upper and lower topology is d-sober. This is seen as follows: Although $F_\infty := \{(r,\infty) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}\} = \tau_+ \backslash \{\emptyset\}$ is completely prime and not the neighbourhood filter of any real number, there is no F_- such that the pair (F_∞, F_-) is a d-point. Indeed, all F_- have the form $F_s := \{(-\infty,r) \mid r > s\}$ or $F_{-\infty} := \tau_- \backslash \{\emptyset\}$; F_s can not be paired with F_∞ because $(-\infty,s+1) \cap (s+1,\infty) = \emptyset$ but neither $(-\infty,s+1) \notin F_s$ nor $(s+1,\infty) \notin F_\infty$. The same argument shows that $F_{-\infty}$ can not be paired with F_∞.*

Example 4.12 *Consider $X = [0,1] \setminus \{\frac{1}{2}\}$ with the usual upper and lower topology. This is a d-sober space.*

Indeed, the only pair of completely prime filters that could cause trouble is

$$F_+ \;:=\; \{(r,1]\setminus\{\tfrac{1}{2}\} \mid r < \tfrac{1}{2}\} \quad F_- \;:=\; \{[0,r)\setminus\{\tfrac{1}{2}\} \mid r > \tfrac{1}{2}\}$$

but this does not give rise to a d-point: $X = [0,\frac{1}{2}) \cup (\frac{1}{2},1]$ *but neither* $[0,\frac{1}{2}) \in F_-$ *nor* $(\frac{1}{2},1] \in F_+$.

Both of these examples are order-separated. The fact that they are sober generalises to all such spaces.

Theorem 4.13 *Order-separated bitopological spaces are sober.*

Proof. Order separation clearly implies that the canonical map $\eta\colon X \to \mathsf{spec}\,\Omega X$ is injective; the real issue is surjectivity. So assume that (F_+, F_-) is a point of ΩX. Consider the two sets

$$V_+ \;:= \bigcup \{O_+ \in \tau_+ \mid O_+ \notin F_+\} \quad V_- \;:= \bigcup \{O_- \in \tau_- \mid O_- \notin F_-\}$$

and their complements V_+^c, V_-^c. Because of condition ($\mathsf{dp}_{\mathsf{tot}}$), $V_+ \cup V_-$ cannot be the whole space, in other words, the intersection $V_+^c \cap V_-^c$ is non-empty.

Next we show that every element of V_+^c is below every element of V_-^c in the specialisation order $\leq \;=\; \leq_+ \cap \geq_-$. Indeed, if $x \in V_+^c$, $y \in V_-^c$, and $x \not\leq y$, then by order separation there is a partial predicate (O_+, O_-) with $x \in O_+$ and $y \in O_-$. By definition of V_+, V_- we have $O_+ \in F_+$ and $O_- \in F_-$, contradicting condition ($\mathsf{dp}_{\mathsf{con}}$) of d-points.

Finally, let a be an element in the intersection $V_+^c \cap V_-^c$. We show that F_+ is the neighbourhood filter of a in τ_+. Assume $a \in O_+$; this implies $O_+ \not\subseteq V_+$ and the latter is equivalent to $O_+ \in F_+$. For the converse we start at $O_+ \not\subseteq V_+$, which gives us an element $b \in V_+^c \cap O_+$ about which we already know that $b \leq a$. It follows that $b \leq_+ a$ and hence $a \in O_+$. ∎

We see this result as being analogous to the well-known fact that T_2 spaces are sober in the traditional sense.

Corollary 4.14 *Compact regular bitopological spaces are d-sober.*

A perhaps less well-known (but easy to prove) fact is that sub-spaces of sober T_1 spaces are sober. Recalling our understanding of "bi-T_1", this, too, has an analogue in the bitopological setting:

Theorem 4.15 *Let $(X;\tau_+,\tau_-)$ be d-sober and $\leq_+\cap\leq_- =$ '$=$'. Let further Y be any subset of X. Then $(Y;\tau_+\restriction_Y,\tau_-\restriction_Y)$ is d-sober, too.*

Proof. (i) Let a be an element of X. Since $\mathsf{cl}_+(a) = {\downarrow_+} a$, $\mathsf{cl}_-(a) = {\downarrow_-} a$, and $\leq_+\cap\leq_- =$ '$=$', $\mathsf{cl}_+(a) \cap \mathsf{cl}_-(a) = \{a\}$. Now, $V_+ := X \setminus \mathsf{cl}_+(a)$ is in τ_+ but not in $\mathcal{N}_+(a)$. Likewise, $V_-(a) := X \setminus \mathsf{cl}_-(a) \in \tau_- \setminus \mathcal{N}_-(a)$. Furthermore, $V_+(a) \cup V_-(a) = X \setminus \{a\}$.

(ii) Assume $a \in X \setminus Y$. Then $V_+(a)\restriction_Y = V_+(a) \cap Y \in \tau_+\restriction_Y$ and $V_-(a)\restriction_Y = V_-(a) \cap Y \in \tau_-\restriction_Y$. Furthermore, $V_+(a)\restriction_Y \cup V_-(a)\restriction_Y = Y$.

(iii) Let (F_+^Y, F_-^Y) be a d-point on Y. Because of (ii), either $V_+(a)\restriction_Y \in F_+^Y$ or $V_-(a)\restriction_Y \in F_-^Y$.

(iv) The embedding $e\colon Y \to X$ is bicontinuous, hence e^{-1} restricts to frame homomorphisms $\tau_+ \to \tau_+\restriction_Y$ and $\tau_- \to \tau_-\restriction_Y$. Concretely, $e^{-1}(O) = O \cap Y$. It follows that the inverse image map to the frame homomorphisms maps a completely prime filter F_+^Y in $\tau_+\restriction_Y$ to a completely prime filter F_+ in τ_+. Concretely, $F_+ = \{O \in \tau_+ \mid O \cap Y \in F_+^Y\}$.

(v) For the d-point (F_+^Y, F_-^Y) consider (F_+, F_-) on X. This is again a d-point: $O_+ \cap O_- = \emptyset$ implies $(O_+ \cap Y) \cap (O_- \cap Y) = \emptyset$, so either $O_+ \cap Y \notin F_+^Y$ or $O_- \notin F_+$. The covering condition is proved analogously.

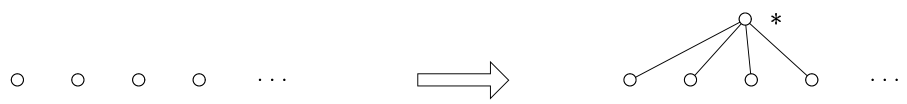
(vi) By the assumption of X being d-sober, $(F_+, F_-) = (\mathcal{N}_+(a), \mathcal{N}_-(a))$ for some $a \in X$. Case 1: $a \in Y$. Then $F_+^Y = \mathcal{N}_+(a)\restriction_Y = \{O \cap Y \mid O \in \mathcal{N}_+(a)\}$ because $O \in F_+$ can by definition only happen if $O \cap Y \in F_+^Y$. Conversely, $U \in F_+^Y$ implies that there exists $O \in \tau_+$ such that $U = O \cap Y$ by definition of the sub-space topology. This O then belongs to F_+. It follows that F_+^Y is the neighbourhood filter of a in Y. Likewise for F_-^Y.

Case 2: $a \notin Y$. We know from (ii) that either $V_+(a) \cap Y \in F_+^Y$ or $V_-(a) \cap Y \in F_-^Y$. This implies $V_+(a) \in F_+$ or $V_-(a) \in F_-$. This is a contradiction, though, because neither $V_+(a)$ nor $V_-(a)$ are neighbourhoods of a in X.

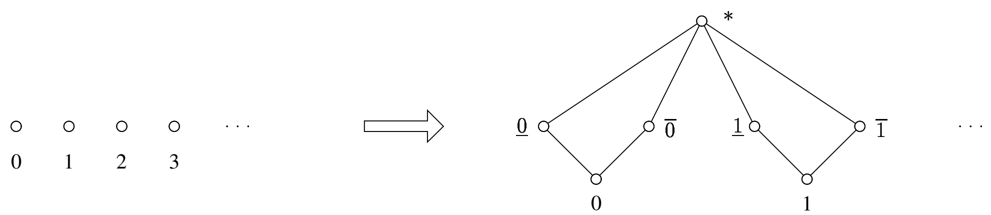
(vii) The conclusion is that all d-points of ΩY are pairs of neighbourhood filters of points in Y. ∎

It is tempting to conclude from this rather surprising result that every space which satisfies $\leq_+ \cap \leq_- =$ '$=$' is d-sober, as it can be rendered as a sub-space of its d-sobrification. However, the condition does not survive d-sobrification:

Example 4.16 *Consider the co-finite topology* τ *on* $\mathbb{N}$*. Its specialisation order is trivial, yet the sobrification (in the usual sense) adds a new point* $*$ *(corresponding to the filter* $\mathcal{C}$ *of all non-empty co-finite subsets) which sits above all other elements:*



The d-sobrification of $(\mathbb{N}; \tau, \tau)$ *adds* $*$ *(represented as the d-point* $(\mathcal{C}, \mathcal{C})$*) and triplicates each natural number: apart from* $n = (\mathcal{N}(n), \mathcal{N}(n))$ *there are also* $\underline{n} := (\mathcal{N}(n), \mathcal{C})$ *and* $\overline{n} := (\mathcal{C}, \mathcal{N}(n))$*. The order* $\leq_+ \cap \leq_-$ *between these is quite rich:*



Each sub-structure $\{*, \underline{n}, n, \overline{n}\}$ *carries the same bitopology as the space* $\mathbb{S}.\mathbb{S}$ *on the bottom of Figure 1.*

4.2 Hofmann-Mislove

We conclude this section with a discussion of Hofmann-Mislove type theorems for bitopological spaces. For motivation and comparison we look at an ordinary topological space $(X; \tau)$ first. The collection $\mathcal{N}(A)$ of open neighbourhoods of any subset $A \subseteq X$ is always a filter in the frame τ, but in general there are far more filters than neighbourhood filters. In three cases we can say more:

1. If A is an open set then $\mathcal{N}(A) = \uparrow A$, that is, the neighbourhood filter is principal. All principal filters in τ arise in this way.

2. If $A = \uparrow x$ for x an element of X then $\mathcal{N}(A)$ is completely prime. The space is sober if and only if all completely prime filters arise uniquely in this way.

3. If A is compact then $\mathcal{N}(A)$ is Scott-open. The Hofmann-Mislove Theorem states that in a sober space all Scott-open filters arise in this way.

Consider $A \subseteq X$ for a bitopological space $(X; \tau_+, \tau_-)$. The collection $\mathcal{S}_+A := \bigcap_{a \in A} \mathcal{N}_+(a)$ is equal to the set of all τ_+-neighbourhoods of A, which from the point of logic corresponds to all those predicates φ for which every elements of A satisfies φ. On the negative side, one should consider $\mathcal{U}_-A := \bigcup_{a \in A} \mathcal{N}_-(a)$ which is all those predicates φ for which some element of A fails φ. A moment's consideration will convince the reader that the pair $(\mathcal{S}_+A, \mathcal{U}_-A)$ still satisfies the axioms ($\mathsf{dp}_{\mathsf{con}}$) and ($\mathsf{dp}_{\mathsf{tot}}$) for d-points. However, the collection $\mathcal{S}_+A$ is merely a filter in τ_+, while $\mathcal{U}_-A$ is a completely prime upper set (but not necessarily a filter). As with ordinary topological spaces we can ask which special cases of such pairs can be characterised by properties of the subset A. D-sobriety, obviously, is case (2) generalised to bitopological spaces; case (1) takes the following form:

Proposition 4.17 *For any topological space $(X; \tau)$, there is a bijection between closed sets and completely prime upper subsets of τ.*

Proof. For a closed set A, let $\mathcal{U}_{\mathcal{A}} := \{\mathcal{O} \in \tau \mid \mathcal{A} \cap \mathcal{O} \neq \emptyset\}$. For a completely prime upper set $\mathcal{U}$, let $A_{\mathcal{U}} := X \setminus \bigcup\{O \in \tau \mid O \notin \mathcal{U}\}$. ∎

Proposition 4.18 *Let $(X; \tau_+, \tau_-)$ be a bitopological space. Let $\mathcal{U}_-$ be a completely prime upper set in τ_-, and A the corresponding τ_--closed set according to the previous proposition. Then the τ_+-neighbourhood filter $\mathcal{S}_+$ of A satisfies:*

$$\begin{array}{lll} (\mathsf{hm}_{\mathsf{tot}}) & \textit{for all} & O_+ \in \tau_+, O_- \in \tau_- : \\ & & O_+ \cup O_- = X \quad \Longrightarrow \quad O_+ \in \mathcal{S}_+ \quad \textit{or} \quad O_- \in \mathcal{U}_-; \\ (\mathsf{hm}'_{\mathsf{con}}) & \textit{for all} & O_+ \in \mathcal{S}_+ : \quad O_+ \cup V_- = X \\ & \textit{where} & V_- := X \setminus A = \bigcup \{O_- \in \tau_- \mid O_- \notin \mathcal{U}_-\}. \end{array}$$

Moreover, $\mathcal{S}_+$ *is uniquely determined by* (hm_{tot}) *and* ($\text{hm}'_{\mathsf{con}}$).

Proof. The first part is immediate.

For uniqueness suppose $\mathcal{S}'$ is a filter in τ_+ satisfying (hm_{tot}) and ($\text{hm}'_{\mathsf{con}}$). From ($\text{hm}'_{\mathsf{con}}$), $A_- \subseteq O_+$ holds for each $O_+ \in \mathcal{S}'$. Hence $\mathcal{S}' \subseteq \mathcal{S}_+$. For the reverse inclusion, suppose $A_- \subseteq O_+$. Therefore $O_+ \cup V_- = X$, and since $V_- \notin \mathcal{U}_-$ we must have $O_+ \in \mathcal{S}'$ by (hm_{tot}). ∎

From the preceding discussion, the reader may have expected to find the following instead of ($\text{hm}'_{\mathsf{con}}$):

$$(\text{hm}_{\mathsf{con}}) \quad \text{for all} \quad O_+ \in \tau_+, O_- \in \tau_- : \\ O_+ \cap O_- = \emptyset \quad \Longrightarrow \quad O_+ \notin \mathcal{S}_+ \quad \text{or} \quad O_- \notin \mathcal{U}_-.$$

However, this easily follows from (hm_{tot}) and ($\text{hm}'_{\mathsf{con}}$): if $O_+ \in \mathcal{S}_+$ then $O_+ \cup V_- = X$ and if also $O_+ \cap O_- = \emptyset$ then $O_- \subseteq V_-$ must follow. Without additional assumptions on X, however, the stronger condition is needed for uniqueness.

Proposition 4.19 *With the terminology of the previous proposition:*

1. $\mathcal{S}_+$ *is Scott-open if and only if* $\mathsf{sat}_+(A_-)$ *is* τ_+*-compact.*
2. *If* $(X; \tau_+, \tau_-)$ *satisfies* $\leq_+ \subseteq \geq_-$ *then* $A_- = \mathsf{sat}_+(A_-)$.

Proof. Part (1) is trivial. For the second claim assume $x \in \mathsf{sat}_+(A_-)$. Since $\mathsf{sat}_+(A_-) = {\uparrow_+}(A_-)$ there is $y \in A_-$ with $x \geq_+ y$, so by assumption $x \leq_- y$ and $x \in \mathsf{cl}_-(y)$. This forces $x \in A_-$. ∎

We doubt that Proposition 4.18 together with part (1) of the preceding result qualifies as a bitopological version of the Hofmann-Mislove Theorem, since its proof is so easy and d-sobriety is not required. We obtain a more satisfactory result when we assume regularity (Definition 2.13). First of all, it is not hard to show that every regular bitopological space is order-separated and therefore sober. The interesting bit for us, however, is that condition ($\text{hm}_{\mathsf{con}^*}$) can be replaced by ($\text{hm}_{\mathsf{con}}$):

Theorem 4.20 *Let* $(X; \tau_+, \tau_-)$ *be a regular bitopological space. There is a one-to-one correspondence between* *(i) subsets A of X which are τ_--closed and τ_+-compact; and*

(ii) pairs $(\mathcal{S}_+, \mathcal{U}_-)$ where $\mathcal{S}_+$ is a Scott-open filter in τ_+ and $\mathcal{U}_-$ is a completely prime upper set in τ_-, satisfying $(\text{hm}_{\mathsf{con}})$ *and* $(\text{hm}_{\mathsf{tot}})$.

Proof. We show that $(\text{hm}_{\mathsf{con}^*})$ follows from regularity and $(\text{hm}_{\mathsf{con}})$. To this end, let $O_+ \in \mathcal{S}_+$. It is the directed union of open sets well-inside it, and so by Scott-openness some $O'_+ \lhd O_+$ belongs to $\mathcal{S}_+$, too. The witness $O_- \in \tau_-$ satisfies $O'_+ \cap O_- = \emptyset$, so by $(\text{hm}_{\mathsf{con}})$ cannot belong to $\mathcal{U}_-$, in other words, it is a subset of $V_- = \bigcup \tau_- \setminus \mathcal{U}_-$. By definition, we also have $O_+ \cup O_- = X$ and therefore $O_+ \cup V_- = X$ as required. ∎

5 Reasonable d-frames and spatiality

We say that a d-frame $\mathcal{L}$ is *spatial* if $\mathcal{L}$ is isomorphic to $\Omega\, X$ for some bitopological space X. As with d-sobriety, spatiality has internal characterisations.

For any d-frame $\mathcal{L} = (L; tt, ff; \mathsf{con}, \mathsf{tot})$, the co-unit $\epsilon_{\mathcal{L}}$ is determined by the two frame homomorphisms $\epsilon_+ \colon L_+ \to (\Omega\, \mathsf{spec}\, \mathcal{L})_+$ and $\epsilon_- \colon L_- \to (\Omega\, \mathsf{spec}\, \mathcal{L})_-$ defined by $x \mapsto \Phi_+(x)$ and $y \mapsto \Phi_-(y)$. Clearly, both of these are surjective and so $\epsilon_{\mathcal{L}}$ itself is surjective.

Theorem 5.1 *For a d-frame $\mathcal{L}$, the following are equivalent:*

1. *$\mathcal{L}$ is spatial.*
2. *$\mathcal{L}$ is isomorphic to* $\Omega\, \mathsf{spec}\, \mathcal{L}$.
3. *The co-unit $\epsilon_{\mathcal{L}}$ is an isomorphism.*
4. *The co-unit is injective and reflects* con *and* tot.
5. *$\mathcal{L}$ satisfies the following four conditions:*

$$\begin{array}{rl}
(\text{s}_+) & \forall x \not\sqsubseteq x' \in L_+\ \exists (F_+, F_-) \in \mathsf{spec}\, \mathcal{L}.\ x \in F_+, x' \notin F_+; \\
(\text{s}_-) & \forall y \not\sqsubseteq y' \in L_-\ \exists (F_+, F_-) \in \mathsf{spec}\, \mathcal{L}.\ y \in F_-, y' \notin F_-; \\
(\text{s}_{\mathsf{con}}) & \forall \alpha \notin \mathsf{con}\ \exists (F_+, F_-) \in \mathsf{spec}\, \mathcal{L}.\ \alpha_+ \in F_+, \alpha_- \in F_-; \\
(\text{s}_{\mathsf{tot}}) & \forall \alpha \notin \mathsf{tot}\ \exists (F_+, F_-) \in \mathsf{spec}\, \mathcal{L}.\ \alpha_+ \notin F_+, \alpha_- \notin F_-;
\end{array}$$

Proof. Clearly, the d-frame $\Omega\, X$ associated with a bitopological space X satisfies the four other conditions. So (1) $\Rightarrow$ (5). Also (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are trivial. As $\epsilon_{\mathcal{L}}$ is a surjective frame homomorphism that preserves con and tot, if it is also injective and reflects con and tot, then it is an isomorphism.

For (5) $\Rightarrow$ (4), observe that conditions (s_+) and (s_-) imply that the assignments $x \mapsto \Phi_+(x)$ and $y \mapsto \Phi_-(y)$ are injective. Thus $\epsilon_{\mathcal{L}}$ is injective, and we only need to check that the two predicates are reflected. If we assume that $\Phi_+(\alpha_+) \cap \Phi_-(\alpha_-) = \emptyset$ then we know that for every abstract point (F_+, F_-), either $F_+ \notin \Phi_+(\alpha_+)$ or $F_- \notin \Phi_-(\alpha_-)$, which by definition means $\alpha_+ \notin F_+$ or $\alpha_- \notin F_-$. By Rule ($\mathrm{s}_{\mathsf{con}}$) it follows that α must belong to con. Reflection of tot is shown analogously. ▮

We collect some properties of the con- and tot-predicate on spatial d-frames.

Lemma 5.2 *Let $(L; tt, ff; \mathsf{con}, \mathsf{tot})$ be a spatial d-frame, and $\alpha \sqsubseteq \beta$. Then*

$$\begin{array}{rcl} \alpha \in \mathsf{tot} & \Longrightarrow & \beta \in \mathsf{tot} \\ \beta \in \mathsf{con} & \Longrightarrow & \alpha \in \mathsf{con} \end{array}$$

Proof. We show the contrapositive: $\beta \notin \mathsf{tot}$ implies by ($\mathrm{s}_{\mathsf{tot}}$) that there is an abstract point (F_+, F_-) with $\beta_+ \notin F_+$ and $\beta_- \notin F_-$, so the same is true for α_+ and α_-, which shows that α can not belong to tot.

The proof for con is analogous. ▮

Lemma 5.3 *Let $(L; tt, ff; \mathsf{con}, \mathsf{tot})$ be a spatial d-frame. Then $tt \in \mathsf{con}$, $tt \in \mathsf{tot}$, $ff \in \mathsf{con}$, $ff \in \mathsf{tot}$ and*

$$\begin{array}{rcl} \alpha, \beta \in \mathsf{tot} & \Longrightarrow & (\alpha \wedge \beta) \in \mathsf{tot} \textit{ and } (\alpha \vee \beta) \in \mathsf{tot} \\ \alpha, \beta \in \mathsf{con} & \Longrightarrow & (\alpha \wedge \beta) \in \mathsf{con} \textit{ and } (\alpha \vee \beta) \in \mathsf{con} \end{array}$$

Proof. Suppose $tt \notin \mathsf{con}$. Then there is a d-point (F_+, F_-) such that $tt_- = 0 \in F_-$. This is impossible as F_- is a completely prime filter. The other three memberships are proved similarly.

For closure of tot under $\vee$ (and by a symmetric argument, $\wedge$), we show the contrapositive: Suppose $\alpha = \langle x, y \rangle$ and $\beta = \langle x', y' \rangle$ and assume $\alpha \vee \beta = \langle x \sqcup$

$x', y \sqcap y'\rangle \notin \mathsf{tot}$, then by ($\mathrm{s}_{\mathsf{tot}}$) there is an abstract point (F_+, F_-) with $x \sqcup x' \notin F_+$, $y \sqcap y' \notin F_-$ from which we get that both x and x' do not belong to F_+ and either y or y' does not belong to F_-. It follows that either $\langle x, y\rangle \notin \mathsf{tot}$ or $\langle x', y'\rangle \notin \mathsf{tot}$. The proofs for con are analogous, using the primality of F_+ and F_-. ∎

Lemma 5.4 *Let $\mathcal{L}$ be a spatial d-frame. Then* con *is Scott closed with respect to* $\sqsubseteq$.

Proof. If D were a directed subset of con with $\bigsqcup^{\uparrow} D \notin \mathsf{con}$, there would be an abstract point (F_+, F_-) with $(\bigsqcup^{\uparrow} D)_+ \in F_+$ and $(\bigsqcup^{\uparrow} D)_- \in F_-$. Both filters are completely prime. So for some $\alpha \in D$, $\alpha_+ \in F_+$ and for some $\beta \in D$, $\beta_- \in F_-$. But D is directed, so we can choose $\alpha = \beta$. This contradicts the definition of abstract points. We already checked that con is closed downward. ∎

Lemma 5.5 *Let* $(L; \mathit{tt}, \mathit{ff}; \mathsf{con}, \mathsf{tot})$ *be a spatial d-frame,* $\alpha \in \mathsf{con}$ *and* $\beta \in \mathsf{tot}$. *Then*

$$\begin{array}{ccc} \alpha_+ = \beta_+ & \Longrightarrow & \alpha \sqsubseteq \beta \\ \alpha_- = \beta_- & \Longrightarrow & \alpha \sqsubseteq \beta \end{array}$$

Proof. Toward a contradiction for the first implication suppose $\alpha_- \not\sqsubseteq \beta_-$. By spatialilty, there is an abstract point (F_+, F_-) for which $\alpha_- \in F_-$ and $\beta_- \notin F_-$. But $\alpha \in \mathsf{con}$ implies $\alpha_+ \notin F_+$ and $\beta \in \mathsf{tot}$ implies $\beta_+ \in F_+$; so $\beta_+ \not\sqsubseteq \alpha_+$, contradicting the assumption. ∎

We believe that the properties expressed in the lemmas in this section are independent of each other (except for the trivial fact that con being Scott closed implies con is closed downward), but we do not have a proof for it.

5.1 Reasonable d-frames

The properties stated in Lemmas 5.2 through 5.5 constitute a good restriction on general d-frames without necessarily requiring spatiality. All of them, except for Lemma 5.4 are first-order properties. The one non-first-order property is nevertheless constructive, as it simply involves closure of con under directed suprema.

$(\mathsf{con}-\downarrow)$	$\alpha \sqsubseteq \beta \;\&\; \beta \in \mathsf{con} \quad \Longrightarrow \quad \alpha \in \mathsf{con}$	
$(\mathsf{tot}-\uparrow)$	$\alpha \sqsubseteq \beta \;\&\; \alpha \in \mathsf{tot} \quad \Longrightarrow \quad \beta \in \mathsf{tot}$	
$(\mathsf{con}-\mathit{tt})$	$\mathit{tt} \in \mathsf{con}$	
$(\mathsf{con}-\mathit{ff})$	$\mathit{ff} \in \mathsf{con}$	
$(\mathsf{con}-\wedge)$	$\alpha \in \mathsf{con} \;\&\; \beta \in \mathsf{con} \quad \Longrightarrow \quad (\alpha \wedge \beta) \in \mathsf{con}$	
$(\mathsf{con}-\vee)$	$\alpha \in \mathsf{con} \;\&\; \beta \in \mathsf{con} \quad \Longrightarrow \quad (\alpha \vee \beta) \in \mathsf{con}$	
$(\mathsf{tot}-\mathit{tt})$	$\mathit{tt} \in \mathsf{tot}$	
$(\mathsf{tot}-\mathit{ff})$	$\mathit{ff} \in \mathsf{tot}$	
$(\mathsf{tot}-\wedge)$	$\alpha \in \mathsf{tot} \;\&\; \beta \in \mathsf{tot} \quad \Longrightarrow \quad (\alpha \wedge \beta) \in \mathsf{tot}$	
$(\mathsf{tot}-\vee)$	$\alpha \in \mathsf{tot} \;\&\; \beta \in \mathsf{tot} \quad \Longrightarrow \quad (\alpha \vee \beta) \in \mathsf{tot}$	
$(\mathsf{con}-\bigsqcup^{\uparrow})$	$A \subseteq \mathsf{con}$ directed w.r.t. $\sqsubseteq \quad \Longrightarrow \quad \bigsqcup^{\uparrow} A \in \mathsf{con}$	
$(\mathsf{con}-\mathsf{tot})$	$\alpha \in \mathsf{con}, \beta \in \mathsf{tot}, \quad (\alpha_+ = \beta_+ \quad \text{or} \quad \alpha_- = \beta_-) \quad \Longrightarrow \quad \alpha \sqsubseteq \beta$	

Table 1: The defining properties of reasonable d-frames.

From now on, we will generally concentrate on d-frames that satisfy these conditions.

Definition 5.6 *A d-frame which satisfies the properties stated in Lemmas 5.2 through 5.5, is called* reasonable. *For ease of reference they are collected and named in Table 1. The category of reasonable d-frames is denoted by* **rdFrm**.

Taking Lemmas 5.2 through 5.5 together, we see that the adjunction between bitopological spaces and d-frames co-restricts to reasonable d-frames. So there is no real loss in restricting our attention to **rdFrm**.

A reasonable d-frame need not be spatial: take a frame L without any points and consider $(L, L; \mathsf{con}, \mathsf{tot})$ where $\langle x, y\rangle \in \mathsf{con}$ if $x \sqcap y = 0$, and $\langle x, y\rangle \in \mathsf{tot}$ if $x \sqcup y = 1$. It is a trivial exercise to prove that the resulting d-frame is reasonable, but it obviously can't have any points. Below we will combine reasonableness with compactness and regularity, and then spatiality will follow.

Proposition 5.7 *The forgetful functor from* **rdFrm** *to* **Set** *has a left adjoint.*

Proof. The free reasonable d-frame over a set A is $(FA, FA; \mathsf{con}, \mathsf{tot})$ where FA is the free frame over A. Generators are the pairs (a, a), $a \in A$. The two relations are chosen minimally: $\langle x, y\rangle \in \mathsf{con}$ if and only if $x = 0$ or $y = 0$; $\langle x, y\rangle \in \mathsf{tot}$ if and only if $x = 1$ or $y = 1$. The conditions for a reasonable d-frame are proved by case analysis. ∎

As an example, the structure labelled 3.3 in Figure 1 is the free reasonable d-frame generated by a one-element set.

5.2 Biframes

We return to the question of how d-frames are related to the biframes of Banaschewski, Brümmer, and Hardie. If (L_0, L_1, L_2) is a biframe then we can define a d-frame by setting $\Delta(L_0, L_1, L_2)$ to be $(L_1, L_2; \mathsf{con}, \mathsf{tot})$ where $\langle x, y\rangle \in \mathsf{con} \;:\Leftrightarrow\; x \sqcap y = 0$, and $\langle x, y\rangle \in \mathsf{tot} \;:\Leftrightarrow\; x \sqcup y = 1$, exploiting the fact that L_1 and L_2 are subsets of the frame L_0. Clearly, this extends to a functor Δ from biframes to d-frames. The following is an easy exercise:

Proposition 5.8 *Every d-frame derived from a biframe is reasonable.*

One can think of Δ as a forgetful functor from biframes to d-frames. Specifically, it forgets everything about L_0 except for the pairs of elements in $L_1 \times L_2$ that meet at 0 or join at 1.

Theorem 5.9 *The functor Δ has a left adjoint.*

Proof. Given $\mathcal{L} = (L; tt, ff; \mathsf{con}, \mathsf{tot})$, we generate a frame L_0 using the set

$$G := \{\ulcorner x \urcorner \mid x \in L_+ \cup L_-\}$$

subject to the relations

$$\begin{array}{rcll}
\ulcorner x \urcorner & \sqsubseteq & \ulcorner x' \urcorner & \text{for all } \; x \sqsubseteq x' \in L_+ \;\text{ or }\; x \sqsubseteq x' \in L_- \\
\bigsqcap_{a \in M} \ulcorner a \urcorner & \sqsubseteq & \ulcorner \bigsqcap M \urcorner & \text{for all } \; M \subseteq_{\text{fin}} L_+ \;\text{ or }\; M \subseteq_{\text{fin}} L_- \\
\ulcorner \bigsqcup A \urcorner & \sqsubseteq & \bigsqcup_{a \in A} \ulcorner a \urcorner & \text{for all } \; A \subseteq L_+ \;\text{ or }\; A \subseteq L_- \\
\ulcorner x \urcorner \sqcap \ulcorner y \urcorner & \sqsubseteq & 0 & \text{for all } \; \langle x, y\rangle \in \mathsf{con} \\
1 & \sqsubseteq & \ulcorner x \urcorner \sqcup \ulcorner y \urcorner & \text{for all } \; \langle x, y\rangle \in \mathsf{tot}
\end{array}$$

Note the similarity of this construction with the coproduct of frames; the only difference is in the last two rules which ensure that the two predicates are respected. As with the coproduct, here too we find that L_0 contains subframes $\ulcorner L_+ \urcorner$ and $\ulcorner L_- \urcorner$ given by the equivalence classes of the generators from L_+ and L_-. Thus the triple $(L_0, \ulcorner L_+ \urcorner, \ulcorner L_- \urcorner)$ is a biframe, and $x \mapsto \ulcorner x \urcorner$ restricts to frame homomorphisms from L_+ to $\ulcorner L_+ \urcorner$ and from L_- to $\ulcorner L_- \urcorner$. By the last two rules, these maps preserve con and tot, so we have a d-frame homomorphism from $\mathcal{L}$ to $\Delta(L_0, \ulcorner L_+ \urcorner, \ulcorner L_- \urcorner)$.

If (K_0, K_1, K_2) is a biframe and h a d-frame homomorphism h from $\mathcal{L}$ to $\Delta(K_0, K_1, K_2)$ we get a map from the set of generators of L_0 to K_0 by setting $\ulcorner x \urcorner \mapsto h(x)$. Since this map clearly respects the above relations, it extends uniquely to a frame homomorphism; the result sends equivalence classes of generators from L_1 to K_1 and similarly for L_2. ∎

An alternative construction of the free biframe highlights, once again, the interplay between logic and information. For this we use a known folk theorem that a cartesian product $L_1 \times L_2$ of frames is also a coproduct of L_1 and L_2 in the category of meet semilattices. In terms of a d-frame, the injections from L_+ and L_- take the form $x \mapsto x \sqcup \mathit{ff}$ and $y \mapsto y \sqcup \mathit{tt}$, respectively.

For a given d-frame $\mathcal{L} = (L; \mathit{tt}, \mathit{ff}; \mathsf{con}, \mathsf{tot})$, we consider the collection J_0 of subsets $A \subseteq L$ satisfying

(bif−con)	$\mathsf{con} \subseteq A$;
(bif−tot)	$\alpha \in \mathsf{tot}, \langle \alpha_+ \sqcap \beta_+, \beta_- \rangle \in A, \langle \beta_+, \beta_- \sqcap \alpha_- \rangle \in A \implies \beta \in A$;
(bif−$\sqsubseteq$)	A is Scott closed with respect to with respect to $\sqsubseteq$;
(bif−$\leq$)	A is a sub-lattice with respect to $\leq$;

Considering L as a $\sqcap$-semi-lattice, we define families $C(\alpha)$ of subsets of L indexed by members of L as follows:

1. $\alpha \in \mathsf{con}$ and $\beta \sqsubseteq \alpha$ implies $\emptyset \in C(\beta)$;

2. $\alpha \in \mathsf{tot}$ implies $\{\langle \alpha_+ \sqcap \beta_+, \beta_- \rangle, \langle \beta_+, \beta_- \sqcap \alpha_- \rangle\} \in C(\beta)$;

3. $\alpha \sqsubseteq \bigsqcup^{\uparrow} B$ implies $\{\beta \sqcap \alpha \mid \beta \in B\} \in C(\alpha)$;

4. $\emptyset \in C(tt)$;

5. $\emptyset \in C(f\!f)$;

6. $\{\langle x, y\rangle, \langle x', y\rangle\} \in C(\langle x \sqcup x', y\rangle)$;

7. $\{\langle x, y\rangle, \langle x, y'\rangle\} \in C(\langle x, y \sqcup y'\rangle)$;

These families form a *covering* as per [Joh82, Section II.2.11]. That is, they are stable under meets: $S \in C(\alpha)$ implies that $\{\sigma \sqcap \beta \mid \sigma \in S\} \in C(\alpha \sqcap \beta)$. Define a C-ideal to be a subset I of L which is closed downward and for which $S \subseteq I$ and $S \in C(\alpha)$ imply $\alpha \in I$. Consider $A \in J_0$. By definition it is downward closed and satisfies the closure conditions for C-ideals – the last two conditions being special cases of the closure under $\vee$ and $\wedge$. So J_0 is contained in the collection of C-ideals. Conversely, the first three conditions on C-ideals are essentially (bif$-$con), (bif$-$tot), and (bif$-\sqsubseteq$). To see that a C-ideal is also closed under (bif$-\leq$), recall that $\langle x, y\rangle \vee \langle x', y'\rangle = \langle x \sqcup x', y \sqcap y'\rangle$ and that a C-ideal is closed downward. Thus J_0 is the collection of C-ideals.

Following Johnstone's general construction, the set J_0 of C-ideals is guaranteed to form a frame with three useful properties: First, letting $A(\alpha)$ denote the principle C-ideal containing α, the resulting frame is generated by the sets $A(\alpha)$. Second, the map $\alpha \mapsto A(\alpha)$ preserves the finite meets of L. Third, J_0 is free with respect to transferring covers to joins. That is, if K is a frame and h is a meet semi-lattice morphism from L to K for which $S \in C(\alpha)$ implies $\bigsqcup h(S) = h(\alpha)$, then h extends uniquely to a frame homomorphism $H\colon J_0 \to K$ so that $h(\alpha) = H(A(\alpha))$ for all $\alpha \in L$.

Because L, as a meet semi-lattice, is a coproduct of L_+ and L_-, it is generated by the sub-semilattices L^+ and L^-, where L^+ consists of elements of the form $\alpha^+ := \alpha \sqcup f\!f$ and similarly for L^-. Define J_1 and J_2 to be the images of L^+ and L^- with respect to $A(-)$. The last five clauses in the definition of C ensure that J_1 and J_2 are subframes of J_0, and the map $x \mapsto A(x^+)$ from L_+ to J_1 is a frame homomorphism, and similarly for J_2. Thus (J_0, J_1, J_2) is a biframe. The first two clauses ensure that the map $\alpha \mapsto (A(\alpha^+), A(\alpha^-))$ is a d-frame homomorphism from $\mathfrak{L}$ to $\Delta(J_0, J_1, J_2)$.

Given a biframe (K_0, K_1, K_2) and a d-frame homomorphism h from $\mathcal{L}$ to $\Delta(K_0, K_1, K_2)$, the map h cuts down to two semilattice homomorphisms from L^+ and L^- into K_0. So these extend uniquely to a semilattice homomorphism h' from L to K_0. Specifically, $h'(\langle x, y\rangle) = h_+(x) \sqcap h_-(y)$.

Now it is fairly easy to check that h' transfers covers to joins. For example, $\langle x, y\rangle \in \mathsf{con}$ implies that $h_+(x) \sqcap h_-(y) = 0$ (in K_0), so for $\beta \sqsubseteq \langle x, y\rangle$, $h'(\beta) = 0$. The other conditions are just as routine, except perhaps the condition involving tot. Suppose $\alpha = \langle x, y\rangle \in \mathsf{tot}$. So, $h_+(x) \sqcup h_-(y) = 1$ in K_0. For any other $\beta = \langle x', y'\rangle$, we have

$$
\begin{aligned}
h'(\langle \alpha_+ \sqcap \beta_+, \beta_-\rangle) &= h_+(x) \sqcap h_+(x') \sqcap h_-(y') \\
h'(\langle \beta_+, \alpha_- \sqcap \beta_-\rangle) &= h_+(x') \sqcap h_-(y) \sqcap h_-(y').
\end{aligned}
$$

So the join of these is simply $h_+(x') \sqcup h_-(y') = h'(\beta)$.

Finally, because h' is a semilattice homomorphism from L to K_0 that transfers covers to joins, it extends uniquely to a frame homomorphism H from J_0 to K_0. This sends $x \in L_+$ to $H(A(\langle x, 1\rangle)) = h'(x, 1) = h_+(x) \in K_1$. Likewise, H sends elements of L_- to K_2, so it is a biframe homomorphism.

One can wonder whether the d-frame that one obtains from (J_0, J_1, J_2) (equivalently, $(L_0, \ulcorner L_+ \urcorner, \ulcorner L_- \urcorner)$) is isomorphic to the original $\mathcal{L} = (L; \mathit{tt}, \mathit{ff}; \mathsf{con}, \mathsf{tot})$. This need not be the case, even if $\mathcal{L}$ is reasonable:

Lemma 5.10 *Let $\mathcal{L}$ be (isomorphic to) $\Delta(\mathcal{B})$ for a biframe $\mathcal{B}$. Then $\mathcal{L}$ satisfies the cut rules*[14] *listed in Table 2.*

Proof. The cut rules in $\Delta(\mathcal{B})$ are simply instances of laws that hold in any frame. ∎

Counterexample 5.11 *The finitary cut rule* ($\mathsf{cut}_{\mathsf{tot}}$) *does not follow from reasonableness. Consider the d-frame in which both L_+ and L_- are the powersets $\mathcal{P}(\{0, 1\})$. We take $\langle x, y\rangle \in$* con *if and only if $|x| + |y| \leq 2$ and $\langle x, y\rangle \in$* tot

[14] The "logical" terminology for tt and ff and for the operations $\wedge$ and $\vee$ was justified by Proposition 3.2. Similar terminology here will be justified in Section 7.

$$
\begin{array}{ll}
(\mathrm{cut}_{\mathsf{tot}}) & \langle x \sqcup a, y\rangle \in \mathsf{tot}\ \&\ \langle x, y \sqcup b\rangle \in \mathsf{tot}\ \&\ \langle a, b\rangle \in \mathsf{con} \\
& \qquad \Longrightarrow \quad \langle x, y\rangle \in \mathsf{tot} \\
(\mathrm{cut}_{\mathsf{con}}) & \langle x \sqcap a, y\rangle \in \mathsf{con}\ \&\ \langle x, y \sqcap b\rangle \in \mathsf{con}\ \&\ \langle a, b\rangle \in \mathsf{tot} \\
& \qquad \Longrightarrow \quad \langle x, y\rangle \in \mathsf{con} \\
(\mathrm{CUT}_r) & \langle x, y \sqcup \bigsqcup_{i\in I} b_i\rangle \in \mathsf{tot}\ \&\ \forall i \in I.\ \langle x \sqcup a_i, y\rangle \in \mathsf{tot}\ \&\ \langle a_i, b_i\rangle \in \mathsf{con} \\
& \qquad \Longrightarrow \quad \langle x, y\rangle \in \mathsf{tot} \\
(\mathrm{CUT}_l) & \langle x \sqcup \bigsqcup_{i\in I} a_i, y\rangle \in \mathsf{tot}\ \&\ \forall i \in I.\ \langle x, y \sqcup b_i\rangle \in \mathsf{tot}\ \&\ \langle a_i, b_i\rangle \in \mathsf{con} \\
& \qquad \Longrightarrow \quad \langle x, y\rangle \in \mathsf{tot}
\end{array}
$$

Table 2: The finitary and infinitary cut rules.

if and only if $x = \{0, 1\}$ *or* $y = \{0, 1\}$. *The axioms* (con−∨) *and* (con−∧) *can be checked by a case analysis. To see that* (con−tot) *holds, note that if* $|x| + |y| \leq 2$ *and* $x = \{0, 1\}$, *then* $y = \emptyset$ *and similarly for their roles reversed. The other axioms are trivially satisfied. So this d-frame is reasonable.*

Now consider the pairs $(\{0\} \sqcup \{1\}, \{1\})$, $(\{0\}, \{0\} \sqcup \{1\})$ *and* $(\{1\}, \{0\})$. *These meet the pre-conditions of the cut rule, which would require* $(\{0\}, \{1\}) \in$ tot. *On the other hand, it is easy to check that the other finitary cut rule,* $(\mathrm{cut}_{\mathsf{con}})$, *holds in this d-frame.*

The order dual of this example shows that $(\mathrm{cut}_{\mathsf{con}})$ *also does not follow from reasonableness. Together, these show that the two finitary cut rules are independent of each other.*

Counterexample 5.12 *To show that the infinitary cut rules are independent and strictly stronger than* $(\mathrm{cut}_{\mathsf{tot}})$ *requires a more subtle counter-example. Consider the d-frame consisting of pairs* (U, V) *where* U *is an open in the relative Sorgenfrey topology on the interval* $[0, 1]$ *and* V *is an open in the standard topology on* $[0, 1]$. *So* U *is a disjoint union of sets of the forms* $[a, b)$, (a, b) *and* $(a, 1]$ *where* $0 \leq a < b \leq 1$ *Take* $(U, V) \in$ con *if and only if* $U \cap V = \emptyset$ *and take* $(U, V) \in$ tot *if and only if there exists a pair* $(A, B) \in$ con *so that* $U \cup A = [0, 1]$ *and* $V \cup B = [0, 1]$. *Clearly,* con *is exactly the same as in the spectrum of* $[0, 1]$ *with the two given topologies, so it satisfies the conditions for reasonableness. Also,* tot *is closed upward and* $(U, V) \in$ tot *implies* $U \cup V = [0, 1]$. *So* con *and*

tot *satisfy the axiom* ($\mathsf{con}-\mathsf{tot}$). *Finally,* tot *is closed under the logical operations: if* $(U,V),(U',V') \in \mathsf{tot}$*, then there exist pairs* (A,B) *and* (A',B') *in* con *as witnesses; thus* $(A \cap A', B \cup B')$ *witnesses that* $(U \cup U', V \cap V')$ *also belongs to* tot.

Furthermore, suppose $(X \cup A, Y),(X, Y \cup B) \in \mathsf{tot}$. *Let* $(C,D),(C',D') \in \mathsf{con}$ *be the needed witnesses. Then* $(A \cup C) \cap C'$ *and* $(B \cup C') \cap D'$ *are disjoint and witness that* $\langle X,Y\rangle \in \mathsf{tot}$. *So this d-frame satisfies the first of the finitary cut rules.*

Next, consider $X = [\frac{1}{2},1]$, $Y = [0,\frac{1}{2})$, $A_n = [0,\frac{n+1}{2n})$ *and* $B_n = (\frac{n-1}{2n},1]$ *for positive integers* n. *Clearly,* $X \cup \bigcup_n A_n = [0,1]$ *and for each* n*,* $Y \cup B_n = [0,1]$. *So these fulfil the conditions for the rule* (CUT_l). *But we claim that* $\langle X,Y\rangle \notin \mathsf{tot}$. *Consider any open* D *in the standard topology for which* $Y \cup D = [0,1]$. *Then* D *must cover some open set of the form* $(\frac{1}{2}-\epsilon,1]$. *No open* C *in the Sorgenfrey topology that is disjoint from* D *can cover the complement of* X.

This example does double duty by showing that the two infinitary cut rules are independent: because the standard topology is compact, (CUT_r) *reduces to* ($\mathrm{cut}_{\mathsf{tot}}$).

Spatiality is respected by the translation to biframes and back. For this discussion, we need to consider the dual adjoint functors between **biTop** and **biFrm**. We use subscripts d and b to distinguish these from the dual adjunction between **biTop** and **dFrm**. That is, $\Omega_b(X;\tau_+,\tau_-) = (\tau_+ \vee \tau_-, \tau_+, \tau_-)$ yields a biframe, whereas $\Omega_d(X;\tau_+,\tau_-)$ yields a d-frame, similarly for spec.

Checking definitions, one sees immediately that $\Omega_d = \Delta \circ \Omega_b$. An equally easy exercise in type checking for the adjunctions involved shows that spec_d is naturally isomorphic to $\mathsf{spec}_b \circ J$. This leads to the following corollary.

Corollary 5.13 *If a d-frame* $\mathcal{L}$ *is spatial, then it is derived from a biframe.*

This also provides an alternative proof of Lemma 4.6.

We have found no application for ($\mathrm{cut}_{\mathsf{con}}$) in anything that follows, but note that in the alternative construction of the free biframe over a d-frame, the condition ($\mathrm{bif}-\mathsf{tot}$) in the definition of the elements of J_0 is formally similar to ($\mathrm{cut}_{\mathsf{con}}$). In fact, if a reasonable d-frame satisfies ($\mathrm{cut}_{\mathsf{con}}$), then the set con itself is the least element of J_0. So for a reasonable d-frame $\mathcal{L}$, ($\mathrm{cut}_{\mathsf{con}}$) is a necessary and sufficient condition for con to be reflected in the translation from $\mathcal{L}$ to a biframe and back. It is tempting to claim "by symmetry" that ($\mathrm{cut}_{\mathsf{tot}}$) is likewise sufficient for the reflection of tot, but a closer look shows that the Scott closure of con is needed. Since tot is an upper set and can not be Scott closed (except in a trivial d-frame), the symmetry fails.

6 Regularity and compactness

Definition 2.13 can easily be adapted for d-frames:

Definition 6.1 *Let $(L; tt, ff; \mathsf{con}, \mathsf{tot})$ be a reasonable d-frame. For two elements $x, x' \in L_+$ we say that x' is* well-inside *x (and write $x' \lhd x$) if there is $y \in L_-$ such that $\langle x', y\rangle \in \mathsf{con}$ and $\langle x, y\rangle \in \mathsf{tot}$. To avoid lengthy verbiage, we will usually write $r_{x' \lhd x}$ for the "witnessing" element y (although it is not uniquely determined). On L_- the well-inside relation is defined analogously.*

A d-frame is called regular *if every $x \in L_+$ is the supremum of elements well-inside it, and similarly for elements of L_-.*

We note that the elements well-inside a fixed element x of a reasonable d-frame form a directed set; this follows from (con–$\vee$) and (tot–$\vee$). That they are all below x is a consequence of (con–tot). Finally, $1 \lhd 1$ is always true.

As an exercise in reasoning with the logical structure of a d-frame $\mathcal{L}$, consider the following definition: for elements α and β, say that α is *well-inside* β (and write $\alpha \lhd \beta$) if and only if $\alpha_+ \lhd \beta_+$ and $\alpha_- \lhd \beta_-$. Then one can easily show that the elements well-inside β form a directed set with β as an upper bound, and that L is regular if and only if every β is the supremum of elements well-inside β. Moreover, $\alpha \lhd \beta$ holds if and only if there is a γ so that $\alpha \wedge \gamma \in \mathsf{con}$, $\alpha \vee \gamma \in \mathsf{con}$, $\beta \wedge \gamma \in \mathsf{tot}$ and $\beta \vee \gamma \in \mathsf{tot}$.

Regularity allows us to derive a lot more information about d-points. This will come in handy later, so it is useful to formulate a couple of lemmas.

Lemma 6.2 *Let $\mathcal{L}$ be a reasonable d-frame and $x \in L_+$. Define*

$$\mathsf{P}(x) := \{b \in L_- \mid \exists a \not\sqsubseteq x.\ \langle a, b\rangle \in \mathsf{con}\} \ \textit{and} \ \mathsf{C}(x) := \{b \in L_- \mid \langle x, b\rangle \notin \mathsf{tot}\}$$

1. $\mathsf{P}(x) \subseteq \mathsf{C}(x)$;
2. *If $\mathcal{L}$ is regular then* $\bigsqcup \mathsf{P}(x) = \bigsqcup \mathsf{C}(x)$.

Proof. (1) is a direct consequence of ($\mathsf{con}-\mathsf{tot}$): if we have $\langle a, b\rangle \in \mathsf{con}$ and $\langle x, b\rangle \in \mathsf{tot}$ then $a \sqsubseteq x$ follows.

For (2) let $b' \lhd b \in \mathsf{C}(x)$. The witness $r_{b' \lhd b}$ cannot be below x as otherwise we could conclude $\langle x, b\rangle \in \mathsf{tot}$ from $\langle r_{b'\lhd b}, b\rangle \in \mathsf{tot}$. We also have $\langle r_{b'\lhd b}, b'\rangle \in \mathsf{con}$ and so find that $b' \in \mathsf{P}(x)$. By regularity, $\bigsqcup \mathsf{P}(x)$ is above b itself. It follows that $\bigsqcup \mathsf{P}(x) \sqsupseteq \bigsqcup \mathsf{C}(x)$, and by (1) the two suprema are in fact the same. ∎

Lemma 6.3 *Let $\mathcal{L} = (L; tt, ff; \mathsf{con}, \mathsf{tot})$ be a reasonable d-frame and $v_+ \in L_+$, $v_- \in L_-$. Consider the following statements:*

(i) $v_- = \max \mathsf{C}(v_+)$ *and* $v_+ = \max \mathsf{C}(v_-)$;

(ii) $(L_+ \setminus {\downarrow} v_+, L_- \setminus {\downarrow} v_-)$ *is a d-point;*

(iii) $\langle v_+, v_-\rangle \notin \mathsf{tot}$, $v_- \sqsupseteq \bigsqcup^{\uparrow} \mathsf{P}(v_+)$, *and* $v_+ \sqsupseteq \bigsqcup^{\uparrow} \mathsf{P}(v_-)$;

(iv) $\langle v_+, v_-\rangle \notin \mathsf{tot}$ *and* $v_- \sqsupseteq \bigsqcup^{\uparrow} \mathsf{P}(v_+)$;

(v) $\langle v_+, v_-\rangle$ *is a maximal element of* $(L_+ \times L_-) \setminus \mathsf{tot}$.

The following are true:

1. *(i) ⇒ (ii) ⇒ (iii) ⇒ (iv), and (i) ⇒ (v).*
2. *If $\mathcal{L}$ is regular then (iv) ⇒ (i).*
3. *If $\mathcal{L}$ satisfies the* ($\mathrm{cut}_{\mathsf{tot}}$) *rule then (v) ⇒ (ii).*

Proof. Part (1), (i) $\Rightarrow$ (ii): If $\langle x, y\rangle \in \mathsf{tot}$ then either $x \not\sqsubseteq v_+$ or $y \not\sqsubseteq v_-$ as otherwise we would have $\langle v_+, v_-\rangle \in \mathsf{tot}$ by (tot–$\uparrow$). If $\langle x, y\rangle \in \mathsf{con}$ and $x \not\sqsubseteq v_+$ then $y \in \mathsf{P}(v_+) \subseteq \mathsf{C}(v_+)$ by the previous lemma; hence $y \sqsubseteq v_-$. Thus we have shown that the pair $(L_+ \setminus {\downarrow} v_+, L_- \setminus {\downarrow} v_-)$ satisfies conditions (dp$_{\mathsf{tot}}$) and (dp$_{\mathsf{con}}$) for d-points and it remains to show that we have two completely prime filters. This will hold if v_+ and v_- are $\sqcap$-irreducible. So let $v_- = y \sqcap y'$; by (tot–$\vee$) either $\langle v_+, y\rangle \notin \mathsf{tot}$ or $\langle v_+, y'\rangle \notin \mathsf{tot}$, which means that either $y = v_-$ or $y' = v_-$.

(ii) $\Rightarrow$ (iii): If $x \not\sqsubseteq v_+$ and $\langle x, y\rangle \in \mathsf{con}$ then $y \sqsubseteq v_-$ by (dp$_{\mathsf{con}}$). So we have $v_- \sqsupseteq \bigsqcup \mathsf{P}(v_+)$. $\langle v_+, v_-\rangle \notin \mathsf{tot}$ follows from (dp$_{\mathsf{tot}}$). The set $\mathsf{P}(v_+)$ is directed because $L_+ \setminus {\downarrow} v_+$ is a filter and (con–$\wedge$) is assumed for reasonable d-frames.

(iii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (v) are trivial.

Part (2), (iv) $\Rightarrow$ (i): On the side of L_- we already have $v_- \sqsupseteq \bigsqcup \mathsf{C}(v_+)$ by the previous lemma. For the other side side, assume $x \not\sqsubseteq v_+$. By regularity there is $x' \lhd x$ with $x' \not\sqsubseteq v_+$. Because of $\langle x', r_{x' \lhd x}\rangle \in \mathsf{con}$ we have $r_{x' \lhd x} \sqsubseteq v_-$ by assumption, and then from $\langle x, r_{x' \lhd x}\rangle \in \mathsf{tot}$ we infer $\langle x, v_-\rangle \in \mathsf{tot}$ by (tot–$\uparrow$). It follows that $\mathsf{C}(v_-) \subseteq {\downarrow} v_+$. Together with $\langle v_+, v_-\rangle \notin \mathsf{tot}$ this is exactly (i).

Part (3), (v) $\Rightarrow$ (ii): As in (i) $\Rightarrow$ (ii) we get that v_+ and v_- are $\sqcap$-prime, and that condition (dp$_{\mathsf{tot}}$) is satisfied for $(L_+ \setminus {\downarrow} v_+, L_- \setminus {\downarrow} v_-)$. In order to show (dp$_{\mathsf{con}}$) assume $\langle x, y\rangle \in \mathsf{con}$. If we had $x \not\sqsubseteq v_+$ and $y \not\sqsubseteq v_-$ then by (the contrapositive of) the (cut$_{\mathsf{tot}}$) rule we would have either $\langle v_+, v_- \sqcup y\rangle \notin \mathsf{tot}$ or $\langle v_+ \sqcup x, v_-\rangle \notin \mathsf{tot}$, contradicting the maximality of $\langle v_+, v_-\rangle$. ∎

Proposition 6.4 *For $\mathcal{L}$ a regular d-frame, the bitopological space* $\mathsf{spec}\,\mathcal{L}$ *is order-separated (cf. Definition 4.9).*

Proof. The specialisation orders $\leq_+$ and $\leq_-$ on the spectrum are given by inclusion of first, respectively, second component of d-points. Let (F_+, F_-), (G_+, G_-) be two points with $F_+ \not\subseteq G_+$. There is some $x \in F_+ \setminus G_+$ and by regularity some $x' \lhd x$ also belongs to $F_+ \setminus G_+$. The corresponding witness $r_{x' \lhd x}$ must lie in $G_- \setminus F_-$ because of (dp$_{\mathsf{con}}$) and (dp$_{\mathsf{tot}}$). So $\leq_+$ and $\geq_-$ agree. Also, the same witnesses show that $(F_+, F_-) \in \Phi_+(x')$, $(G_+, G_-) \in \Phi_-(r_{x' \lhd x})$, and $\Phi_+(x') \cap \Phi_-(r_{x' \lhd x}) = \emptyset$. ∎

The frame-theoretic version of the Hofmann-Mislove Theorem, cf. [GHK$^+$03, Corollary V-5.4], states that a Scott-open filter in a frame is equal to the intersection of a compact collection of completely prime filters. Assuming regularity and one of the infinitary cut rules, we have an analogue for d-frames:

Lemma 6.5 *Let $\mathcal{L}$ be a regular d-frame that satisfies the infinitary rule* (CUT$_r$). *Assume that $\mathcal{S}_+$ is a Scott-open filter in L_+ and $\mathcal{U}_- = L_- \setminus {\downarrow} u_-$ is a completely prime upper set in L_- such that:*

$$\begin{array}{rlcl} (\mathrm{hm}_{\mathsf{con}}) & \alpha \in \mathsf{con} & \Longrightarrow & \alpha_+ \notin \mathcal{S}_+ \ \textit{or} \ \alpha_- \notin \mathcal{U}_- \\ (\mathrm{hm}_{\mathsf{tot}}) & \alpha \in \mathsf{tot} & \Longrightarrow & \alpha_+ \in \mathcal{S}_+ \ \textit{or} \ \alpha_- \in \mathcal{U}_- \end{array}$$

Then the following are true:

1. $u_- = \bigsqcup^{\uparrow}\{b \mid \exists a \in \mathcal{S}_+ .\, \langle a, b\rangle \in \mathsf{con}\}$, *that is, $\mathcal{U}_-$ is uniquely determined by $\mathcal{S}_+$.*
2. $\mathcal{S}_+ = \{a \mid \langle a, u_-\rangle \in \mathsf{tot}\}$, *that is, $\mathcal{S}_+$ is uniquely determined by $\mathcal{U}_-$.*
3. $x \sqsubseteq \mathcal{S}_+ \;\Leftrightarrow\; (x, u_-) \in \mathsf{con}$.
4. *For any point $(F_+, F_-) \in \mathsf{spec}\,\mathcal{L}$,* $\quad \mathcal{S}_+ \subseteq F_+ \;\Leftrightarrow\; F_- \subseteq \mathcal{U}_-$.
5. *$\mathcal{S}_+$ is the intersection of all F_+ where (F_+, F_-) is a point and $\mathcal{S}_+ \subseteq F_+$.*
6. *$\mathcal{U}_-$ is the union of all F_- where (F_+, F_-) is a point and $F_- \subseteq \mathcal{U}_-$.*
7. *The set $A := \{(F_+, F_-) \mid \mathcal{S}_+ \subseteq F_+\} = \{(F_+, F_-) \mid F_- \subseteq \mathcal{U}_-\}$ is $\mathcal{T}_+$-compact saturated and $\mathcal{T}_-$-closed in the bitopological space $(\mathsf{spec}\,\mathcal{L}; \mathcal{T}_+, \mathcal{T}_-)$.*

Proof. (1) The element u_- can not be any smaller because of (hm$_{\mathsf{con}}$). For the converse assume $y \lhd u_-$. The corresponding witness $r_{y \lhd u_-}$ belongs to $\mathcal{S}_+$ by (hm$_{\mathsf{tot}}$) and so $y \in \{b \mid \exists a \in \mathcal{S}_+ .\, \langle a, b\rangle \in \mathsf{con}\}$. By regularity, then, $u_- \sqsubseteq \bigsqcup^{\uparrow}\{b \mid \exists a \in \mathcal{S}_+ .\, \langle a, b\rangle \in \mathsf{con}\}$.

(2) Because of (hm$_{\mathsf{tot}}$) it is clear that $\mathcal{S}_+$ must contain all $a \in L_+$ with $\langle a, u_-\rangle \in \mathsf{tot}$. For the converse let $x \in \mathcal{S}_+$. By regularity and Scott-openness of $\mathcal{S}_+$ there

is $x' \lhd x$ still in $\mathcal{S}_+$. The corresponding witness $r_{x' \lhd x}$ must be below u_- because of (hm$_{\mathsf{con}}$), but then $\langle x, u_- \rangle \in \mathsf{tot}$ by ($\mathsf{tot}{-}\uparrow$).

(3) Assume $x \sqsubseteq a$ for all $a \in \mathcal{S}_+$. By ($\mathsf{con}{-}\downarrow$) we have $(x, b) \in \mathsf{con}$ for all $b \in \{b \mid \exists a \in \mathcal{S}_+ . (a, b) \in \mathsf{con}\}$, so $(x, u_-) \in \mathsf{con}$ by ($\mathsf{con}{-}\bigsqcup^{\uparrow}$) and part (1). For the converse, remember that $(a, u_-) \in \mathsf{tot}$ for all $a \in \mathcal{S}_+$ by (2), so $(x, u_-) \in \mathsf{con}$ implies $x \sqsubseteq a$ by ($\mathsf{con}{-}\mathsf{tot}$).

(4) Let $v_+ = \bigsqcup(L_+ \setminus F_+)$. From $\mathcal{S}_+ \subseteq F_+$ and (hm$_{\mathsf{con}}$) we get $\mathsf{P}(v_+) \supseteq (L_- \setminus \mathcal{U}_-)$, so $v_- = \bigsqcup \mathsf{P}(v_+) \sqsupseteq u_-$ and hence $F_- \subseteq \mathcal{U}_-$.

Starting with the right hand side, $F_- \subseteq \mathcal{U}_-$, we let $v_- = \bigsqcup(L_- \setminus F_-)$. From (hm$_{\mathsf{con}}$) we get $\mathsf{P}(v_-) \cap \mathcal{S}_+ = \emptyset$. So $v_+ = \bigsqcup^{\uparrow} \mathsf{P}(v_-) \notin \mathcal{S}_+$ and hence $\mathcal{S}_+ \subseteq F_+$.

(5) Assume that $x \notin \mathcal{S}_+$. Because $\mathcal{S}_+$ is assumed to be Scott-open, we can apply Zorn's Lemma to obtain a maximal element v_+ above x that does not belong to $\mathcal{S}_+$. The set $F_+ := L_+ \setminus {\downarrow} v_+$ is a completely prime filter that separates x from $\mathcal{S}_+$, and it remains to show that it is the first component of a d-point. According to Lemma 6.3 the right candidate is $F_- = L_- \setminus {\downarrow} v_-$ where $v_- = \bigsqcup^{\uparrow} \mathsf{P}(v_+) = \bigsqcup \mathsf{C}(v_+)$. Note that $u_- \sqsubseteq v_-$ as $u_- \in \mathsf{C}(v_+)$ by (hm$_{\mathsf{tot}}$). Using Lemma 6.3(iv) we only need to show that $\langle v_+, v_- \rangle \notin \mathsf{tot}$. For this we employ (CUT$_r$): for all $\langle a, b \rangle \in \mathsf{con}$ with $a \in F_+$ we have $\langle v_+ \sqcup a, v_- \rangle \in \mathsf{tot}$ by (2); if it was the case that $\langle v_+, v_- \rangle = \langle v_+, u_- \sqcup \bigsqcup^{\uparrow} \mathsf{P}(v_+) \rangle \in \mathsf{tot}$, then $\langle v_+, u_- \rangle \in \mathsf{tot}$ would follow, contradicting (hm$_{\mathsf{tot}}$).

For part (6) let $y \in \mathcal{U}_-$. By regularity and the assumption that $\mathcal{U}_-$ is completely prime, some $y' \lhd y$ also belongs to $\mathcal{U}_-$. The witness $r_{y' \lhd y}$ is not in $\mathcal{S}_+$ because of $\langle r_{y' \lhd y}, y' \rangle \in \mathsf{con}$ and assumption (hm$_{\mathsf{con}}$). By part (5) there is a point (F_+, F_-) that separates $r_{y' \lhd y}$ from $\mathcal{S}_+$. By (4) we have that $F_- \subseteq \mathcal{U}_-$ and because of $\langle r_{y' \lhd y}, y \rangle \in \mathsf{tot}$ it must also be the case that $y \in F_-$.

Finally, consider the last claim; the two descriptions of A are equal because of (4). Any $\mathcal{T}_+$-open neighbourhood of A has the form $\Phi_+(x)$ with $x \in \mathcal{S}_+$ by (5). It follows that A is $\mathcal{T}_+$-compact. Only the maximality of u_- in $L_- \setminus \mathcal{U}_-$ is required to see that A is the complement of $\Phi_-(u_-)$. ∎

Theorem 6.6 *For a regular d-frame $\mathcal{L} = (L; tt, ff; \mathsf{con}, \mathsf{tot})$ that satisfies the infinitary cut rule* (CUT$_r$) *there is a one-to-one correspondence between*

1. *maps q from L to the four-element d-frame $\mathbf{2.2}$ which preserve tt, $\bigsqcup^{\uparrow}$,* con, tot, *and the logical operation $\wedge$, and*

2. *subsets A of* spec $\mathcal{L}$ *which are compact saturated in the positive and closed in the negative topology.*

Proof. Given a map q as described in part (1), consider $\mathcal{S}_+ = q^{-1}(\mathit{tt}) \cap L_+$ and $\mathcal{U}_- = q^{-1}(\mathit{ff}) \cap L_-$. It is straightforward to show that the pair $(\mathcal{S}_+, \mathcal{U}_-)$ satisfies the assumptions of Lemma 6.5. The translation in the opposite direction is equally easy. ∎

A few comments on this result are in order: Given a *consistent predicate* φ, that is, $\varphi \in$ con, then the value of q at φ can only be tt, ff, or $\perp$. The first outcome indicates that *all* elements of A satisfy φ, the second that *some* element of A fails φ, and the last that neither holds (which is a possibility because a consistent predicate does not need to be Boolean). This means that q acts like a *universal quantifier* for partial predicates.

Generally, one would expect a universal quantifier to preserve tt but not necessarily ff, because A could be the empty set. Also, one would expect it to preserve conjunction ($\wedge$) but not disjunction ($\vee$), and certainly one would not want it to be inconsistent (returning $\top$) for a consistent predicate, or to be undecided (returning $\perp$) for a total predicate, that is, one expects it to preserve con and tot.

The preservation of $\bigsqcup^{\uparrow}$ can be seen as a *computability* condition on the universal quantifier: If a (partial) predicate φ is the directed supremum of (partial) predicates φ_i, and if the universal quantifier applied to φ returns a definite answer, that is, either tt or ff, then computability requires the same answer to be obtained from an approximant φ_i already.

All in all, then, Theorem 6.6 is a generalisation of Martín Escardó's theory of computable quantification on topological spaces, [Esc04], to a logic in which predicates are allowed to have value ff as well as tt.

Let us now turn to a notion of compactness for d-frames.

Definition 6.7 *A d-frame is called* compact *if whenever $\langle \bigsqcup X, \bigsqcup Y\rangle \in$* tot, *then $\langle \bigsqcup X', \bigsqcup Y'\rangle \in$* tot *for some finite subsets $X' \subseteq X$ and $Y' \subseteq Y$.*

Lemma 6.8 *A reasonable d-frame $\mathcal{L}$ is compact if and only if the set* tot *is Scott-open in L.*

Definition 6.9 *A distributive continuous lattice is called* stably continuous *if its way-below relation is* multiplicative*, that is,* $1 \ll 1$ *and* $x \ll y, y' \Rightarrow x \ll y \sqcap y'$.[15]

Lemma 6.10 *For $\mathcal{L}$ a compact regular d-frame, the well-inside relation on L_+ (respectively, L_-) is the same as the way-below relation. Furthermore, L_+ and L_- are stably continuous lattices.*

Proof. Let $x' \lhd x$; we show that $x' \ll x$ also holds. Indeed, assume $x \sqsubseteq \bigsqcup^{\uparrow} A$; then $\langle \bigsqcup^{\uparrow} A, r_{x' \lhd x} \rangle \in \mathsf{tot}$ by (tot–$\uparrow$). Compactness implies that $\langle a_0, r_{x' \lhd x} \rangle \in \mathsf{tot}$ for some $a_0 \in A$, and since $\langle x', r_{x' \lhd x} \rangle \in \mathsf{con}$, $x' \sqsubseteq a_0$ follows from (con–tot).

On the other hand, $x' \ll x$ implies $x' \sqsubseteq a$ for some $a \lhd x$ because of regularity, so $x' \lhd x$ as well.

Closure of the well-inside relation against infima on the right follows from (con–$\wedge$) and (tot–$\wedge$); $1 \lhd 1$ holds in any d-frame. ∎

Theorem 6.11 *Compact regular d-frames are spatial.*

Proof. We check the conditions (s_+), (s_-), ($\mathrm{s}_{\mathsf{con}}$), and ($\mathrm{s}_{\mathsf{tot}}$) of Theorem 5.1. Let $x' \not\sqsubseteq x \in L_+$. Since L_+ is a continuous lattice, there is a Scott-open filter $\mathcal{S}_+$ that contains x' but not x. Let v_+ be maximal above x outside $\mathcal{S}_+$, and set $v_- = \bigsqcup^{\uparrow} \mathsf{P}(v_+)$. As the complement of tot is Scott-closed, and $\{v_+\} \times \mathsf{P}(v_+)$ is a directed subset of it, we have $\langle v_+, v_- \rangle \notin \mathsf{tot}$. By Lemma 6.3(iv) we have a d-point that separates x' from x.

For the condition ($\mathrm{s}_{\mathsf{tot}}$) assume that $\langle x, y \rangle \notin \mathsf{tot}$. Pick a maximal element v_+ above x such that $\langle v_+, y \rangle \notin \mathsf{tot}$. As in the paragraph above, the element $v_- = \bigsqcup^{\uparrow} \mathsf{P}(v_+)$ partners up with v_+ to define a d-point. By construction, $x \sqsubseteq v_+$ and $y \sqsubseteq v_-$.

For ($\mathrm{s}_{\mathsf{con}}$) let $\langle x, y \rangle \notin \mathsf{con}$. By regularity and the fact that con is Scott-closed, (con–$\bigsqcup^{\uparrow}$), we must have x' well-inside x with $\langle x', y \rangle \notin \mathsf{con}$. The witness $r_{x' \lhd x}$

[15] These structures were called *arithmetic lattices* in [JS96].

can't be above y as otherwise $\langle x', y\rangle \in \mathsf{con}$ would follow by (con–$\downarrow$). So $y \not\sqsubseteq r_{x' \triangleleft x}$ and we can apply the first part of the proof to obtain a d-point (F_+, F_-) with $y \in F_-$, $r_{x'\triangleleft x} \notin F_-$. The latter fact implies that x must belong to F_+ because $\langle x, r_{x'\triangleleft x}\rangle \in \mathsf{tot}$. ∎

Corollary 6.12 *The spectra of compact regular d-frames are exactly the compact regular bitopological spaces.*

The next three results are an immediate consequence of spatiality but it is of some interest that they can in fact be derived without using the Axiom of Choice.

Proposition 6.13 *Compact regular d-frames satisfy the cut rules* (cut$_{\mathsf{tot}}$)*,* (CUT$_r$)*, and* (CUT$_l$)*.*

Proof. First of all, the two infinitary cut rules reduce to (cut$_{\mathsf{tot}}$) because of compactness, so this is all that we need to show. Assume, then, that $\langle x \sqcup a, y\rangle \in \mathsf{tot}$, $\langle x, y \sqcup b\rangle \in \mathsf{tot}$, and $\langle a, b\rangle \in \mathsf{con}$. By regularity and compactness there are $x' \triangleleft x$ and $y' \triangleleft y$ such that $\langle x' \sqcup a, y\rangle \in \mathsf{tot}$ and $\langle x, y' \sqcup b\rangle \in \mathsf{tot}$ are still valid. A semi-formal derivation is best suited for the somewhat involved argument that follows:

1	$\langle x' \sqcup a, y\rangle \in \mathsf{tot}$	assumption
2	$\langle x, y' \sqcup b\rangle \in \mathsf{tot}$	assumption
3	$\langle a, b\rangle \in \mathsf{con}$	assumption
4	$\langle x, r_-\rangle \in \mathsf{tot}$	regularity, $r_- = r_{x' \lhd x}$
5	$\langle x', r_-\rangle \in \mathsf{con}$	regularity
6	$\langle r_+, y\rangle \in \mathsf{tot}$	regularity, $r_+ = r_{y' \lhd y}$
7	$\langle r_+, y'\rangle \in \mathsf{con}$	regularity
8	$\langle x, r_- \sqcup y'\rangle \in \mathsf{tot}$	from (4) by ($\mathsf{tot}$$-$$\uparrow$)
9	$\langle x, y' \sqcup (b \sqcap r_-)\rangle \in \mathsf{tot}$	from (8) and (2) by ($\mathsf{tot}$$-$$\vee$)
10	$\langle (x' \sqcup a) \sqcap r_+, y\rangle \in \mathsf{tot}$	from (1) and (6) by ($\mathsf{tot}$$-$$\wedge$)
11	$\langle a \sqcap r_+, y' \sqcup b\rangle \in \mathsf{con}$	from (3) and (7) by ($\mathsf{con}$$-$$\wedge$)
12	$\langle x' \sqcap r_+, y' \sqcup r_-\rangle \in \mathsf{con}$	from (5) and (7) by ($\mathsf{con}$$-$$\wedge$)
13	$\langle (a \sqcup x'\rangle \sqcap r_+, y' \sqcup (b \sqcap r_-)) \in \mathsf{con}$	from (11) and (12) by ($\mathsf{con}$$-$$\vee$)
14	$\langle a \sqcup x'\rangle \sqcap r_+ \sqsubseteq x$	from (13) and (9) by ($\mathsf{con}$$-$$\mathsf{tot}$)
15	$\langle x, y\rangle \in \mathsf{tot}$	from (14) and (10) by ($\mathsf{tot}$$-$$\uparrow$)

∎

Lemma 6.14 *Let $\mathcal{L}$ be a compact regular d-frame.*

1. *To every Scott-open filter $\mathcal{S}_+$ in L_+ there exists a unique completely prime upper set $\mathcal{U}_-$ in L_- such that the conditions* ($\mathrm{hm}_{\mathsf{con}}$) *and* ($\mathrm{hm}_{\mathsf{tot}}$) *of Lemma 6.5 are satisfied.*

2. *To every completely prime upper set $\mathcal{U}_-$ in L_- there exists a unique Scott-open filter $\mathcal{S}_+$ in L_+ such that* ($\mathrm{hm}_{\mathsf{con}}$) *and* ($\mathrm{hm}_{\mathsf{tot}}$) *are satisfied.*

3. *The translations in (1) and (2) are inverses of each other.*

Proof. Lemma 6.5(1) states that there is a unique candidate for $\mathcal{U}_-$, namely $L_- \setminus {\downarrow} u_-$, where $u_- = \bigsqcup^{\uparrow}\{b \in L_- \mid \exists a \in \mathcal{S}_+.\ \langle a, b\rangle \in \mathsf{con}\}$. Condition ($\mathrm{hm}_{\mathsf{con}}$) is satisfied by construction, so let us look at ($\mathrm{hm}_{\mathsf{tot}}$). If $\langle x, y\rangle \in \mathsf{tot}$ with $y \sqsubseteq u_-$, then let $y' \lhd y$ with $\langle x, y'\rangle \in \mathsf{tot}$, too. The existence of y' is guaranteed by regularity and compactness. By Lemma 6.10, we have $y' \ll y$, and so $y' \sqsubseteq b$ for some $b \in \{b \in L_- \mid \exists a \in \mathcal{S}_+.\ \langle a, b\rangle \in \mathsf{con}\}$. Now we know that $\langle a, b\rangle \in \mathsf{con}$ for some $a \in \mathcal{S}_+$, and since $a \sqsubseteq x$ by (con−tot), we obtain $x \in \mathcal{S}_+$ as required.

For the second statement let $u_- = \bigsqcup(L_- \setminus \mathcal{U}_-)$ and define $\mathcal{S}_+ = \{a \in L_+ \mid \langle a, u_-\rangle \in \mathsf{tot}\}$ as prescribed by 6.5(2). This is a filter by (tot−$\wedge$); it is Scott-open because of compactness. If $\langle a, y\rangle \in \mathsf{con}$ for some $a \in \mathcal{S}_+$, then $y \sqsubseteq u_-$ by (con−tot).

Part (3) is an immediate consequence of uniqueness. ∎

The preceding lemma allows us to conclude a frame-theoretic analogue of Theorem 2.16 characterising stably compact spaces in bitopological terms:

Theorem 6.15 *A d-frame $\mathcal{L} = (L; tt, ff; \mathsf{con}, \mathsf{tot})$ is compact regular if and only if the following conditions are satisfied:*

(i) L_+ is a stably continuous lattice;

(ii) L_- is isomorphic to the Lawson dual *of L_+, that is, the set of Scott-open filters of L_+ ordered by inclusion;*

(iii) $\langle x, y\rangle \in \mathsf{con}$ if and only if $x \sqsubseteq a$ for all $a \in F_y$, where F_y is the Scott-open filter associated with y according to (ii);

(iv) $\langle x, y\rangle \in \mathsf{tot}$ if and only if $x \in F_y$.

Furthermore, d-frame homomorphisms from $\mathcal{L}$ to another compact regular d-frame $\mathcal{L}'$ are in one-to-one correspondence to frame maps from L_+ to L'_+ which preserve the way-below relation.

Proof. "Only if:" We showed in Lemma 6.10 that L_+ is a stably continuous lattice. From the preceding lemma we get the order-reversing bijections $y \leftrightarrow \mathcal{U}_-(y) := L_+ \setminus \downarrow y$ and $\mathcal{U}_-(y) \leftrightarrow \mathcal{S}_+(y)$ which together establish an order-preserving bijection between L_+ and its Lawson dual. If $\langle x, y\rangle \in \mathsf{con}$ then by studying the construction in the proof of 6.14(2) one sees that for all $a \in \mathcal{S}_+(y)$, $\langle a, y\rangle \in \mathsf{tot}$, so $x \sqsubseteq a$ by (con−tot). Conversely, $x \sqsubseteq a$ for all $a \in \mathcal{S}_+(y)$ implies $\langle x, b\rangle \in \mathsf{con}$ for all b belonging to $\{b \in L_- \mid \exists a \in \mathcal{S}_+.\ \langle a, b\rangle \in \mathsf{con}\}$, by (con−$\downarrow$).

This in turn implies $\langle x, y\rangle \in \mathsf{con}$ by (con–$\bigsqcup^{\uparrow}$). The equivalence in (iv) is true by construction of $\mathcal{S}_+(y)$.

"If:" In order to show regularity, assume that $x \not\sqsubseteq x'$ in L_+. By continuity, there is $a \ll x$ with $a \not\sqsubseteq x'$, so there is a Scott-open filter F such that $x \in F$ and $a \sqsubseteq F$, in other words, $\langle x, y\rangle \in \mathsf{tot}$ and $\langle a, y\rangle \in \mathsf{con}$ for the element $y \in L_-$ associated with F. Compactness is trivial as the filter associated to $\bigsqcup^{\uparrow} y_j$ is the directed union of the filters associated with each y_j.

Regarding morphisms, we have that the two components of d-frame homomorphisms preserve $\lhd$, so by 6.10 they preserve $\ll$. For the converse, let h_+ be a $\ll$-preserving frame homomorphism from L_+ to L'_+, the first components of two compact regular d-frames $\mathcal{L}$ and $\mathcal{L}'$. We set $h_-(y) := {\uparrow} h_+(F_y)$. The result is again a Scott-open filter by the preservation of $\ll$. The frame operations of L_- viewed as the collection of Scott-open filters of L_+ are: intersection for $\sqcap$, directed union for $\bigsqcup^{\uparrow}$, and $\{a \sqcap a' \mid a \in F, a' \in F'\}$ for $\sqcup$ (multiplicativity of $\ll$ is used in this formula). With this knowledge, their preservation by h_- is easily checked. ∎

7 Partial frames

We have alluded several times to the fact that con corresponds to the *consistent predicates* on the spectrum of the d-frame. From a logical point of view, these are preferable to general elements of the d-frame as they do not give conflicting answers. In this section we will demonstrate that it is possible to replace a reasonable d-frame by its set of consistent predicates without any loss of expressivity.

To emphasise that we are dealing with a new structure in its own right, we will denote the set con by $P_{\mathcal{L}}$ (and the resulting structure by $\mathcal{P}_{\mathcal{L}}$).

Clearly, the totality relation tot has been just as important as con so far, and we need a way to represent it within $P_{\mathcal{L}}$. This is the rationale behind the following definition. For $\varphi, \psi \in P_{\mathcal{L}}$ we set

$$\varphi \prec \psi \;:\Leftrightarrow\; (\varphi_- \sqcup \psi_+) \in \mathsf{tot}$$

and say that φ *strongly implies* ψ. In the context of pairs, $\langle v, w\rangle \prec \langle x, y\rangle$ holds if

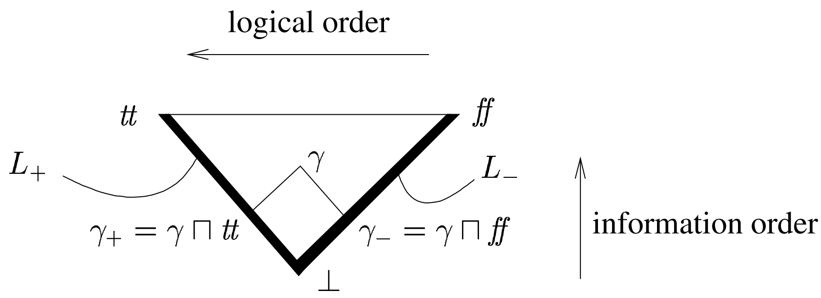


Figure 3: The structure of the set of partial predicates associated with a d-frame.

and only if $\langle x, w\rangle \in \mathsf{tot}$. Altogether, there is quite a bit of structure on $P_{\mathcal{L}}$:

- Binary infima ($\sqcap$) are inherited from $\mathcal{L}$; they stay in $P_{\mathcal{L}}$ because of (con–$\downarrow$).
- By (con–$\bigsqcup^{\uparrow}$), $P_{\mathcal{L}}$ is closed under directed sups $\bigsqcup^{\uparrow}$.
- Because of (con–tt), (con–ff), (con–$\wedge$) and (con–$\vee$), $P_{\mathcal{L}}$ contains the two constants and is closed under $\wedge$ and $\vee$.
- We also have the constant $\perp \in P_{\mathcal{L}}$.
- $\sqcap$ and $\bigsqcup^{\uparrow}$ induce the *information order* $\sqsubseteq$; $\wedge$ and $\vee$ induce the *logical order* $\leq$. The *strong implication* $\prec$ was introduced above.

Figure 3 presents a graphical representation of the set of consistent predicates as a sub-structure of a d-frame.

Proposition 7.1 *Let* $\mathcal{L} = (L; tt, ff; \mathsf{con}, \mathsf{tot})$ *be a reasonable d-frame.*

1. $(P_{\mathcal{L}}; \sqsubseteq)$ *has binary meets, directed suprema, and a least element (denoted by* $\sqcap$, $\bigsqcup^{\uparrow}$, *and* $\perp$, *respectively). Meets distribute over directed suprema.*
2. $(P_{\mathcal{L}}; \wedge, \vee, tt, ff)$ *is a distributive lattice.*

3. *The relation $\prec$ is contained in $\leq$; it is transitive and interacts with the logical operations as follows:*[16]

$$\begin{array}{llcl}
(\mathit{ff}-\prec) & \mathit{ff} \prec \gamma & & \\
(\prec-\mathit{tt}) & \gamma \prec \mathit{tt} & & \\
(\vee-\prec) & \gamma \prec \delta,\ \gamma' \prec \delta & \Longleftrightarrow & \gamma \vee \gamma' \prec \delta \\
(\prec-\wedge) & \gamma \prec \delta,\ \gamma \prec \delta' & \Longleftrightarrow & \gamma \prec \delta \wedge \delta'
\end{array}$$

4. *The following mixed laws hold:*

$$\begin{array}{ll}
(\sqcap-=) & \gamma \sqcap \delta = (\gamma \wedge \bot) \vee (\delta \wedge \bot) \vee (\gamma \wedge \delta) \\
(\prec-\sqsubseteq) & \gamma \sqsubseteq \gamma',\ \delta \sqsubseteq \delta',\ \gamma \prec \delta \quad \Longrightarrow \quad \gamma' \prec \delta'
\end{array}$$

Proof. (1) We have a dcpo by ($\mathsf{con}-\bigsqcup^{\uparrow}$). The element $\bot$ is a member of $P_{\mathfrak{L}}$ because $\bot \sqsubseteq \mathit{tt} \in \mathsf{con}$. Meet-continuity is inherited from the frame L.

(2) This is immediate from the "logical" axioms.

Regarding (3), let $\gamma \prec \delta$; so $\langle \delta_+, \gamma_- \rangle = \gamma_- \sqcup \delta_+ \in \mathsf{tot}$ by definition. Since $\gamma \in \mathsf{con}$, $\gamma_+ \sqsubseteq \delta_+$ follows from ($\mathsf{con}-\mathsf{tot}$). In the same way, one concludes $\gamma_- \sqsupseteq \delta_-$ from $\delta \in \mathsf{con}$. Transitivity uses a similar argument: if $\gamma \prec \delta \prec \epsilon$ then $\delta_+ \sqsubseteq \epsilon_+$ and $\gamma_- \sqcup \delta_+ \in \mathsf{tot}$. So $\gamma_- \sqcup \epsilon_+ \in \mathsf{tot}$ by ($\mathsf{tot}-\uparrow$).

Since $\mathit{ff} = \mathsf{tot}$ in any reasonable d-frame, we have $\langle \gamma_+, 1 \rangle \in \mathsf{tot}$ by ($\mathsf{tot}-\uparrow$). This shows ($\mathit{ff}-\prec$). Regarding ($\vee-\prec$), assume $\gamma \prec \delta$ and $\gamma' \prec \delta$. This means $\gamma_- \sqcup \delta_+ \in \mathsf{tot}$ and $\gamma'_- \sqcup \delta_+ \in \mathsf{tot}$ by definition. By ($\mathsf{tot}-\vee$) and distributivity of $\sqcup$ and $\sqcap$ over $\vee$, we have $(\gamma \vee \gamma')_- \sqcup \delta_+ \in \mathsf{tot}$ as well. The reverse direction is an application of ($\mathsf{tot}-\uparrow$). The proofs of the other rules are similar.

(4) The first law is precisely the second part of Proposition 3.2. The other law follows from ($\mathsf{tot}-\uparrow$). ∎

For two partial predicates φ and φ' to be related in the information order, that is, $\varphi \sqsubseteq \varphi'$, means that φ' will always give the same answer as φ, whenever the latter gives an answer at all, and may answer where φ doesn't. Rule ($\prec-\sqsubseteq$) can

[16] We re-use the labels of Definition 2.18 because the axioms are formally the same, but note that in the present situation we neither have ($\prec-\vee$) nor ($\wedge-\prec$). On the other hand, here we have $\prec\ \subseteq\ \leq$ which is not a requirement for strong proximity lattices. The exact relationship between the two is explored in Section 8.1.

therefore be read as saying that with $\varphi \prec \psi$, every (information order) refinement of φ implies every refinement of ψ. Hence the terminology "strong implication."

Definition 7.2 *A structure $\mathcal{P} = (P; \sqcap, \bigsqcup^{\uparrow}, \bot; \wedge, \vee, tt, ff; \prec)$ which satisfies the properties of Proposition 7.1 is called a* partial frame. *Morphisms between partial frames preserve all operations and the strong implication relation. The resulting category is denoted by* **pFrm**.

Before we embark on the proof that every partial frame arises as the structure $\mathcal{P}_{\mathcal{L}}$ of a reasonable d-frame, we note some consequences of the axioms.

Lemma 7.3 *Let $\mathcal{P}$ be a partial frame.*

1. *Whenever $\gamma \prec \delta$, then also $\gamma \wedge \gamma' \prec \delta \vee \delta'$ for arbitrary γ', δ'.*
2. *Any one of the operations $\sqcap$, $\wedge$, and $\vee$ distributes over any other.*
3. *The operations $\wedge$ and $\vee$ are $\sqsubseteq$-monotone in each argument, and $\sqcap$ is $\leq$-monotone.*
4. *If $\varphi, \psi \sqsubseteq \gamma$ then $(\varphi \vee \gamma) \wedge (\psi \vee \gamma) \wedge (\varphi \vee \psi)$ is the least upper bound of $\{\varphi, \psi\}$ in the information order, denoted by $\sup_{\sqsubseteq} \{\varphi, \psi\}$. Furthermore, for any δ, $\delta \sqcap \sup_{\sqsubseteq} \{\varphi, \psi\} = \sup_{\sqsubseteq} \{\delta \sqcap \varphi, \delta \sqcap \psi\}$.*
5. *$(P; \sqsubseteq)$ is bounded-complete, that is, every bounded subset has a supremum. Furthermore, for every $\gamma \in P$, the set $\downarrow_{\sqsubseteq}\gamma = \{\delta \mid \delta \sqsubseteq \gamma\}$ is a frame.*
6. *In the frame $\downarrow_{\sqsubseteq} tt$, logical order and information order coincide; in particular, the $\wedge$ coincides with $\sqcap$, and $\vee$ with $\sup_{\sqsubseteq}$. In the frame $\downarrow_{\sqsubseteq} ff$, the logical order is the opposite of the information order; in particular, $\wedge$ coincides with $\sup_{\sqsubseteq}$, and $\vee$ with $\sqcap$.*
7. *The logical constants tt and ff satisfy the laws:*

$$
\begin{aligned}
\gamma \sqcap tt &= \gamma \vee \bot \\
\gamma \sqcap ff &= \gamma \wedge \bot \\
\bot &= tt \sqcap ff \\
\gamma &= \sup_{\sqsubseteq} \{\gamma \sqcap tt,\ \gamma \sqcap ff\}
\end{aligned}
$$

8. *Information order and strong implication are related by:*

$$\begin{array}{rcl} \gamma \prec \delta,\ \gamma' \prec \delta & \Longrightarrow & \gamma \sqcap \gamma' \prec \delta \\ \gamma \prec \delta,\ \gamma \prec \delta' & \Longrightarrow & \gamma \prec \delta \sqcap \delta' \\ \gamma \prec \delta & \Longleftrightarrow & \gamma \sqcap \mathit{ff} \prec \delta \sqcap \mathit{tt} \end{array}$$

9. $\gamma \leq \delta$ *if and only if* $\gamma \sqcap \mathit{tt} \sqsubseteq \delta \sqcap \mathit{tt}$ *and* $\gamma \sqcap \mathit{ff} \sqsupseteq \delta \sqcap \mathit{ff}$.

10. *The operations* $\wedge$ *and* $\vee$ *are Scott-continuous in each argument.*

Proof. The first statement is a consequence of ($\vee{-}\prec$) and ($\prec{-}\wedge$), because $\gamma = \gamma \vee (\gamma \wedge \gamma')$ and $\delta = \delta \wedge (\delta \vee \delta')$.

The second statement is part of Proposition 3.2 but in any case only requires some straightforward computations in the distributive lattice $(P; \wedge, \vee)$.

The third statement holds because of (2), e.g., if $\gamma \sqsubseteq \gamma'$ then $\gamma \sqcap \gamma' = \gamma$ and $\gamma \wedge \delta = (\gamma \sqcap \gamma') \wedge \delta = (\gamma \wedge \delta) \sqcap (\gamma' \wedge \delta)$, hence $\gamma \wedge \delta \sqsubseteq \gamma' \wedge \delta$.

(4) To see that the given expression is an upper bound, compute

$$\begin{array}{l} \varphi \sqcap ((\varphi \vee \gamma) \wedge (\psi \vee \gamma) \wedge (\varphi \vee \psi)) = \\ ((\varphi \sqcap \varphi) \vee (\varphi \sqcap \gamma)) \wedge ((\varphi \sqcap \psi) \vee (\varphi \sqcap \gamma)) \wedge ((\varphi \sqcap \varphi) \vee (\varphi \sqcap \psi)) = \\ (\varphi \vee \varphi) \wedge ((\varphi \sqcap \psi) \vee \varphi) \wedge (\varphi \vee (\varphi \sqcap \psi)) = \varphi \wedge ((\varphi \sqcap \psi) \vee \varphi) = \varphi \end{array}$$

Next, if δ is an upper bound for φ and ψ, then $(\varphi \vee \gamma) \wedge (\psi \vee \gamma) \wedge (\varphi \vee \psi) \sqsubseteq (\delta \vee \gamma) \wedge (\delta \vee \gamma) \wedge (\delta \vee \delta) = (\delta \wedge \gamma) \vee \delta = \delta$. Finally, let δ be an arbitrary element of P; we get

$$\begin{array}{l} \delta \sqcap \sup_{\sqsubseteq} \{\varphi, \psi\} = \delta \sqcap ((\varphi \vee \gamma) \wedge (\psi \vee \gamma) \wedge (\varphi \vee \psi)) = \\ ((\delta \sqcap \varphi) \vee (\delta \sqcap \gamma)) \wedge ((\delta \sqcap \psi) \vee (\delta \sqcap \gamma)) \wedge ((\delta \sqcap \varphi) \vee (\delta \sqcap \psi)) = \\ \sup_{\sqsubseteq} \{\delta \sqcap \varphi,\ \delta \sqcap \psi\} \end{array}$$

where the last step holds because $\delta \sqcap \gamma$ is an upper bound for $\{\delta \sqcap \varphi,\ \delta \sqcap \psi\}$.

(5) The supremum of the empty set is $\perp$. A nonempty set is the directed union of its finite subsets, so bounded-completeness follows from the previous item and the fact that $(P; \sqsubseteq)$ is a dcpo. It is then automatic that the elements below some fixed bound form a complete lattice; the frame distributivity law is

satisfied because infima distribute over directed suprema by assumption, and over finite suprema by the previous item.

(6) Let $\varphi, \varphi' \sqsubseteq \mathit{tt}$. By (4) we have $\sup_{\sqsubseteq}\{\varphi, \varphi'\} = (\varphi \vee \mathit{tt}) \wedge (\varphi' \vee \mathit{tt}) \wedge (\varphi \vee \varphi') = \mathit{tt} \wedge \mathit{tt} \wedge (\varphi \vee \varphi') = \varphi \vee \varphi'$ and we can conclude that information order and logical order coincide. In the frame $\downarrow_{\sqsubseteq}\mathit{ff}$ the analogous calculation reads $\sup_{\sqsubseteq}\{\varphi, \varphi'\} = (\varphi \vee \mathit{ff}) \wedge (\varphi' \vee \mathit{ff}) \wedge (\varphi \vee \varphi') = \varphi \wedge \varphi' \wedge (\varphi \vee \varphi') = \varphi \wedge \varphi'$ from which it follows that the logical order is the opposite of the information order.

For (7) we only need to compute according to ($\sqcap{-}{=}$): $\gamma \sqcap \mathit{tt} = (\gamma \wedge \bot) \vee (\mathit{tt} \wedge \bot) \vee (\gamma \wedge \mathit{tt}) = (\gamma \wedge \bot) \vee \bot \vee \gamma = \gamma \vee \bot$, etc. The last equation requires the formula for the supremum given in (4), with γ as the upper bound: $\sup_{\sqsubseteq}\{\gamma \sqcap \mathit{tt}, \gamma \sqcap \mathit{ff}\} = ((\gamma \vee \bot) \wedge \gamma) \vee ((\gamma \wedge \bot) \wedge \gamma) \vee ((\gamma \vee \bot) \wedge (\gamma \wedge \bot) = \gamma \vee (\gamma \wedge \bot) = \gamma$.

(8) One first uses (1) to obtain $\gamma \wedge \bot \prec \delta$, $\gamma' \wedge \bot \prec \delta$, and $\gamma \wedge \gamma' \prec \delta$ from the assumptions. The left hand sides are put together by ($\vee{-}{\prec}$), and one obtains the expression for $\gamma \sqcap \gamma'$ there. The second implication is completely analogous if one remembers the equality $(\gamma \wedge \bot) \vee (\delta \wedge \bot) \vee (\gamma \wedge \delta) = (\gamma \vee \bot) \wedge (\delta \vee \bot) \wedge (\gamma \vee \delta)$ quoted in Proposition 3.2.

In the last law in (8), the direction from left to right is an application of what we just showed plus ($\mathit{ff}{-}{\prec}$) and (${\prec}{-}\mathit{tt}$). The other direction follows from (${\prec}{-}{\sqsubseteq}$).

(9) The direction from left to right is trivial by (3) and (6). For the converse we use (6) to get $(\gamma \wedge \delta) \sqcap \mathit{tt} = (\gamma \sqcap \mathit{tt}) \wedge (\delta \sqcap \mathit{tt}) = (\gamma \sqcap \mathit{tt}) \sqcap (\delta \sqcap \mathit{tt}) = \gamma \sqcap \mathit{tt}$ and $(\gamma \wedge \delta) \sqcap \mathit{ff} = (\gamma \sqcap \mathit{ff}) \wedge (\delta \sqcap \mathit{ff}) = \sup_{\sqsubseteq}\{\gamma \sqcap \mathit{ff},\ \delta \sqcap \mathit{ff}\} = \gamma \sqcap \mathit{ff}$. Two applications of the last law in (7) complete the proof.

(10) Let Γ be a $\sqsubseteq$-directed set of elements of P. We have

$$
\begin{array}{ll}
\mathit{tt} \sqcap \bigsqcup^{\uparrow}_{\gamma\in\Gamma}(\varphi \wedge \gamma) \;= & (\sqcap \text{ distributes over } \bigsqcup^{\uparrow} \text{and } \wedge) \\
\bigsqcup^{\uparrow}_{\gamma\in\Gamma}((\mathit{tt} \sqcap \varphi) \wedge (\mathit{tt} \sqcap \gamma)) \;= & (6) \\
\bigsqcup^{\uparrow}_{\gamma\in\Gamma}((\mathit{tt} \sqcap \varphi) \sqcap (\mathit{tt} \sqcap \gamma)) \;= & (\text{distributivity}) \\
(\mathit{tt} \sqcap \varphi) \sqcap (\mathit{tt} \sqcap \bigsqcup^{\uparrow}_{\gamma\in\Gamma} \gamma) \;= & (6) \\
(\mathit{tt} \sqcap \varphi) \wedge (\mathit{tt} \sqcap \bigsqcup^{\uparrow}_{\gamma\in\Gamma} \gamma) \;= & (\text{distributivity}) \\
\mathit{tt} \sqcap (\varphi \wedge \bigsqcup^{\uparrow}_{\gamma\in\Gamma} \gamma) &
\end{array}
$$

A similar computation shows $\mathit{ff} \sqcap \bigsqcup^{\uparrow}_{\gamma\in\Gamma}(\varphi \wedge \gamma) = \mathit{ff} \sqcap (\varphi \wedge \bigsqcup^{\uparrow}_{\gamma\in\Gamma} \gamma)$ and the last law in (7) completes the argument. ∎

Proposition 7.4 *For $\mathcal{P}$ a partial frame consider the structure $\mathcal{L}_{\mathcal{P}} := (L_+, L_-; \mathsf{con}, \mathsf{tot})$ where*

$$\begin{array}{rcl} L_+ & := & [\bot, tt] = \{\varphi \in P \mid \bot \sqsubseteq \varphi \sqsubseteq tt\} \\ L_- & := & [\bot, ff] = \{\psi \in P \mid \bot \sqsubseteq \psi \sqsubseteq ff\} \\ (\varphi, \psi) \in \mathsf{con} & :\Leftrightarrow & \exists \gamma \in P.\ \varphi, \psi \sqsubseteq \gamma \\ (\varphi, \psi) \in \mathsf{tot} & :\Leftrightarrow & \psi \prec \varphi \end{array}$$

Then

1. *$\mathcal{L}_{\mathcal{P}}$ is a reasonable d-frame and $\mathcal{P}_{\mathcal{L}_{\mathcal{P}}} \cong \mathcal{P}$.*
2. *If $\mathcal{L}$ is a reasonable d-frame then $\mathcal{L}_{\mathcal{P}_{\mathcal{L}}} \cong \mathcal{L}$.*

Proof. (1) L_+ and L_- are frames by Lemma 7.3(5). The pairs $\langle tt, \bot\rangle$ and $\langle \bot, ff\rangle$ are in con because $\{tt, \bot\}$ and $\{\bot, ff\}$ are (trivially) bounded. They are in tot by $(tt{-}\prec)$ and $(\prec{-}ff)$. Condition $(\mathsf{con}{-}\downarrow)$ is trivially satisfied, and $(\mathsf{tot}{-}\uparrow)$ reduces to $(\prec{-}\sqsubseteq)$. For $(\mathsf{con}{-}\wedge)$ assume that $\{\varphi, \psi\}$ is bounded by γ and $\{\varphi', \psi'\}$ by γ'. Then $\varphi \sqcap \varphi' = \varphi \wedge \varphi' \sqsubseteq \gamma \wedge \gamma'$ as φ and φ' are elements below tt. Likewise, $\sup_{\sqsubseteq}\{\psi, \psi'\} = \psi \wedge \psi' \sqsubseteq \gamma \wedge \gamma'$. So $\gamma \wedge \gamma'$ is a bound for $\{\varphi \sqcap \varphi',\ \sup_{\sqsubseteq}\{\psi, \psi'\}\}$. Condition $(\mathsf{con}{-}\vee)$ is shown in the same way. For $(\mathsf{tot}{-}\wedge)$ assume $\psi \prec \varphi$ and $\psi' \prec \varphi'$. We get $\sup_{\sqsubseteq}\{\psi, \psi'\} \prec \varphi$ and $\sup_{\sqsubseteq}\{\psi, \psi'\} \prec \varphi'$ by $(\prec{-}\sqsubseteq)$, and then $\sup_{\sqsubseteq}\{\psi, \psi'\} \prec \varphi \sqcap \varphi'$ by 7.3(8). Next consider $(\mathsf{con}{-}\bigsqcup^{\uparrow})$; by assumption, each pair (γ, δ) in A is bounded, and since we are in a bounded-complete dcpo, the suprema exist and form a directed set. The supremum of the latter is an upper bound for $\bigsqcup^{\uparrow} A = (\bigsqcup^{\uparrow}\{\gamma \mid \exists \delta.\ \langle \gamma, \delta\rangle \in A\}, \bigsqcup^{\uparrow}\{\delta \mid \exists \gamma.\ \langle \gamma, \delta\rangle \in A\})$. Finally, consider $(\mathsf{con}{-}\mathsf{tot})$, so let $\langle \varphi, \psi\rangle \in \mathsf{tot}$ and $\langle \varphi', \psi\rangle \in \mathsf{con}$. This means $\psi \prec \varphi$ and that the supremum of ψ and φ' exists. We get $\sup_{\sqsubseteq}\{\psi, \varphi'\} \prec \varphi$ by $(\prec{-}\sqsubseteq)$ which implies $\sup_{\sqsubseteq}\{\psi, \varphi'\} \leq \varphi$ or $\sup_{\sqsubseteq}\{\psi, \varphi'\} \wedge \varphi = \sup_{\sqsubseteq}\{\psi, \varphi'\}$. Taking the meet with tt on both sides yields for the left hand side $(\sup_{\sqsubseteq}\{\psi, \varphi'\} \wedge \varphi) \sqcap tt = (\sup_{\sqsubseteq}\{\psi, \varphi'\} \sqcap tt) \wedge (\varphi \sqcap tt) = \sup_{\sqsubseteq}\{\bot, \varphi'\} \wedge \varphi = \varphi' \wedge \varphi$ and for the right hand side just φ', so indeed $\varphi' \sqsubseteq \varphi$ as desired.

To see that the structure $\mathcal{P}_{\mathcal{L}_{\mathcal{P}}}$ is isomorphic to $\mathcal{P}$, consider the translations

$$\begin{aligned} F\colon \mathcal{P} \to \mathcal{P}_{\mathcal{L}_{\mathcal{P}}}, &\quad \gamma \mapsto \langle \gamma \sqcap tt, \gamma \sqcap ff \rangle \\ G\colon \mathcal{P}_{\mathcal{L}_{\mathcal{P}}} \to \mathcal{P}, &\quad \langle \varphi, \psi \rangle \mapsto \sup_{\sqsubseteq} \{\varphi, \psi\} \end{aligned}$$

G is well-defined because the pairs $(\varphi, \psi) \in \mathsf{con}$ are bounded and Lemma 7.3(4) applies. We have $G \circ F = \mathsf{Id}_P$ by 7.3(7), and $F \circ G = \mathsf{Id}_{\mathcal{D}(P)_p}$ by 7.3(4) and (7). F and G preserve the information order, so $\perp$, $\sqcap$, and $\bigsqcup^{\uparrow}$ are preserved by $F \circ G$ and $G \circ F$. They also preserve the logical order: $\gamma \leq \delta$ in $\mathcal{P}$ is equivalent to $\gamma \wedge \delta = \gamma$, so $F(\gamma) = (\gamma \sqcap tt, \gamma \sqcap ff) = ((\gamma \wedge \delta) \sqcap tt, (\gamma \wedge \delta) \sqcap ff) = ((\gamma \sqcap tt) \wedge (\delta \sqcap tt), (\gamma \sqcap ff) \wedge (\delta \sqcap ff)) = ((\gamma \sqcap tt) \sqcap (\delta \sqcap tt), \sup_{\sqsubseteq} \{\gamma \sqcap ff,\ \delta \sqcap ff\}) = F(\gamma) \wedge F(\delta)$, in other words, $F(\gamma) \leq F(\delta)$. For G, assume $(\varphi, \psi) \leq (\varphi', \psi')$ which is equivalent to $\varphi \sqsubseteq \varphi'$ and $\psi \sqsupseteq \psi'$. We get $\sup_{\sqsubseteq} \{\varphi, \psi\} \leq \sup_{\sqsubseteq} \{\varphi', \psi'\}$ by the characterisation of $\leq$ in 7.3(9). Finally, consider strong implication: $(\varphi, \psi) \prec (\varphi', \psi')$ in $\mathcal{P}_{\mathcal{L}_{\mathcal{P}}}$ is by definition equivalent to $(\varphi', \psi) \in \mathsf{tot}$ in $\mathcal{L}_{\mathcal{P}}$ which is, again by definition, equivalent to $\psi \prec \varphi'$ in $\mathcal{P}$. The latter implies $\sup_{\sqsubseteq} \{\varphi, \psi\} \prec \sup_{\sqsubseteq} \{\varphi', \psi'\}$ by ($\prec$–$\sqsubseteq$), and is also implied by it because of Lemma 7.3(8): $\psi = ff \sqcap \sup_{\sqsubseteq} \{\varphi, \psi\} \prec tt \sqcap \sup_{\sqsubseteq} \{\varphi', \psi'\} = \varphi'$.

(2) It is clear that $\mathcal{L}_{\mathcal{P}_{\mathcal{L}}}$ returns $L_+ \times L_-$ which is isomorphic to L, so we only need to check the two relations. Now, in $\mathcal{P}_{\mathcal{L}}$ we have $\langle x, y \rangle \in [\perp, tt]$ if and only if $y = 0$, and $\langle x, y \rangle \in [\perp, ff]$ if and only if $x = 0$. The pair $(\langle x, 0 \rangle, \langle 0, y \rangle)$ belongs to con in $\mathcal{L}_{\mathcal{P}_{\mathcal{L}}}$ if and only if it is bounded in $\mathcal{P}_{\mathcal{L}}$, which happens if and only if $\langle x, y \rangle \in \mathsf{con}$ in $\mathcal{L}$. For the covering relation, $(\langle x, 0 \rangle, \langle 0, y \rangle)$ belongs to tot in $\mathcal{L}_{\mathcal{P}_{\mathcal{L}}}$ if and only if $\langle 0, y \rangle \prec \langle x, 0 \rangle$ in $\mathcal{P}_{\mathcal{L}}$, which happens if and only if $\langle x, y \rangle \in \mathsf{tot}$ in $\mathcal{L}$.
∎

Theorem 7.5 *The categories* **rdFrm** *and* **pFrm** *are equivalent.*

Proof. We extend the construction of the previous proposition to morphisms, so let $h = (h_+, h_-)$ be a d-frame map from $\mathcal{L}$ to $\mathcal{L}'$. For $\alpha \in \mathsf{con}$ we set $\mathcal{P}(h)(\alpha) = \langle h_+(\alpha_+), h_-(\alpha_-) \rangle$. This is a well-defined function from $\mathcal{P}_{\mathcal{L}}$ to $\mathcal{P}_{\mathcal{L}'}$ because h preserves the con-relation. It is easy to check that it preserves all partial frame operations, and the preservation of $\prec$ follows because h preserves tot. Thus $\mathcal{P}(h)$ is a morphism in **pFrm**.

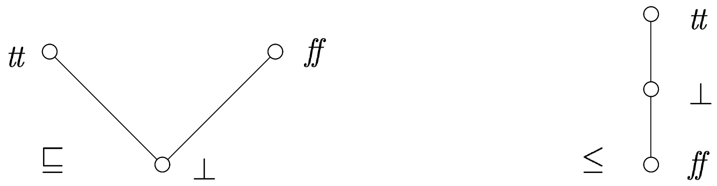


Figure 4: The dualising object 2.2 as the partial frame $\mathbb{B}_\bot := \mathcal{P}_{2.2}$. Note that strong implication contains only those pairs that are required by ($ff-\prec$) and ($\prec-tt$), in particular, $\bot \prec \bot$ does not hold.

Vice versa, if $h\colon \mathcal{P} \to \mathcal{P}'$ in **pFrm**, we let h_+ and h_- be the restrictions to $[\bot, tt]$ and $[\bot, ff]$, respectively. Apart from $\bot$, tt, ff, $\sqcap$, and $\bigsqcup^\uparrow$ they also preserve $\sup_\sqsubseteq$ because of Lemma 7.3(4), so they are frame homomorphisms. Since h is monotone with respect to $\sqsubseteq$, bounded pairs are mapped to bounded pairs, hence (h_+, h_-) preserves the con-relation on $\mathcal{L}_\mathcal{P}$. The preservation of the tot-relation is consequence of $\prec$ being preserved by h.

The translations are inverses of each other because every element γ of a partial frame is the supremum of $\gamma \sqcap tt$ and $\gamma \sqcap ff$ by 7.3(7), and suprema are preserved because they are computed from the logical operations, 7.3(4). ∎

Because of this equivalence we can from now on pretend that any partial frame is given concretely as the set con of a reasonable d-frame with the operations defined as at the beginning of this section.

The effect of the equivalence on the dualising object 2.2 in **dFrm** is to chop off the top element. Figure 4 depicts the resulting partial frame $\mathbb{B}_\bot := \mathcal{P}_{2.2}$ in the information order and the logical order.

Points on a d-frame $\mathcal{L}$ are given by pairs (F_+, F_-) or by **dFrm**-maps into the dualising object; the equivalent for the partial frame $\mathcal{P}_\mathcal{L}$ is a pair (G_+, G_-) of subsets of $P_\mathcal{L}$ that satisfies:

- G_+ and G_- are disjoint, non-empty, and Scott-open with respect to $\sqsubseteq$.
- With respect to $\leq$, G_+ is a prime filter and G_- a prime ideal.
- Whenever $\varphi \prec \psi$ then either $\varphi \in G_-$ or $\psi \in G_+$.

These properties follow immediately from the characterisation of G_+ and G_- as the inverse image of tt and ff, respectively, under the **pFrm**-equivalent to the morphism into 2.2. It is also clear that every pair of subsets with these properties defines a **pFrm**-morphism into $\mathcal{P}_{2.2}$.

The forgetful functor from **pFrm** to **Set** does not factor through the equivalence with d-frames, as the underlying set in general is smaller. Nonetheless, the construction of a left adjoint is similar to that in the proof of Proposition 5.7. For a set A we again consider the free frame FA and the d-frame $\mathcal{L}(A) := (FA, FA; \mathsf{con}, \mathsf{tot})$ where tot is chosen minimally to satisfy ($\mathsf{tot}-\mathit{tt}$), ($\mathsf{tot}-\mathit{ff}$) and ($\mathsf{tot}-\uparrow$), that is $\langle x, y\rangle \in \mathsf{tot}$ iff $x = 1$ or $y = 1$, but con is now the smallest Scott-closed subset of $FA\times FA$ that contains all pairs (a, a), $a \in A$, and is closed under the logical operations. It is obvious that the axioms for a reasonable d-frame are satisfied, except possibly ($\mathsf{con}-\mathsf{tot}$). For this we need to analyse FA and $\mathcal{L}(A)$ more carefully.

First of all, the set $FA \setminus \{1\}$ contains the generators and is closed under all frame operations except empty meet. One sees this by studying the concrete construction of the free frame or by considering the extension of the assignment $A \mapsto 0$ from FA to the two-element frame $2 := 0 < 1$; only the top element of FA is mapped to 1 by it. In the frame $FA\times FA$ the subset $S := \{\langle x, y\rangle \mid x \neq 1 \neq y\}$ is likewise closed under binary meets and arbitrary sups, and it contains the diagonal elements (a, a), $a \in A$. In particular, it is Scott-closed. S is further closed under the logical operations but lacks the constants $\mathit{tt} = (1, 0)$ and $\mathit{ff} = (0, 1)$. Adding them does not change Scott-closedness (think of taking the union with $\downarrow\mathit{tt} \cup \downarrow\mathit{ff}$) but binary suprema can no longer be taken. The logical operations are fine, though, as one can check by a case distinction. By this consideration we see that the only elements of con that lie above an element of tot are tt and ff. So ($\mathsf{con}-\mathsf{tot}$) is valid.

For the extension property let $\mathcal{P}$ be any partial frame and f a function from A to P in **Set**. We also have $f_+ \colon a \mapsto f(a) \sqcap \mathit{tt}$ and $f_- \colon a \mapsto f(a) \sqcap \mathit{ff}$. These lift to frame homomorphisms $h_+ \colon FA \to [\bot, \mathit{tt}]$ and $h_- \colon FA \to [\bot, \mathit{ff}]$, respectively. The pair (h_+, h_-) is a d-frame homomorphism from $\mathcal{L}(A)$ to the d-frame

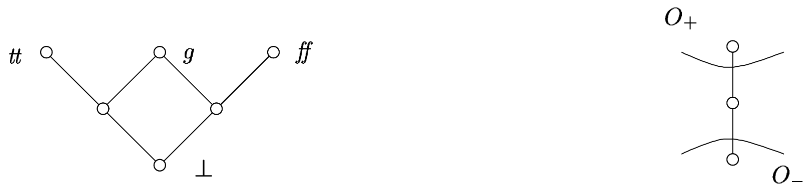


Figure 5: The free partial frame over one generator and its Stone dual.

associated with $\mathcal{P}$, because $(h_+ \times h_-)^{-1}(p)$ is Scott-closed, contains the diagonal elements, and is closed under the logical operations, and so preserves con; it preserves tot because this was chosen minimally.

Figure 5 shows the free partial frame over the one-element set $A = \{g\}$ on the left. Its Stone dual is the three-element bitopological space shown on the right. As we should expect, it has the same number of elements as the dualising partial frame shown in Figure 4.

We conclude this section with a characterisation of three special cases of d-frames.

Proposition 7.6 *The equivalence of Theorem 7.5 cuts down to one between reasonable d-frames that satisfy axioms* ($\mathrm{cut}_{\mathsf{tot}}$), ($\mathrm{Cut}_l$) *or* ($\mathrm{Cut}_r$) *and partial frames that satisfy the following corresponding* Gentzen-style cut rules*:*

$$\begin{array}{rl}
(\text{G-cut}_{\mathsf{tot}}) & \gamma \prec \delta \vee \varphi \ \textit{ and } \ \varphi \wedge \gamma' \prec \delta' \\
& \qquad \Longrightarrow \quad \gamma \wedge \gamma' \prec \delta \vee \delta' \\
(\text{G-Cut}_r) & \gamma \prec \delta \vee (\bigsqcup \Phi) \ \textit{ and } \ \forall \varphi \in \Phi . \varphi \wedge \gamma' \prec \delta' \\
& \qquad \Longrightarrow \quad \gamma \wedge \gamma' \prec \delta \vee \delta' \\
(\text{G-Cut}_l) & \forall \varphi \in \Phi . \gamma \prec \delta \vee \varphi \ \textit{ and } \ (\bigsqcup \Phi) \wedge \gamma' \prec \delta' \\
& \qquad \Longrightarrow \quad \gamma \wedge \gamma' \prec \delta \vee \delta'
\end{array}$$

Proof. Start with a d-frame $\mathcal{L}$ that satisfies ($\mathrm{cut}_{\mathsf{tot}}$), and assume the two hypotheses in Gentzen's rule for elements $\gamma, \gamma', \delta, \delta', \varphi \in P_{\mathcal{L}}$. Unwinding the definition of $\prec$ and using (tot–$\uparrow$), we get $\gamma \prec \delta \vee \varphi \iff \langle \delta_+ \sqcup \varphi_+, \gamma_- \rangle \in \mathsf{tot} \Rightarrow \langle \delta_+ \sqcup \delta'_+ \sqcup \varphi_+, \gamma_- \sqcup \gamma'_- \rangle \in \mathsf{tot}$ and $\varphi \wedge \gamma \prec \delta \iff \langle \delta_+, \varphi_- \sqcup \gamma_- \rangle \in \mathsf{tot} \Rightarrow \langle \delta_+ \sqcup \delta'_+, \varphi_- \sqcup \gamma_- \sqcup \gamma'_- \rangle \in \mathsf{tot}$. Since $\varphi \in \mathsf{con}$, we conclude by ($\mathrm{cut}_{\mathsf{tot}}$) that $\langle \delta_+ \sqcup \delta'_+, \gamma_- \sqcup \gamma'_- \rangle \in \mathsf{tot}$ or $\gamma \wedge \gamma' \prec \delta \vee \delta'$.

For the converse assume that the partial frame $\mathcal{P}$ satisfies Gentzen's (finitary) cut rule, and let $\psi, \alpha \sqsubseteq tt$ and $\varphi, \beta \sqsubseteq ff$, and assume $\langle \psi \sqcup \alpha, \varphi \rangle \in \mathsf{tot}$, $\langle \psi, \varphi \sqcup \beta \rangle \in \mathsf{tot}$ and $\langle \alpha, \beta \rangle \in \mathsf{con}$. By definition of the two predicates on $\mathcal{L}_{\mathcal{P}}$, Proposition 7.4, this means $\varphi \prec \psi \vee \alpha$, $\varphi \wedge \beta \prec \psi$ and $\alpha, \beta \sqsubseteq \gamma$ for some $\gamma \in P$. By ($\prec$–$\sqsubseteq$) we infer $\varphi \prec \psi \vee \gamma$, $\varphi \wedge \gamma \prec \psi$ from which Gentzen's cut rule allows us to conclude $\varphi \prec \psi$, or, equivalently, $(\psi_+, \varphi_-) \in \mathsf{tot}$.

The proofs for the infinitary rules are similar, noting that $\gamma \prec \delta \vee (\bigsqcup \Phi)$ if and only if $\langle \delta_+ \sqcup (\bigsqcup \Phi_+), \gamma_- \rangle \in \mathsf{tot}$ and similarly for $\bigsqcup$ on the left. ∎

Next we characterise partial frames that arise from regular d-frames (Definition 6.1). We begin with the following observation which is an immediate consequence of the definitions.

Lemma 7.7 *Let $\mathcal{P}$ be a partial frame. The following are equivalent:*

1. $\varphi \prec \psi$
2. $\varphi_+ \lhd \psi_+$ *with witness* φ_-
3. $\psi_- \lhd \varphi_-$ *with witness* ψ_+

The following technical lemma relates logical order to information order, and is used in the characterisation of regularity in partial frames and later in Section 9 when we consider symmetric d-frames.

Lemma 7.8 *In any d-frame, if I is directed and F is filtered with respect to $\leq$, and for each $\alpha \in I$ and $\beta \in F$, $\alpha \leq \beta$, then $\{\alpha \sqcap \beta \mid \alpha \in I, \beta \in F\}$ is directed with respect to $\sqsubseteq$.*

Proof. First note that $\alpha \leq \beta$ implies $\alpha \sqcap \beta = \langle \alpha_+, \beta_- \rangle$. Next, if $\alpha, \alpha' \leq \beta, \beta'$, then $(\alpha \vee \alpha') \sqcap (\beta \wedge \beta') = \alpha_+ \sqcup \alpha'_+ \sqcup \beta_- \sqcup \beta'_- \sqsupseteq \alpha \sqcap \beta, \alpha' \sqcap \beta'$ by Proposition 3.2. ∎

Lemma 7.9 *For any γ in a partial frame, the set $\{\varphi \sqcap \psi \mid \varphi \prec \gamma \prec \psi\}$ is directed. Hence the supremum exists (denoted by $\gamma_\circ$). Moreover, $\gamma_\circ \sqsubseteq \gamma$.*

Proof. Because $\varphi, \varphi' \prec \gamma \prec \psi, \psi'$ implies $\varphi \vee \varphi' \prec \gamma \prec \psi \wedge \psi'$ by ($\vee$–$\prec$) and ($\prec$–$\wedge$), Lemma 7.8 applies. From $\varphi \prec \gamma \prec \psi$ we obtain $\varphi_+ \sqsubseteq \gamma_+ \sqsubseteq \psi_+$ and $\psi_- \sqsubseteq \gamma_- \sqsubseteq \varphi_-$, so $\varphi \sqcap \psi = \langle \varphi_+, \psi_- \rangle \sqsubseteq \gamma$. ∎

Proposition 7.10 *A d-frame $\mathcal{L}$ is regular if and only if $\gamma = \gamma_\circ$ for every element γ of the associated partial frame $\mathcal{P}_\mathcal{L}$.*

Proof. By Lemma 7.7, we have that the sets $\{\varphi_+ \mid \varphi \prec \gamma\}$ and $\{\varphi_+ \mid \varphi_+ \lhd \gamma_+\}$ are the same, and likewise $\{\psi_- \mid \gamma \prec \psi\} = \{\psi_- \mid \psi_- \lhd \gamma_-\}$. The equivalence between regularity and $\gamma = \gamma_\circ$ is now obvious. ∎

Now, compactness (Definition 6.7):

Proposition 7.11 *A reasonable d-frame $\mathcal{L}$ is compact if and only if for any two $\sqsubseteq$-directed collections $(\varphi_i)_{i\in I}$ and $(\psi_j)_{j\in J}$ of elements in the associated partial frame $\mathcal{P}_\mathcal{L}$ it is true that $\bigsqcup^{\uparrow}_{i\in I} \varphi_i \prec \bigsqcup^{\uparrow}_{j\in J} \psi_j$ implies $\varphi_i \prec \psi_j$ for some $i \in I$, $j \in J$.*

Proof. The forward direction follows because $\bigsqcup^{\uparrow}_{i\in I} \varphi_i \prec \bigsqcup^{\uparrow}_{j\in J} \psi_j$ is equivalent to $\langle \bigsqcup^{\uparrow}_{j\in J} (\psi_j)_+, \bigsqcup^{\uparrow}_{i\in I} (\varphi_i)_- \rangle \in \mathsf{tot}$. For the reverse direction note that instead of arbitrary suprema in the definition of compactness we can always use directed suprema, and that the assumption $\langle \bigsqcup^{\uparrow} A, \bigsqcup^{\uparrow} B \rangle \in \mathsf{tot}$ can be rephrased as $\bigsqcup^{\uparrow}_{b\in B} \langle 0, b\rangle \prec \bigsqcup^{\uparrow}_{a\in A} \langle a, 0\rangle$. ∎

Another way of expressing the characterisation in this proposition is to say that $\prec$ as a subset of $P_\mathcal{L} \times P_\mathcal{L}$ is Scott-open.

8 From partial frames to distributive lattices

8.1 Removing the information order

As Proposition 7.1 and Lemma 7.3 demonstrate, partial frames have a very rich algebraic-relational structure. It is desirable to explore under which conditions a simpler structure can take their place without any loss of expressivity. Specifically, we will aim to get rid of all aspects of the information order, that is, binary

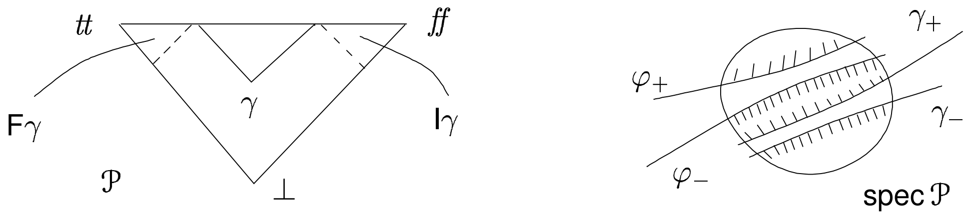


Figure 6: Representing the elements of a partial frame by ideal-filter pairs.

meets $\sqcap$, directed joins $\bigsqcup^{\uparrow}$, and the least element $\perp$, leaving only the logic-related structure (including strong implication).

We take our cue from Proposition 7.10 and define for an element γ of a partial frame $\mathcal{P}$,

$$\mathsf{I}\gamma := \{\varphi \in P \mid \varphi \prec \gamma\} \qquad \mathsf{F}\gamma := \{\psi \in P \mid \gamma \prec \psi\}$$

It is clear that $\mathsf{I}\gamma$ is an ideal (in the logical sense), that is, a $\leq$-lower set that is closed under $\vee$, and that $\mathsf{F}\gamma$ is a filter, that is, a $\leq$-upper set that is closed under $\wedge$. By ($\prec - \sqsubseteq$) it is also clear that $\gamma \sqsubseteq \gamma'$ implies $\mathsf{I}\gamma \subseteq \mathsf{I}\gamma'$ and $\mathsf{F}\gamma \subseteq \mathsf{F}\gamma'$, and by 7.10 the converse holds if and only if the partial frame is regular. Figure 6 illustrates the situation. It is worthwhile to note that the *positive* part γ_+ of a partial predicate is captured by $\mathsf{I}\gamma$, which is located on the *negative* side of $\mathcal{P}$. The sketch of the spectrum in Figure 6 indicates why this is the right approach, in that $\varphi \prec \gamma$ is the condition that guarantees that $\varphi_+ \sqsubseteq \gamma_+$.

So under the assumption of regularity the problem that we set ourselves appears to be solved; the operations $\wedge$, $\vee$, tt, and ff, and the relation $\prec$ encode all information about the partial frame. However, it would be very cumbersome to write down the conditions under which a given structure $(P; \wedge, \vee, tt, ff; \prec)$ can be guaranteed to have been derived from a partial frame, and would amount to a re-introduction of the information order via the sets $\mathsf{I}\gamma$ and $\mathsf{F}\gamma$. Specifying the morphisms would also require that the conditions that pertain to the information order be encoded.

Instead, our strategy is different; we will use the fact that there is a *construction* that, when given a strong proximity lattice (Definition 2.18), always returns a compact regular partial frame, and define morphisms between strong proximity lattices that always yield homomorphisms between the associated partial frames. Furthermore, every compact regular partial frame can be shown to arise in this way (up to isomorphism), and overall this will lead to an equivalence of categories.

Lemma 8.1 *If* $\mathcal{P}$ *is a compact regular partial frame, then* $\mathcal{X}_{\mathcal{P}} := (P; \wedge, \vee, tt, ff; \prec)$ *is a strong proximity lattice that additionally satisfies* $\prec \subseteq \leq$.

Proof. Comparing Definition 2.18 with Proposition 7.1(3), we see that there are only two additional conditions to check, ($\wedge-\prec$) and ($\prec-\vee$). We show the first:

Assume $\alpha \wedge \gamma \prec \delta$; by regularity we can write α as $\bigsqcup^{\uparrow}\{\varphi \sqcap \psi \mid \varphi \prec \alpha \prec \psi\}$, so by compactness $(\varphi \sqcap \psi) \wedge \gamma \prec \delta$ for some $\varphi \prec \alpha \prec \psi$, which by ($\prec-\sqsubseteq$) implies that ψ is an interpolant. ∎

Recall that a *round ideal* of a strong proximity lattice X is a nonempty subset $I \subseteq X$ which is closed under $\vee$, downward closed with respect to $\prec$, and which contains an element $\varphi' \succ \varphi$ for every $\varphi \in I$. A *round filter* is defined dually. It was shown in [JS96] that the set of round ideals (respectively, round filters) forms a stably continuous lattice. For our purposes we will need the concrete definition of infimum and supremum in these lattices:

$$I \sqcap I' = I \cap I' \qquad I \sqcup I' = \{\gamma \mid \exists \varphi \in I, \varphi' \in I'.\ \gamma \prec \varphi \vee \varphi'\}$$
$$F \sqcap F' = F \cap F' \qquad F \sqcup F' = \{\gamma \mid \exists \varphi \in F, \varphi' \in F'.\ \gamma \succ \varphi \wedge \varphi'\}$$

We say that (I, F) is a *round ideal-filter pair* if for all $\varphi \in I$, $\psi \in F$, $\varphi \prec \psi$. Because of interpolation and transitivity of $\prec$, $(\mathsf{I}\gamma, \mathsf{F}\gamma)$ is a round ideal-filter pair for any $\gamma \in X$.

Proposition 8.2 *Let* $(X; \wedge, \vee, tt, ff; \prec)$ *be a strong proximity lattice. The set* $\mathcal{P}_{\mathcal{X}}$

of round ideal-filter pairs carries the structure of a compact regular partial frame:

$$\begin{array}{rcl}
\bot & := & (\mathsf{I}f\!f, \mathsf{F}t\!t) \\
(I,F) \sqcap (I',F') & := & (I \cap I', F \cap F') \\
\bigsqcup^{\uparrow}_{k\in K}(I_k,F_k) & := & (\bigcup^{\uparrow}_{k\in K} I_k, \bigcup^{\uparrow}_{k\in K} F_k) \\
(I,F) \wedge (I',F') & := & (I \cap I', F \sqcup F') \\
(I,F) \vee (I',F') & := & (I \sqcup I', F \cap F') \\
t\!t & := & (X, \mathsf{F}t\!t) \\
f\!f & := & (\mathsf{I}f\!f, X) \\
(I,F) \prec (I',F') & := & F \cap I' \neq \emptyset
\end{array}$$

Proof. This is very easy to show, except perhaps regularity. For this observe that for a given round ideal-filter pair (I,F) and elements $\varphi \in I$, $\psi \in F$ we have $(\mathsf{I}\varphi, \mathsf{F}\varphi) \prec (I,F) \prec (\mathsf{I}\psi, \mathsf{F}\psi)$ because of roundness, and hence $(\mathsf{I}\varphi, \mathsf{F}\varphi) \sqcap (\mathsf{I}\psi, \mathsf{F}\psi) = (\mathsf{I}\varphi, \mathsf{F}\psi)$. The (directed) supremum of these pairs yields (I,F), again by roundness. ∎

We remark that the characterisation of $\prec$ can also be given as

$$(I,F) \prec (I',F') \quad := \quad \exists \varphi \in F, \psi \in I'.\, \varphi \prec \psi$$

Proposition 8.3 *Every compact regular partial frame $\mathcal{P}$ is isomorphic to its partial frame of round ideal-filter pairs.*

Proof. We begin by showing that every round ideal-filter pair (I,F) is of the form $(\mathsf{I}\gamma, \mathsf{F}\gamma)$ for some $\gamma \in P$. For this let $\gamma := \bigsqcup^{\uparrow}\{\varphi \sqcap \psi \mid \varphi \in I, \psi \in F\}$. If $\varphi' \prec \gamma$ then by compactness $\varphi' \prec \varphi \sqcap \psi \sqsubseteq \varphi$ for some $\varphi \in I$, hence $\varphi' \in I$ already. This shows $\mathsf{I}\gamma \subseteq I$. To see that the other inclusion holds, let $\varphi \in I$ and $\varphi' \succ \varphi$ by roundness. For any $\psi \in F$ we have $\varphi \prec \psi$, so $\varphi \prec \varphi' \sqcap \psi$ by Lemma 7.3(8). Since $\varphi' \sqcap \psi \sqsubseteq \gamma$, we get $\varphi \in \mathsf{I}\gamma$ by ($\prec$–$\sqsubseteq$).

The proof is complete if we can show that the function $\gamma \mapsto (\mathsf{I}\gamma, \mathsf{F}\gamma)$ preserves the partial frame structure. We check each line of the definition in Proposition 8.2:

- $\varphi \prec \bot$ implies $\varphi \prec f\!f$ because $\bot \sqsubseteq f\!f$, and $\varphi \prec f\!f$ implies $\varphi \prec f\!f \sqcap t\!t = \bot$ by ($\prec$–$t\!t$) and 7.3(8).
- $\varphi \prec \gamma \sqcap \delta$ if and only if $\varphi \prec \gamma$ and $\varphi \prec \delta$ by 7.3(8) and ($\prec$–$\sqsubseteq$).

- $\varphi \prec \bigsqcup^{\uparrow}_{k\in K} \gamma_k$ if and only if $\varphi \prec \gamma_k$ for some $k \in K$ by compactness and ($\prec$–$\sqsubseteq$).

- $\varphi \prec \gamma \wedge \delta$ if and only if $\varphi \prec \gamma$ and $\varphi \prec \delta$ by ($\prec$–$\wedge$).

- $\varphi \prec \gamma \vee \delta$ if and only if there exist $\gamma' \prec \gamma$, $\delta' \prec \delta$ with $\varphi \prec \gamma' \vee \delta'$. The "if" direction is ($\vee$–$\prec$), ($\prec$–$\vee$), and transitivity of $\prec$, "only if" is Lemma 8.1 and axiom ($\prec$–$\vee$) of strong proximity lattices.

- $\mathit{ff} \prec \varphi$ and $\varphi \prec \mathit{tt}$ are always true.

- $\gamma \prec \delta$ if and only if there is φ with $\gamma \prec \varphi \prec \delta$.

∎

Strong proximity lattices don't have to be isomorphic (in the straightforward sense) if their associated partial frames of round ideal-filter pairs are isomorphic. Indeed, any $(\wedge, \vee)$-sub-lattice X of a given compact regular partial frame $\mathcal{P}$ that is *dense*, in the sense that for any $\gamma \prec \delta$ in $\mathcal{P}$, there is $\varphi \in X$ with $\gamma \prec \varphi \prec \delta$, will produce a partial frame of round ideal-filter pairs that is isomorphic to $\mathcal{P}$. This is actually a good thing because it allows us not only to drop the information order but also to restrict to a "basis" of a given compact regular partial frame.

If $h\colon \mathcal{P} \to \mathcal{P}'$ is a homomorphism of partial frames and if X and X' are dense sub-lattices of $\mathcal{P}$ and $\mathcal{P}'$, respectively, then the restriction of h to X has no reason to return results in X'. However, because of density (and regularity), an image $h(\gamma)$ in P' can still be written as $\bigsqcup^{\uparrow}\{\varphi \sqcap \psi \mid \varphi, \psi \in X', \varphi \prec h(\gamma) \prec \psi\}$, so this suggests to replace h by a relation that associates with a given $\gamma \in X$ the sets $X' \cap \mathsf{I}h(\gamma)$ and $X' \cap \mathsf{F}h(\gamma)$. From previous work, [MJ02, JKM01], we know that the situation is well captured by *consequence relations*:

Definition 8.4 *Let $\mathcal{X}, \mathcal{X}'$ be strong proximity lattices. A subset $\vdash\ \subseteq X \times X'$ is* *called a* consequence relation *if the following conditions are satisfied:*

$$
\begin{array}{lll}
(\vdash\!-\!\prec) & \vdash\, = \,\prec \circ \vdash \;\; \textit{and} \;\; \vdash\, = \,\vdash \circ \prec' & \\
(\mathit{ff}\!-\!\vdash) & \mathit{ff} \vdash \delta & \\
(\vdash\!-\!\mathit{tt}) & \gamma \vdash \mathit{tt} & \\
(\vee\!-\!\vdash) & \gamma \vdash \delta,\ \gamma' \vdash \delta \quad \Longleftrightarrow \quad \gamma \vee \gamma' \vdash \delta & \\
(\vdash\!-\!\wedge) & \gamma \vdash \delta,\ \gamma \vdash \delta' \quad \Longleftrightarrow \quad \gamma \vdash \delta \wedge \delta' & \\
(\vdash\!-\!\vee) & \gamma \vdash \delta \vee \psi \quad \Longrightarrow \quad \exists \psi' \in X'.\ \psi' \prec' \psi \;\; \textit{and} \;\; \gamma \vdash \delta \vee \psi' & \\
(\wedge\!-\!\vdash) & \gamma \wedge \varphi \vdash \delta \quad \Longrightarrow \quad \exists \varphi' \in X.\ \varphi \prec \varphi' \;\; \textit{and} \;\; \gamma \wedge \varphi' \vdash \delta &
\end{array}
$$

A pair of consequence relations $\vdash^\diamond \subseteq X' \times X$, $\vdash_\diamond \subseteq X \times X'$ *is called* adjoint *if* $\prec \subseteq \vdash_\diamond ; \vdash^\diamond$ *and* $\vdash^\diamond ; \vdash_\diamond \subseteq \prec'$*, where "$;$" denotes relational composition.*

The category **Prox** *has strong proximity lattices as objects and adjoint pairs of consequence relations as morphisms. We choose the direction of a pair* $(\vdash^\diamond, \vdash_\diamond)$ *as going from* $\mathcal{X}$ *to* $\mathcal{X}'$*, that is, with the direction of* $\vdash_\diamond$ *and opposite to the direction of* $\vdash^\diamond$*. Identities are given by the pairs* $(\prec, \prec)$*, and composition is componentwise relational product.*

Proposition 8.5 *Let* $\mathcal{P}, \mathcal{P}'$ *be compact regular partial frames and* $h\colon \mathcal{P} \to \mathcal{P}'$ *a homomorphism. Define relations* $\vdash^h \subseteq P' \times P$ *and* $\vdash_h \subseteq P \times P'$ *by*

$$
\begin{array}{lll}
\delta \vdash^h \gamma & :\Leftrightarrow & \delta \prec' h(\gamma) \\
\gamma \vdash_h \delta & :\Leftrightarrow & h(\gamma) \prec' \delta
\end{array}
$$

This is an adjoint pair of consequence relations between the associated strong proximity lattices $\mathcal{X}_{\mathcal{P}}$ *and* $\mathcal{X}_{\mathcal{P}'}$*.*

The assignments $\mathcal{P} \mapsto \mathcal{X}_{\mathcal{P}}$*,* $h \mapsto (\vdash^h, \vdash_h)$ *constitute a functor* $\mathcal{X}$ *from the category* **cr-pFrm** *of compact regular partial frames to* **Prox***.*

Proof. The conditions for consequence relations follow straightforwardly from the analogous properties of strong implication $\prec$. Of some interest, perhaps, is the argument for the interpolation part of $(\vdash\!-\!\prec)$: In $\delta \vdash^h \gamma$, or equivalently, $\delta \prec' h(\gamma)$, we are allowed to replace γ by $\bigsqcup^{\uparrow}\{\varphi \sqcap \psi \mid \varphi \prec \gamma \prec \psi\}$ because $\mathcal{P}$ is regular. Then Scott-continuity of h and compactness of $\mathcal{P}'$ allow us to conclude that $\delta \prec' h(\varphi) \sqcap h(\psi)$ for some $\varphi \prec \gamma \prec \psi$, and consequently $\delta \prec' h(\varphi)$ or $\delta \vdash^h \varphi \prec \gamma$ by ($\prec$–$\sqsubseteq$).

The adjointness conditions follow from transitivity and interpolativity of $\prec'$, and the assumption that h preserve $\prec$.

For functoriality assume $g\colon \mathcal{P} \to \mathcal{P}'$ and $h\colon \mathcal{P}' \to \mathcal{P}''$. Then $\varphi \prec'' h{\circ}g(\gamma)$ is equivalent to $\varphi \vdash^h g(\gamma)$. Using ($\vdash$–$\prec$) this is equivalent to $\varphi \vdash^h \delta \prec' g(\gamma)$ for some $\delta \in P'$, which is equivalent to $\varphi \vdash^h ; \vdash^g \gamma$ as required. ∎

For a functor $\mathcal{P}$ from **Prox** to **cr-pFrm** we assign to a strong proximity lattice $\mathcal{X}$ the compact regular partial frame $\mathcal{P}_{\mathcal{X}}$ of round ideal-filter pairs, following Proposition 8.2. If $(\vdash^{\diamond}, \vdash_{\diamond})$ is an adjoint pair of consequence relations between strong proximity lattices $\mathcal{X}$ and $\mathcal{X}'$ then we let $\mathcal{P}(\vdash^{\diamond}, \vdash_{\diamond})$ be the function h which assigns to a round ideal-filter pair (I, F) on $\mathcal{X}$ the pair

$$(\{\varphi \mid \exists \psi \in I.\ \varphi \vdash^{\diamond} \psi\},\ \{\delta \mid \exists \gamma \in F.\ \gamma \vdash_{\diamond} \delta\})$$

in other words, the inverse image of I under the relation $\vdash^{\diamond}$, and the forward image of F under $\vdash_{\diamond}$. From the properties of consequence relations one readily derives that the result consists of a round ideal and a round filter. Regarding the connection between the two, let $\varphi \vdash^{\diamond} \psi$ for some $\psi \in I$, and $\gamma \vdash_{\diamond} \delta$ for some $\gamma \in F$. Since $\psi \prec \gamma$, we get $\varphi \vdash^{\diamond} \gamma \vdash_{\diamond} \delta$ and hence $\varphi \prec' \delta$ by the second adjointness condition. The conditions for a homomorphism of partial frames are easily checked, given the characterisations in Proposition 8.2. For example, if $(I, F) = \bot = (\mathsf{I}\mathit{ff}, \mathsf{F}\mathit{tt})$ in $\mathcal{P}_{\mathcal{X}}$ one shows that $\{\varphi \mid \exists \psi \prec \mathit{ff}.\ \varphi \vdash^{\diamond} \psi\} = \mathsf{I}\mathit{ff}$: containment of the latter in the former is clear because $\mathsf{I}\mathit{ff}$ is the smallest round ideal. For the other inclusion one uses that $\varphi \vdash^{\diamond} \mathit{ff}$ implies $\varphi \vdash^{\diamond} \mathit{ff} \vdash_{\diamond} \mathit{ff}$, hence $\varphi \prec' \mathit{ff}$ by the second adjointness condition. The analogous fact about $\mathsf{F}\mathit{tt}$ is proved similarly. Another case of some interest is the preservation of $\prec$; assume $(I, F) \prec (I', F')$, that is, $F \cap I' \neq \emptyset$. By roundness one finds $\varphi \prec \psi$ in $F \cap I'$ which implies $\varphi \vdash_{\diamond} \gamma \vdash^{\diamond} \psi$ for some $\gamma \in X'$. This element is the witness that $h(I, F) \prec h(I', F')$.

Theorem 8.6 *The functors $\mathcal{X}$: **cr-pFrm** $\to$ **Prox** and $\mathcal{P}$: **Prox** $\to$ **cr-pFrm** constitute an equivalence of categories.*

Proof. The natural isomorphism between a compact regular partial frame $\mathcal{P}$ and its collection of round ideal-filter pairs was presented in 8.3. For a strong proximity lattice $\mathcal{X}$, consider the map $h\colon \mathcal{X} \to \mathcal{X}_{\mathcal{P}_\mathcal{X}}$, $\gamma \mapsto (\mathsf{I}\gamma, \mathsf{F}\gamma)$, and define $\vdash^h$ and $\vdash_h$ as before. We already have $\prec\ \subseteq\ \vdash_h;\vdash^h$, so let $\gamma \vdash_h (I, F) \vdash^h \delta$, which by definition reduces to $(\mathsf{I}\gamma, \mathsf{F}\gamma) = h(\gamma) \prec (I, F) \prec h(\delta) = (\mathsf{I}\delta, \mathsf{F}\delta)$. This in turn is equivalent to having elements $\gamma' \in I \cap \mathsf{F}\gamma$, $\delta' \in F \cap \mathsf{I}\delta$, and since $\gamma' \prec \delta'$ is guaranteed, we get $\gamma \prec \delta$ by transitivity. On the other side we know $\vdash^h;\vdash_h\ \subseteq\ \prec$, so consider $(I, F) \prec (I', F')$ which means that there is $\gamma \in F \cap I'$. By roundness we can expand this to $\varphi \prec \gamma \prec \psi$ within $F \cap I'$ which proves $(I, F) \prec h(\gamma) = (\mathsf{I}\gamma, \mathsf{F}\gamma) \prec (I', F')$ or $(I, F) \vdash^h;\vdash_h (I', F')$. ∎

We note that the composition $\mathcal{X} \circ \mathcal{P}\colon$ **Prox** $\to$ **Prox** has a "normalising" effect on a strong proximity lattice $\mathcal{X}$; while $\prec\ \subseteq\ \leq$ is not required to hold in $\mathcal{X}$, it is true in $\mathcal{X}_{\mathcal{P}_\mathcal{X}}$. Nonetheless, our experience in working with strong proximity lattices for describing stably compact spaces has been that the extra freedom afforded by Definition 2.18 is essential. For more detail see [Keg02].

Let us also have a look at how d-points manifest themselves in the category **Prox**. The dualising partial frame (depicted in Figure 4 on page 72) has a dense sub-lattice consisting of tt and ff only; we denote it with $\mathbb{B}$. For a **Prox**-morphism $(\vdash^\diamond, \vdash_\diamond)$ from a strong proximity lattice $\mathcal{X}$ to $\mathbb{B}$, we consider $F = \{\varphi \in X \mid tt \vdash^\diamond \varphi\}$ and $I = \{\varphi \in X \mid \varphi \vdash_\diamond ff\}$. Although this looks superficially similar to the action of $\mathcal{P}(\vdash_\diamond, \vdash^\diamond)$ on the ideal-filter pair $(\{ff\}, \{tt\})$, in fact we are transporting $(\{ff\}, \{tt\})$ in the *opposite* direction, and while it is still true that I is a round ideal and F a round filter, $I \prec F$ can not be shown. Instead we have the following properties, generalising Proposition 2.2:

Proposition 8.7 *Let F and I be subsets of a strong proximity lattice $\mathcal{X}$. The following are equivalent:*

1. *There is a* **Prox**-*morphism $(\vdash^\diamond, \vdash_\diamond)$ from $\mathcal{X}$ to $\mathbb{B}$ such that $F = \{\varphi \in X \mid tt \vdash^\diamond \varphi\}$ and $I = \{\varphi \in X \mid \varphi \vdash_\diamond ff\}$;*

2. *F is a round filter and I is a round ideal disjoint from F such that $\varphi \prec \psi$ implies $\varphi \in I$ or $\psi \in F$;*

3. *among disjoint pairs of a round filter and a round ideal, (F, I) is maximal with respect to component-wise inclusion;*

4. *F is a round prime filter and $I = \{\varphi \mid \exists \psi \notin F.\ \varphi \prec \psi\}$;*

5. *I is a round prime ideal and $F = \{\psi \mid \exists \varphi \notin I.\ \varphi \prec \psi\}$.*

Proof. (1) $\Rightarrow$ (2): $\varphi \prec \psi$ implies $\varphi \vdash_\diamond ; \vdash^\diamond \psi$, so either $\varphi \vdash_\diamond \mathit{ff}$ or $\mathit{tt} \vdash^\diamond \psi$, hence $\varphi \in I$ or $\psi \in F$. For disjointness assume $\gamma \in I \cap F$. This implies $\mathit{tt} \vdash^\diamond \gamma \vdash_\diamond \mathit{ff}$ and hence $\mathit{tt} \prec \mathit{ff}$ by the second adjointness condition, but this is not valid in the dualising proximity lattice $\mathbb{B}$, and so no such γ exists.

(2) $\Rightarrow$ (3): Assume I' is a round ideal containing I and disjoint from F. If $\varphi \in I' \setminus I$ then there is also $\varphi' \succ \varphi$ in $I' \setminus I$ by roundness. By disjointness, neither $\varphi \in I$, nor $\varphi' \in F$, contradicting (2).

(3) $\Rightarrow$ (4): Assume $\varphi \vee \psi \in F$, and let $F' = \{\gamma \mid \varphi \vee \gamma \in F\}$. Then $F \subseteq F'$ and $\psi \in F'$. Furthermore, F' is a round filter: $\varphi \vee \gamma \in F$ implies $\delta \prec \varphi \vee \gamma$ for some $\delta \in F$, and by ($\prec$–$\vee$), $\delta \prec \varphi \vee \gamma'$ for some $\gamma' \prec \gamma$. By definition, $\gamma' \in F'$, too. Closure under meets is shown by a simple application of the distributivity law. If F' is disjoint from I then we can conclude $F' = F$, and hence $\psi \in F$ as desired. If, on the other hand, $\varphi \vee \gamma \in F$ for some $\gamma \in I$, then repeat the construction of F' with γ replacing φ, and φ replacing ψ. This F' is guaranteed to be disjoint from I, and $\varphi \in F$ is shown.

The set $\{\varphi \mid \exists \psi \notin F.\ \varphi \prec \psi\}$ is a round ideal because F is prime. It is also disjoint from F, and so it must be contained in I because the latter is maximal. On the other hand, every element of I is below some other element of I by roundness, which by disjointness does not belong to F. This shows that $I \subseteq \{\varphi \mid \exists \psi \notin F.\ \varphi \prec \psi\}$.

(4) $\Leftrightarrow$ (5) is Proposition 2.20. The argument for (5) $\Rightarrow$ (2) is straightforward. For (2) $\Rightarrow$ (1) we let $\mathit{tt} \vdash^\diamond \varphi$ if and only if $\varphi \in F$ (and $\mathit{ff} \vdash^\diamond \varphi$ always), and $\varphi \vdash_\diamond \mathit{ff}$ if and only if $\varphi \in I$ (and $\varphi \vdash_\diamond \mathit{tt}$ always). The axioms for consequence relations are easy to check, so let us focus on the adjointness conditions. If $\varphi \prec \psi$ then $\varphi \in I$ or $\psi \in F$, hence either $\varphi \vdash_\diamond \mathit{tt} \vdash^\diamond \psi$ or $\varphi \vdash_\diamond \mathit{ff} \vdash^\diamond \psi$. If $\alpha \vdash^\diamond ; \vdash_\diamond \beta$ then it can not be that $\alpha = \mathit{tt}$ and $\beta = \mathit{ff}$ as otherwise there would be an element

in the intersection of I and F. In the three other cases $\alpha \prec \beta$ holds in $\mathbb{B}$. ∎

Since **Prox**-morphisms into $\mathbb{B}$ are in one-to-one correspondence to **pFrm**-morphisms into $\mathbb{B}_\perp$, and since the latter are in one-to-one correspondence to d-frame homomorphisms into 2.2, we see that the duality between strong proximity lattices and stably compact spaces, Theorem 2.22, is a special case of the duality of d-frames and bitopological spaces.

It may also be helpful to identify the topology on the spectrum in the current setting. If $a = (F, I)$ is a point of a strong proximity lattice according to the proposition, and if $\gamma = (I', F')$ is a round ideal-filter pair encoding a partial predicate on the spectrum, then $a \in \Phi_+(\gamma)$ if and only if $F \cap I' \neq \emptyset$, and $a \in \Phi_-(\gamma)$ if and only if $I \cap F' \neq \emptyset$.

8.2 Reflexivity: Distributive Lattices

Definition 8.8 *An element φ of a partial frame $\mathcal{P}$ is called a* total predicate *or* reflexive element *if $\varphi \prec \varphi$. The set of all reflexive elements is denoted by $R_{\mathcal{P}}$.*

This terminology is justified because $\varphi \prec \varphi$ in the partial frame is equivalent to $\varphi \in \mathsf{con} \cap \mathsf{tot}$ in the associated d-frame, so these are elements which always return an answer, and never give conflicting information. Yet another way of putting this is to say that they are classical Boolean predicates.

It follows from Axiom ($\mathsf{con}-\mathsf{tot}$) that with respect to the information order a total predicate is always a maximal element in a partial frame, but not all maximal elements need to be total. Axioms ($\mathsf{con}-\wedge$), ($\mathsf{con}-\vee$), ($\mathsf{tot}-\wedge$), and ($\mathsf{tot}-\vee$) imply that $R_{\mathcal{P}}$ is a sub-lattice of $(P; \wedge, \vee, tt, ff)$. Strong implication restricted to $R_{\mathcal{P}}$ is the same as ordinary implication: The inclusion $\prec\ \subseteq\ \leq$ always holds for a partial frame, and for the reverse we use that $\varphi \leq \psi$ implies $\varphi \prec \varphi = \varphi \wedge \psi$, which implies $\varphi \prec \psi$ by ($\prec-\wedge$). A partial frame homomorphism maps total predicates to total predicates because it is required to respect $\prec$. Hence, restriction to reflexive elements yields a functor $\mathcal{R}$ from **pFrm** to **dLat**.

Definition 8.9 *A* Stone partial frame *is a compact regular partial frame $\mathcal{P}$ for* *which the set of reflexive elements is dense. In other words, for every $\varphi \prec \psi$ in $\mathcal{P}$ there is $\gamma \in R_{\mathcal{P}}$ such that $\varphi \prec \gamma \prec \psi$.*

The density condition for Stone partial frames can be translated to d-frames using the fact that $R_{\mathcal{P}}$ corresponds to $\mathsf{con} \cap \mathsf{tot}$ in the associated d-frame, but the result is cumbersome and does not seem to shed any additional light on the situation.

Theorem 8.10 *Define a functor $\mathcal{P}'$ from* **dLat** *to* **pFrm** *by setting*

$$\begin{array}{rcll} \mathcal{P}'(D) & := & \{(I,F) \mid (I,F) \text{ is an ideal-filter pair in } D\} & \text{for objects, and} \\ \mathcal{P}'(h) & := & (\{a \mid \exists a' \in I.\, a \leq h(a')\}, \{b \mid \exists b' \in F.\, h(b') \leq b\}) & \text{for morphisms} \end{array}$$

Then

1. *$\mathcal{P}'$ is left adjoint to $\mathcal{R}$;*

2. *$\mathcal{R} \circ \mathcal{P}' \cong \mathsf{id}$;*

3. *The image of $\mathcal{P}'$ is contained in* **SpFrm** *and restricted to* **SpFrm***, $\mathcal{P}' \circ \mathcal{R} \cong \mathsf{id}$.*

Proof. A distributive lattice can be viewed as a strong proximity lattice where $\prec = \leq$. The definition of $\mathcal{P}'$ on objects, then, coincides with that of $\mathcal{P}$ in Proposition 8.2, and we obtain that $\mathcal{P}'(D)$ is a compact regular partial frame. Given the concrete description of the operations on $\mathcal{P}'(D)$ in 8.2, it is also clear that $\mathcal{P}'(h)$ is a partial frame homomorphism.

For any $a \in D$, the pair $(\downarrow a, \uparrow a)$ is a reflexive element of $\mathcal{P}'(D)$ and there are no others. Thus we have shown (2).

The reflexive elements of $\mathcal{P}'(D)$ are dense, because $(I,F) \prec (I',F')$ means by definition that there is $a \in F \cap I'$, so $(I,F) \prec (\downarrow a, \uparrow a) \prec (I',F')$. Hence the image of $\mathcal{P}'$ is contained in **SpFrm**. Part (3) now follows from Proposition 8.3.

For Part (1), let h be a distributive lattice homomorphism from D to $\mathbb{R}(\mathcal{P})$ where $\mathcal{P}$ is any partial frame. We get a homomorphism of partial frames h_* from $\mathcal{P}'(D)$ to $\mathcal{P}$ by setting $h_*(I,F) := \bigsqcup^{\uparrow}\{h(a) \sqcap h(b) \mid a \in I, b \in F\}$. Restricting h_* to the reflexive elements of $\mathcal{P}'(D)$, that is, to the ideal-filter pairs $(\downarrow a, \uparrow a)$, we recover h, as $\bigsqcup^{\uparrow}\{h(a') \sqcap h(b) \mid a' \leq a, b \geq a\} = h(a)$ in this case. Vice versa, starting with a partial frame homomorphism $f\colon \mathcal{P}'(D) \to \mathcal{P}$, denote the restriction to the reflexive elements by f^*. We already know that this is a distributive lattice homomorphism from D to $\mathbb{R}(\mathcal{P})$. Extending f^* to ideal-filter pairs we obtain

$$\begin{array}{rcl} (f^*)_*(I,F) & = & \bigsqcup^{\uparrow}\{f^*(a) \sqcap f^*(b) \mid a \in I, b \in F\} \\ & = & \bigsqcup^{\uparrow}\{f(\downarrow a, \uparrow a) \sqcap f(\downarrow b, \uparrow b) \mid a \in I, b \in F\} \\ & = & f(\bigsqcup^{\uparrow}\{(\downarrow a, \uparrow a) \sqcap (\downarrow b, \uparrow b) \mid a \in I, b \in F\} \\ & = & f(I,F) \end{array}$$

where the last step uses the fact that $\mathcal{P}'(D)$ is a Stone partial frame. Thus **pFrm**$(\mathcal{P}'(D), \mathcal{P})$ and **dLat**$(D, \mathbb{R}(\mathcal{P}))$ are (naturally) isomorphic. ∎

Corollary 8.11 *The categories* **dLat** *and* **SpFrm** *are equivalent.*

An alternate view of a distributive lattice is as a special proximity lattice, namely, one in which $\prec \; = \; \leq$. To a partial frame map $h\colon \mathcal{P} \to \mathcal{Q}$ one would then associate the adjoint pair $(\vdash^h, \vdash_h)$. There is no real difference between this and a distributive lattice homomorphism, however, as we can show that the graph of h (restricted to $R_{\mathcal{P}}$ and co-restricted to $R_{\mathcal{Q}}$) is equal to $(\vdash^h)^{-1} \cap \vdash_h$: For $\varphi \in R_{\mathcal{P}}$ we have $h(\varphi) \leq h(\varphi)$, so $\varphi \vdash_h h(\varphi)$ and $h(\varphi) \vdash^h \varphi$. For the reverse inclusion one assumes $\varphi \vdash_h \psi$ and $\psi \vdash^h \varphi$ and obtains $h(\varphi) \leq \psi \leq h(\varphi)$ by definition and the fact that $\prec = \leq$.

The dualising object $\mathbb{B}_\perp$ in **pFrm** is a Stone partial frame, and its set of reflexive elements consists of *tt* and *ff*. Adapting Proposition 8.7 to the reflexive setting, we see that Stone's duality of bounded distributive lattices is a special case of our bitopological duality.

Definition 8.12 *Say that a bitopological space $(X; \tau_+, \tau_-)$ is* totally order disconnected *if $\leq_+ = \geq_-$ and whenever $x \not\leq_+ y$, there is an upper open neighbourhood U of x and lower open neighbourhood V of y so that $U \cap V = \emptyset$ and $U \cup V = X$.*

Corollary 8.13 *The category* **dLat** *is dually equivalent to the category of compact totally order disconnected bitopological spaces.*

Proof. Stone partial frames are spatial and the category **SpFrm** is equivalent to **dLat**. So it suffices to show that the spectra of Stone partial frames are precisely the compact totally order disconnected spaces. Consider points $(F_+, F_-) \not\leq_+ (G_+, G_-)$ of a Stone partial frame. By (partial frame) compact regularity, the order is inclusion in the first component and (equivalently) reverse inclusion in the second. So there is a token $\varphi \in F_+ \setminus G_+$. Because F_+ is round and the reflexive elements are dense, φ can be chosen to be reflexive, from which $\varphi \in G_-$ follows. Thus $\Phi_+(\varphi)$ and $\Phi_-(\varphi)$ are the desired opens.

Now consider a compact totally order disconnected space $(X; \tau_+, \tau_-)$. We show that the reflexive elements of $\mathsf{con}_{\tau_+ \times \tau_-}$ are dense. For two consistent pairs of opens $(U, V), (U', V') \in \tau_+ \times \tau_-$, $(U, V) \prec (U', V')$ holds if and only if $U' \cup V = X$. If $V = X$, then $(U, V) \prec (X; \emptyset) \prec (U', V')$ and $(X, \emptyset)$ is reflexive. Similarly, if $U' = X$, then $(U, V) \prec (\emptyset, X) \prec (U', V')$. If neither of these holds, then $X \setminus V$ and $X \setminus U'$ are non-empty, and for each $x \in X \setminus V$ and $y \in X \setminus U'$, we have $x \not\leq y$. For each such pair of points, choose a disjoint pair $(Y_{x,y}, Z_{x,y}) \in \tau_+ \times \tau_-$ according to total order disconnectedness. For each $x \in X \setminus V$, the sets $Z_{x,y}$ together with U' cover X. By compactness, finitely many Z's suffice to cover $X \setminus U'$. Let Z'_x be the finite union of these $Z_{x,y}$'s and let Y'_x be the intersection of the corresponding $Y_{x,y}$'s. Then Y'_x and Z'_x are disjoint and cover X. Moreover, $x \in Y'_x$. So the sets Y'_x together with V cover X. Again by compactness, finitely many of them suffice. So we can let Y^* be the union of these and Z^* the intersection of the corresponding sets Z'_x. Since each Z'_x covers $X \setminus U'$, we have $Z^* \cup U' = X$. And clearly $Y^* \cup V = X$. Putting this together, $(U, V) \prec (Y^*, Z^*) \prec (U', V')$ and (Y^*, Z^*) is reflexive. ∎

Of course, from Stone's original duality theorem we know that the upper topology of a compact totally order disconnected bitopological space is a spectral space. Notice, however, that Stone's characterisation requires that we restrict to *perfect* maps, so that the topological category **Spec** is not a full sub-category of **Top**. In contrast, the corollary establishes a duality between the category of distributive lattices and a full sub-category of **BiTop**.

9 Negation

9.1 Negation as additional structure

9.1.1 Symmetric d-frames

Negation exchanges true for false and false for true. Where the truth value is unknown it remains unknown, and where a contradiction occurs it remains a contradiction. This is the approach taken by Belnap [Bel77], and it defines an operation $\neg$ on the four-element lattice 2.2 of truth values (depicted in Figure 1 on page 29).

The effect on the bitopological space of models is that the positive extent $\Phi_+(\alpha)$ of a formula α is exchanged with the negative extent $\Phi_-(\alpha)$, in other words, $\Phi_+(\neg\alpha) = \Phi_-(\alpha)$ and $\Phi_-(\neg\alpha) = \Phi_+(\alpha)$. Since the positive extents comprise the topology $\mathcal{T}_+$, and the negative extents the topology $\mathcal{T}_-$, we see that in the presence of negation the two topologies must coincide, or, in the language of Section 4.1, that the bitopological space $(\mathsf{spec}\,\mathcal{L}; \mathcal{T}_+, \mathcal{T}_-)$ is *symmetric*.

In order to add negation to the abstract setting of d-frames $L = L_+ \times L_-$, we therefore require that the frames L_+ and L_- are isomorphic via an explicit bijection $i\colon L_- \to L_+$. The associated negation operation maps $\alpha = \langle \alpha_+, \alpha_- \rangle$ to $\langle i(\alpha_-), i^{-1}(\alpha_+) \rangle$, and will be looked at below. For the moment, we tentatively define a symmetric d-frame to be a structure $\mathcal{L} = (L_+, L_-; \mathsf{con}, \mathsf{tot}; i)$ and require that homomorphisms preserve the symmetry operation i. The latter requirement is necessary as there may be many different isomorphisms between L_- and L_+, and we must make sure that homomorphisms respect the chosen exchange of true and false.

In general, an abstract point of a d-frame $\mathcal{L}$ is given as a homomorphism from $\mathcal{L}$ to 2.2; since now we require that it additionally preserve the symmetry, there will in general be fewer "symmetric points" than general ones. We denote the resulting bitopological space as $\mathsf{spec}_i\,\mathcal{L}$. One shows without difficulty that this is a symmetric bitopological space.

However, there is little else one can show about symmetric d-frames and we doubt that there is much use for the concept at this level of generality. The problem is that there is no link between the symmetry operation and the predicates con and tot. Indeed, from a semantic point of view it seems natural further to require the laws

$$\begin{array}{llcl} (\mathrm{i}_{\mathsf{con}}) & (x,y) \in \mathsf{con} & \Longleftrightarrow & x \sqcap i(y) = 0 \\ (\mathrm{i}_{\mathsf{tot}}) & (x,y) \in \mathsf{tot} & \Longleftrightarrow & x \sqcup i(y) = 1 \end{array}$$

We take this as our official definition.

Definition 9.1 *A* symmetric d-frame *is a d-frame equipped with an isomorphism* $i\colon L_- \to L_+$ *satisfying* ($\mathrm{i}_{\mathsf{con}}$) *and* ($\mathrm{i}_{\mathsf{tot}}$). *The isomorphism* i *will also be referred to as the* symmetry operation.

Homomorphisms of symmetric d-frames are required to preserve the symmetry operation.

At first glance, symmetric d-frames appear to be quite rich structures, but in actual fact, they are nothing else but ordinary frames; the first component L_+ completely specifies (up to isomorphism) the whole structure. This correspondence allows us to compare the concepts introduced in this paper with their classical frame-theoretic counterparts. For example, the following observations are easily checked:

Proposition 9.2 *Let* $\mathcal{L} = (L_+, L_-; tt, ff; \mathsf{con}, \mathsf{tot}; i)$ *be a symmetric d-frame.*

1. F *is a completely prime filter of* L_+ *iff* $(F, i^{-1}(F))$ *is a symmetric d-point of* $\mathcal{L}$.
2. $\mathcal{L}$ *is reasonable and satisfies all cut rules.*
3. $\mathcal{L}$ *is spatial (regular, compact) if and only if* L_+ *is a spatial (regular, compact) frame.*

We can also show that Lemma 6.5 specialises to the usual Hofmann-Mislove Theorem for frames, in the sense that any Scott-open filter $\mathcal{S}$ on a frame L has a partner $\mathcal{U}$ which satisfies conditions ($\mathrm{hm}_{\mathsf{con}}$) and ($\mathrm{hm}_{\mathsf{tot}}$) of 6.5. As shown there, we should examine $L\backslash{\downarrow}u$ where $u = \bigsqcup^{\uparrow}\{b \mid \exists a \in \mathcal{S}.\, a \sqcap b = 0\}$. By construction, ($\mathrm{hm}_{\mathsf{con}}$) holds, and we check ($\mathrm{hm}_{\mathsf{tot}}$): Assume $x \sqcup u = 1 \in \mathcal{S}$, then there are a, b with $a \in \mathcal{S}$ and $a \sqcap b = 0$ such that $x \sqcup b \in \mathcal{S}$, because $\mathcal{S}$ is Scott-open. Now we have $a \sqcap x = (a \sqcap x) \sqcup (a \sqcap b) = a \sqcap (x \sqcup b) \in \mathcal{S}$ because $\mathcal{S}$ is a filter. It follows that x belongs to $\mathcal{S}$ as required by ($\mathrm{hm}_{\mathsf{tot}}$).

From a semantic point of view, the universal way for obtaining a symmetric space from a general bitopological space $(x; \tau_+, \tau_-)$ is to construct the common refinement $\tau_+ \vee \tau_-$. The d-frame analogue of this is the free biframe over a d-frame, for which we gave two constructions in Section 5.2. This yields a left adjoint to the forgetful functor from symmetric d-frames to **dFrm** as follows:

Let $\mathcal{L} = (L_+, L_-; \mathsf{con}, \mathsf{tot})$ be a d-frame and $h = (h_+, h_-)$ a **dFrm** morphism from $\mathcal{L}$ to a symmetric d-frame $\mathcal{M} = (M_+, M_-; \mathsf{con}', \mathsf{tot}'; i')$. Let L_0 be the frame defined by generators and relations as in the proof of Theorem 5.9. We interpret the generators in M_+ by setting $[\![\ulcorner x \urcorner]\!] = h_+(x)$ for $x \in L_+$, and $[\![\ulcorner y \urcorner]\!] = i(h_-(y))$ for $y \in L_-$, and check the relations. For example, if $\langle x, y\rangle \in \mathsf{con}$ then $\langle h_+(x), h_-(y)\rangle \in \mathsf{con}'$ which by symmetry is equivalent to $h_+(x) \sqcap i'(h_-(y)) = 0$ and from this we read $[\![\ulcorner x \urcorner \sqcap \ulcorner y \urcorner]\!] = [\![\ulcorner x \urcorner]\!] \sqcap [\![\ulcorner y \urcorner]\!] = h_+(x) \sqcap i'(h_-(y)) = 0$. Hence $[\![\cdot]\!]$ extends to a unique frame homomorphism $h_0 \colon L_0 \to M_+$.

From L_0 we obtain the symmetric d-frame $\mathcal{L}_0 = (L_0, L_0; \mathsf{con}_0, \mathsf{tot}_0; \mathsf{id})$, where con_0 and tot_0 are equal to $\sqcap^{-1}(0)$ and $\sqcup^{-1}(1)$, respectively, and h_0 we extend to the symmetric d-frame homomorphism $(h_0, i'^{-1} \circ h_0)$ from $\mathcal{L}_0$ to $\mathcal{M}$: $h_0 \circ \mathsf{id} = h_0 = i' \circ (i'^{-1} \circ h_0)$. It extends the **dFrm** morphism (h_+, h_-) we started with because $h_+(x) = [\![\ulcorner x \urcorner]\!] = h_0[\ulcorner x \urcorner] = h_0 \circ \eta_+(x)$ and $h_-(y) = i'^{-1}([\![\ulcorner y \urcorner]\!]) = i'^{-1}(h_0[\ulcorner y \urcorner]) = i'^{-1} \circ h_0 \circ \eta_-(y)$ where $\eta_+ \colon L_+ \to L_0$ and $\eta_- \colon L_- \to L_0$ are the frame homomorphisms that map an element to (the equivalence class of) the corresponding generator of L_0.

9.1.2 D-frames with negation

Instead of symmetry one can alternatively axiomatise the operation $\langle x, y\rangle \mapsto \langle i(y), i^{-1}(x)\rangle$ on $L = L_+ \times L_-$. This, of course, is the d-frame version of Belnap's negation discussed at the beginning of this section; we denote it by $\neg$. One sees without difficulties that it satisfies the following axioms:

$$\begin{array}{rcl} \gamma \sqsubseteq \delta & \Rightarrow & \neg\gamma \sqsubseteq \neg\delta \\ \neg \circ \neg & = & \mathsf{id}_L \\ \gamma \in \mathsf{con} & \Leftrightarrow & \gamma \sqcap \neg\gamma = \bot \\ \gamma \in \mathsf{tot} & \Leftrightarrow & \gamma \sqcup \neg\gamma = \top \end{array}$$

We call a map $\neg$ with these properties on a reasonable d-frame $L = L_+ \times L_-$ a *negation*. It is again easy to see that a negation gives rise to a symmetry between L_- and L_+ when restricted to L_-: It preserves the frame structure because it is monotone and its own inverse. Further, the element ff is mapped to tt, which is the only element which makes the last two rules true for $\gamma = \mathit{ff}$. It follows that the interval $[\bot, \mathit{ff}] = L_-$ is mapped isomorphically to $[\bot, \mathit{tt}] = L_+$.

We define the category $\mathbf{dFrm}_\neg$ to consist of objects $(L; \mathit{tt}, \mathit{ff}; \mathsf{con}, \mathsf{tot}; \neg)$ where $(L; \mathit{tt}, \mathit{ff}; \mathsf{con}, \mathsf{tot})$ is a reasonable d-frame and $\neg$ is a negation on it. The morphisms are d-frame morphisms that preserve $\neg$. We stress that it follows from the discussion above that $\mathbf{dFrm}_\neg$ is equivalent to **Frm**, the category of frames. Nonetheless, we are interested how negation manifests itself on partial frames.

Definition 9.3 *A* partial frame with negation *is a structure* $(P; \sqcap, \bigsqcup^{\uparrow}, \bot; \wedge, \vee, \neg, \mathit{tt}, \mathit{ff}; \prec)$ *which satisfies the conditions for a partial frame (Definition 7.2) and also*

$$\begin{array}{lll} (\neg\text{–}\sqsubseteq) & \gamma \sqsubseteq \delta \implies \neg\gamma \sqsubseteq \neg\delta \\ (\neg\text{–}\neg) & \neg \circ \neg = \mathsf{id}_P \\ (\neg\text{–}\prec) & \lambda \wedge \gamma \prec \delta \iff \lambda \prec \delta \vee \neg\gamma \\ (\neg\text{–}\sqcap) & \gamma \sqcap \neg\delta = \bot \iff \{\gamma, \delta\} \textit{ is bounded w.r.t. } \sqsubseteq \end{array}$$

As well as the other operations, homomorphisms must also respect negation.

Theorem 9.4 *The category* $\mathbf{dFrm}_\neg$ *is equivalent to the category of partial frames with negation.*

Proof. The constructions are the same as the ones we used to prove Proposition 7.4 and Theorem 7.5, and we only need to check that the two notions of negation agree.

Regarding the translation from $\mathbf{dFrm}_\neg$ to partial frames, note that a negation on a d-frame can be restricted to the subset of elements that satisfy the con-predicate. This is so because with $\gamma \sqcap \neg\gamma = \bot$ one also has $\neg(\gamma \sqcap \neg\gamma) = \neg\gamma \sqcap \gamma = \bot$. From the axioms only ($\neg$–$\prec$) and ($\neg$–$\sqcap$) can be in doubt. For the former consider $\lambda \wedge \gamma \prec \delta$; by definition, this is equivalent to $(\lambda \wedge \gamma)_- \sqcup \delta_+ \in \mathsf{tot}$, which by symmetry and the distributivity laws is equivalent to

$$\lambda_- \sqcup \gamma_- \sqcup \delta_+ \sqcup \neg(\lambda_-) \sqcup \neg(\gamma_-) \sqcup \neg(\delta_+) = \top.$$

The right hand side in ($\neg$–$\prec$), $\lambda \prec \delta \vee \neg\gamma$, rewrites to the same equality using the fact that $(\neg\gamma)_+ = \gamma_-$.

In order to check that ($\neg$–$\sqcap$) is valid, let $\gamma \sqcap \neg\delta = \bot$. To show that $\gamma \sqcup \delta \in \mathsf{con}$ we must show $(\gamma \sqcup \delta) \sqcap \neg(\gamma \sqcup \delta) = \bot$. Rewriting the left-hand side by distributivity, we obtain the join of four terms: $(\gamma \sqcap \neg\gamma)$, $\delta \sqcap \neg\delta$, $\gamma \sqcap \neg\delta$ and $\delta \sqcap \neg\gamma$. The first two equal $\bot$ because $\gamma, \delta \in \mathsf{con}$, the second two equal $\bot$ by assumption. For the reverse implication assume $\gamma, \delta \sqsubseteq \lambda \in \mathsf{con}$. Then $\gamma \sqcap \neg\delta \sqsubseteq \lambda \sqcap \neg\lambda = \bot$ since $\lambda \in \mathsf{con}$.

As a homomorphism of symmetric d-frames "does the same" on the two constituent frames L_+ and L_-, the corresponding partial frame homomorphism respects exchange of the two components γ_+, γ_- of an element γ.

In the reverse direction we associate with a given partial frame the (product of the) two frames $L_+ = [\bot, \mathit{tt}]$ and $L_- = [\bot, \mathit{ff}]$. Negation establishes a bijection between these: First of all, $\bot$ is the smallest element, so $\bot \sqsubseteq \neg\bot$, but then $\neg\bot \sqsubseteq \neg\neg\bot = \bot$ by the (information order) monotonicity of negation. Second, $\mathit{tt} \prec \mathit{tt} = \mathit{tt} \vee \mathit{ff}$ implies $\neg\mathit{tt} = \mathit{tt} \wedge \neg\mathit{tt} \prec \mathit{ff}$ by ($\neg$–$\prec$), and $\mathit{ff} \wedge \mathit{tt} = \mathit{ff} \prec \mathit{ff}$ implies $\mathit{ff} \prec \mathit{ff} \vee \neg\mathit{tt} = \neg\mathit{tt}$. Since $\prec \subseteq \leq$, we can conclude $\neg\mathit{tt} = \mathit{ff}$, and hence $\mathit{tt} = \neg\neg\mathit{tt} = \neg\mathit{ff}$, too.

Next we check that the con and tot predicates, as defined in Proposition 7.4, are derived from the frame structure, say of L_+, alone. Assume $\varphi \sqsubseteq \mathit{tt}$, $\psi \sqsubseteq \mathit{ff}$. By definition, we have $(\varphi, \psi) \in \mathsf{con}$ if and only if $\{\varphi, \psi\}$ is bounded, and by ($\neg$–$\sqcap$) this happens if and only if $\varphi \sqcap \neg\psi = \bot$. Regarding tot, we have $(\varphi, \psi) \in \mathsf{tot}$ if and only if $\psi \prec \varphi$ by definition. Using ($\neg$–$\prec$) this is equivalent to $\mathit{tt} \prec \varphi \vee \neg\psi$, and this happens if and only if $\mathit{tt} = \varphi \vee \neg\psi$.

If $h\colon \mathcal{P} \to \mathcal{P}'$ is a homomorphism of partial frames with negation then the restriction to $[\bot, tt]$ and $[\bot, ff]$, respectively, is a pair of frame homomorphisms as required for a map of d-frames. They respect the symmetry operation because that is just negation restricted to $[\bot, ff]$, and h is required to preserve it. ∎

It may be worthwhile to point out that both directions of ($\neg$–$\sqcap$) are indispensable for the equivalence proof. Consider the free partial frame over one generator depicted in Figure 5 on page 74. Reflection at the vertical axis of symmetry satisfies ($\neg$–$\sqsubseteq$), ($\neg$–$\neg$), and ($\neg$–$\prec$) but not ($\neg$–$\sqcap$), and indeed, this structure does not arise from a symmetric d-frame. Next let $L_+ = L_-$ be the four-element Boolean algebra and let the symmetry operation be identity. The subset $P = \{\langle x, y\rangle \in L \mid x = 0 \text{ or } y = 0\}$ is a partial frame and exchanging the components of a pair satisfies all conditions of Definition 9.3 except the direction from left to right in ($\neg$–$\sqcap$).

9.2 Negation as a structural property

9.2.1 Negation on regular d-frames

Let us compare the treatment of negation in d-frames with the classical situation. There, the existence of a negation operation is a purely *structural property* of the logic; in the language of bounded distributive lattices, it is expressed by the (first-order) formula

$$\forall x\; \exists x'.\, x \wedge x' = ff \quad \text{and} \quad x \vee x' = tt$$

One shows that there can be only one x' that is related to a given x in this way, and so if the formula holds in a bounded distributive lattice L, then a negation operation can be defined by setting $\neg x := x'$. The laws of Boolean algebras hold and furthermore, homomorphisms of bounded lattices preserve negation.

For general d-frames the situation is quite different. The mere existence of an isomorphism between L_+ and L_-, even when it satisfies ($\mathrm{i}_{\mathsf{con}}$) and ($\mathrm{i}_{\mathsf{tot}}$), does not mean that it is uniquely determined or that it will be preserved by **dFrm** homomorphisms. Consequently, the usual spectrum $\mathsf{spec}\,\mathcal{L}$ of such a d-frame need not even be a symmetric bitopological space. An example is shown in the lower right hand corner of Figure 1 on page 29, where both L_+ and L_- are the three-element chain, con and tot are minimal (while reasonable), and ($\mathrm{i}_{\mathsf{con}}$) and ($\mathrm{i}_{\mathsf{tot}}$) are satisfied, but τ_+ and τ_- on $2.2 = \mathsf{spec}\,3.3$ are quite different.

Indeed, any frame L gives rise to an example of this nature: add new bottom $0'$ and top $1'$ to L to obtain L', and define con and tot minimally:

$$\begin{array}{rcl} (x,y) \in \mathsf{con} & :\Leftrightarrow & x = 0' \quad \text{or} \quad y = 0' \\ (x,y) \in \mathsf{tot} & :\Leftrightarrow & x = 1' \quad \text{or} \quad y = 1' \end{array}$$

This yields a (reasonable) d-frame $\mathcal{L}' := (L', L'; \mathsf{con}, \mathsf{tot})$ that is symmetric. This example also shows that the symmetry operation i is not uniquely determined by ($\mathrm{i}_{\mathsf{con}}$) and ($\mathrm{i}_{\mathsf{tot}}$): any automorphism of L gives rise to a symmetry on $\mathcal{L}'$.

The example works because of the paucity of con and tot in $\mathcal{L}'$. Regularity is the exact opposite of this situation, and so the following result should not be too much of a surprise:

Proposition 9.5 *Let $\mathcal{L}$ be a symmetric regular d-frame. Then,*

1. *L_+ and L_- are regular as individual frames;*
2. *the symmetry operation is uniquely determined;*
3. *the spectrum of $\mathcal{L}$ is a symmetric bitopological space.*

Furthermore, **dFrm** *homomorphisms between symmetric regular d-frames preserve the symmetry operation.*

Proof. In the context of this proof, we write $\lhd_d$ for the well-inside relation of the d-frame, and $\lhd_+$ for the well-inside relation of the individual frame L_+.

(1) For two elements x', x of L_+ we have $x' \lhd_d x$ iff $\exists r \in L_-.\,(x', r) \in \mathsf{con}, (x, r) \in \mathsf{tot}$ which by ($\mathrm{i}_{\mathsf{con}}$) and ($\mathrm{i}_{\mathsf{tot}}$) is equivalent to $\exists r \in L_-.\, x' \sqcap i(r) = 0, x \sqcup i(r) = 1$ and this just says $x' \lhd_+ x$ with witness $i(r)$ (in L_+).

(2) Let i' be a symmetry operation satisfying ($\mathrm{i}_{\mathsf{con}}$) and ($\mathrm{i}_{\mathsf{tot}}$). We have $x \lhd_+ i'(y)$ iff $\exists r \in L_+.\, x \sqcap r = 0, i'(y) \sqcup r = 1$ iff $\exists r \in L_+.\, x \sqcap r = 0, (r, y) \in \mathsf{tot}$ iff $\exists r \in L_+.\, x \sqcap r = 0, i(y) \sqcup r = 1$. The last formula implies that $x \sqsubseteq i(y)$ and by regularity $i'(y) \sqsubseteq i(y)$ follows.

(3) Let (F_+, F_-) be an abstract point of $\mathcal{L}$. We have $(F_+, F_-) \in \Phi_+(i(y))$ iff $i(y) \in F_+$ iff $\exists x' \lhd_+ i(y).\, x' \in F_+$. If $r \in L_+$ is a witness for $x' \lhd_+ i(y)$ then because of $r \sqcap x' = 0$ we must have $r \notin F_+$ and so $y \in F_-$ follows because $r \sqcup i(y) = 1$ is equivalent to $(r, y) \in \mathsf{tot}$. In other words, we obtain that the abstract point (F_+, F_-) belongs to $\Phi_-(y)$.

In order to show that in the presence of regularity, homomorphisms preserve symmetry, let $h = (h_+, h_-)\colon (L_+, L_-; \mathsf{con}, \mathsf{tot}; i) \to (M_+, M_-; \mathsf{con}', \mathsf{tot}'; i')$ be a d-frame homomorphism. We use an argument similar to that in (2) to show that $h_+ \circ i = i' \circ h_-$. Let $y \in L_-$ be arbitrary. For any $a \in L_+$ such that $a \lhd_+ i(y)$ we have a witness $r \in L_+$ such that $a \sqcap r = 0$ and $i(y) \sqcup r = 1$. The latter is equivalent to $(r, y) \in \mathsf{tot}$, so we get $h_+(a) \sqcap h_+(r) = 0$ and $(h_+(r), h_-(y)) \in \mathsf{tot}'$ in $\mathcal{M}$. The latter is equivalent to $i'(h_-(y)) \sqcup h_+(r) = 1$ and we conclude $h_+(a) \sqsubseteq i'(h_-(y))$. Taking the supremum of all such $a \in L_+$ we obtain

$$h_+(i(y)) \sqsubseteq i'(h_-(y)) \ .$$

By exploiting the regularity of L_-, one shows in exactly the same way that $h_-(i^{-1}(x)) \sqsubseteq i'^{-1}(h_+(x))$ for all $x \in L_+$. This allows us to compute the other inequality

$$i'(h_-(y)) = i'(h_-(i^{-1}(i(y)))) \sqsubseteq i'(i'^{-1}(h_+(i(y)))) = h_+(i(y))$$

and thus complete the proof. ∎

On partial frames, the assumption of regularity renders negation a purely logical concept:

Proposition 9.6 *If $\mathcal{P}$ is a regular partial frame and $\neg\colon P \to P$ is a map that satisfies $(\neg{-}\neg)$ and $(\neg{-}\prec)$, then $(\neg{-}\sqsubseteq)$ and $(\neg{-}\sqcap)$ also hold. Furthermore, the de Morgan rule $\neg(\gamma \vee \delta) = \neg\gamma \wedge \neg\delta$ is valid.*

Proof. Throughout this proof we make heavy use of the fact (shown in Proposition 7.4) that every partial frame can be seen as the set of pairs satisfying the con-predicate in a concrete d-frame, and with the partial frame structure given concretely as described at the beginning of Section 7.

$(\neg{-}\sqsubseteq)$: In a regular partial frame, $\gamma \sqsubseteq \delta$ is equivalent to $\mathsf{I}\gamma \subseteq \mathsf{I}\delta$ and $\mathsf{F}\gamma \subseteq \mathsf{F}\delta$. In any partial frame satisfying $(\neg{-}\neg)$and $(\neg{-}\prec)$, $\gamma \prec \delta$ is equivalent to $\neg\delta \prec \neg\gamma$. So $\mathsf{I}\gamma \subseteq \mathsf{I}\delta$ is equivalent to $\mathsf{F}\neg\gamma \subseteq \mathsf{F}\neg\delta$.

Regarding $(\neg{-}\sqcap)$ we begin by showing that $\lambda \sqcap \neg\lambda = \bot$ is always true in a regular partial frame. From $\varphi \prec \lambda$ we get $\varphi \wedge \neg\lambda \prec \mathit{ff}$, so $\varphi \wedge \neg\lambda \leq \mathit{ff}$ and hence $(\varphi \wedge \neg\lambda)_+ = \varphi_+ \sqcap (\neg\lambda)_+ = 0$ by Lemma 7.3(9). Likewise, $\lambda \prec \psi$ implies $\mathit{tt} \prec \psi \vee \neg\lambda$, so $0 = (\psi \vee \neg\lambda)_- = \psi_- \sqcap (\neg\lambda)_-$. Using the characterisation of regularity in Proposition 7.10, we thus get $\lambda \sqcap \neg\lambda = \bigsqcup^{\uparrow}_{\varphi\prec\lambda\prec\psi}(\varphi \sqcap \psi \sqcap \neg\lambda) = \bigsqcup^{\uparrow}(\varphi_+ \sqcap (\neg\lambda)_+, \psi_- \sqcap (\neg\lambda)_-) = (0,0) = \bot$. Now if $\gamma, \delta \sqsubseteq \lambda$ then we just use $(\neg{-}\sqsubseteq)$ and obtain $\gamma \sqcap \neg\delta \sqsubseteq \lambda \sqcap \neg\lambda = \bot$.

For the reverse direction assume $\gamma \sqcap \neg\delta = \bot$, and also $\varphi \prec \delta \prec \psi$. We have the following relationships:

1. $\gamma_+ \sqcap (\neg\delta)_+ = 0$ in the frame $[\bot, \mathit{tt}]$ by assumption;
2. $\gamma_- \sqcap (\neg\delta)_- = 0$ in the frame $[\bot, \mathit{ff}]$ by assumption;
3. $\varphi_+ \sqsubseteq \psi_+$ because $\varphi \prec \psi$ implies $\varphi \leq \psi$;
4. $\varphi_- \sqsupseteq \psi_-$ for the same reason;
5. $\psi_+ \sqcup (\neg\delta)_+ = 1$ in the frame $[\bot, \mathit{tt}]$ because $\mathit{tt} \prec \psi \vee \neg\delta$ by $(\neg{-}\prec)$;
6. $\varphi_- \sqcup (\neg\delta)_- = 1$ in the frame $[\bot, \mathit{ff}]$ because $\varphi \wedge \neg\delta \prec \mathit{ff}$ by $(\neg{-}\prec)$;
7. $\gamma_+ = \gamma_+ \sqcap 1 = \gamma_+ \sqcap (\psi_+ \sqcup (\neg\delta)_+) = (\gamma_+ \sqcap \psi_+) \sqcup (\gamma_+ \sqcap (\neg\delta)_+) = \gamma_+ \sqcap \psi_+$, using (5) and (1);
8. $\gamma_- = \gamma_- \sqcap 1 = \gamma_- \sqcap (\varphi_- \sqcup (\neg\delta)_-) = (\gamma_- \sqcap \varphi_-) \sqcup (\gamma_- \sqcap (\neg\delta)_-) = \gamma_- \sqcap \varphi_-$, using (6) and (2);
9. $[(\gamma \wedge \psi) \vee \varphi \ \sqcap\ (\gamma \vee \varphi) \wedge \psi]_+ = ((\gamma_+ \sqcap \psi_+) \sqcup \varphi_+) \sqcap ((\gamma_+ \sqcup \varphi_+) \sqcap \psi_+) = (\gamma_+ \sqcup \varphi_+) \sqcap ((\gamma_+ \sqcap \psi_+) \sqcup (\varphi_+ \sqcap \psi_+)) = \gamma_+ \sqcup \varphi_+$, using (3) and (7);
10. $[(\gamma \wedge \psi) \vee \varphi \ \sqcap\ (\gamma \vee \varphi) \wedge \psi]_- = ((\gamma_- \sqcup \psi_-) \sqcap \varphi_-) \sqcap ((\gamma_- \sqcap \varphi_-) \sqcup \psi_-) = ((\gamma_- \sqcap \varphi_-) \sqcup (\psi_- \sqcap \varphi_-)) \sqcap (\gamma_- \sqcup \psi_-) = \gamma_- \sqcup \psi_-$, using (4) and (8).

It follows that the collection $\{(\gamma \wedge \psi) \vee \varphi \ \sqcap\ (\gamma \vee \varphi) \wedge \psi \mid \varphi \prec \delta \prec \psi\} = \{(\gamma_+ \sqcup \varphi_+, \gamma_- \sqcup \psi_-) \mid \varphi \prec \delta \prec \psi\}$ is directed and that the supremum is above γ and δ.

The de Morgan rule is a simple sequence of equivalences, exploiting regularity: $\varphi \prec \neg\gamma \wedge \neg\delta \ \Leftrightarrow\ \varphi \prec \neg\gamma \ \&\ \varphi \prec \neg\delta \ \Leftrightarrow\ \gamma \prec \neg\varphi \ \&\ \delta \prec \neg\varphi \ \Leftrightarrow\ \gamma \vee \delta \prec \neg\varphi \ \Leftrightarrow\ \varphi \prec \neg(\gamma \vee \delta)$. ∎

In the compact regular case we would like to replace the partial frame $\mathcal{P}$ by the strong proximity lattice structure on a dense subset X, as explained in Section 8.1. The negation operation need not map basis elements to basis elements but this should not disturb us as the idempotency of $\neg$ means that it is enough to add the image $\neg(X)$ to X. Because the de Morgan rules are valid, the result will again be a sub-lattice of $\mathcal{P}$. So we define:

Definition 9.7 *A unary operation* $\neg$ *on a strong proximity lattice is called a* negation *if it satisfies* $(\neg{-}\neg)$ *and* $(\neg{-}{\prec})$.

Theorem 9.8 *The category of compact regular partial frames with negation is equivalent to the category of strong proximity lattices with negation.*

Proof. We examine how the constructions that led to Theorem 8.6 interact with negation. Starting with a strong proximity lattice we define a negation on the set of round ideal-filter pairs by setting $\neg(I, F) := (\neg F, \neg I)$. Repeated use of $(\neg{-}{\prec})$ shows that this is again a round ideal-filter pair; for example, if $\neg\gamma, \neg\delta \in \neg F$, then by roundness there is $\lambda \in F$ such that $\lambda \prec \gamma \wedge \delta$, so $\lambda \prec \gamma$ and $\lambda \prec \delta$; it follows that $\neg\gamma \vee \neg\delta \prec \neg\lambda \in \neg F$.

By Proposition 9.6 it remains to check $(\neg{-}{\prec})$: $(I, F) \wedge (I', F') \prec (I'', F'')$ means that there is an element γ in $(F \sqcup F') \cap I''$. By the formula for the join of two round filters there must exist $\delta \in F$, $\delta' \in F'$ with $\delta \wedge \delta' \prec \gamma$, which is equivalent to $\delta \prec \gamma \vee \neg\delta'$. Thus δ is an element of $F \cap (I'' \sqcup \neg F')$, and $(I, F) \prec (I'', F'') \vee (\neg F', \neg I')$ is established. ∎

Proposition 9.9 *A* **Prox** *morphism* $(\vdash^\diamond, \vdash_\diamond)$ *between strong proximity lattices with negation satisfies* $\varphi \vdash_\diamond \psi \iff \neg\psi \vdash^\diamond \neg\varphi$.

Proof. We exploit the equivalence between **Prox** and the category of compact regular d-frames. The desired result follows from the fact that a **dFrm** morphism h between regular d-frames respects negation, as shown in Proposition 9.5. For the associated **Prox** morphism $(\vdash^h, \vdash_h)$ we compute $\psi \vdash^h \varphi \iff \psi \prec h(\varphi) = \neg h(\neg\varphi) \iff h(\neg\varphi) \prec \neg\psi \iff \neg\varphi \vdash_h \neg\psi$. ∎

Corollary 9.10 (Moshier 2004) *The category of strong proximity lattices with negation is dually equivalent to the category of compact Hausdorff spaces.*

Proof. By Proposition 6.4 the spectrum of a strong proximity lattice is order-separated, and since the two topologies are the same in the symmetric setting, this means that it is a Hausdorff space. Everything else remains the same as in the general (i.e., non-symmetric) case. ∎

9.2.2 A proof-theoretic characterisation of negation

The existence of a negation operation can be deduced from a structural property of strong implication. This was first discovered for strong proximity lattices by the second author in [Mos04] but it holds in the more general context of regular partial frames. Recall once again that an element γ of a regular partial frame $\mathcal{P}$ is completely determined by the sets $\mathsf{I}\gamma = \{\varphi \mid \varphi \prec \gamma\}$ and $\mathsf{F}\gamma = \{\psi \mid \gamma \prec \psi\}$, in the sense that $\gamma = \bigsqcup^{\uparrow}\{\varphi \sqcap \psi \mid \varphi \in \mathsf{I}\gamma, \psi \in \mathsf{F}\gamma\}$. If there is a negation operation on $\mathcal{P}$ and if we want to capture an element $\neg\gamma$ in this way, then by $(\neg-\prec)$ we can express $\mathsf{I}\neg\gamma$ and $\mathsf{F}\neg\gamma$ alternatively by $\{\varphi \mid \varphi \wedge \gamma \prec \mathit{ff}\}$ and $\{\psi \mid \mathit{tt} \prec \gamma \vee \psi\}$. Now, these sets can be defined in any partial frame, whether there is a negation or not, so let us call the first one $\bar{\mathsf{I}}\gamma$ and the second $\overline{\mathsf{F}}\gamma$. If we assume that Gentzen's cut rule is valid (cf. Proposition 7.6) then for every $\varphi \in \bar{\mathsf{I}}\gamma$, $\psi \in \overline{\mathsf{F}}\gamma$, $\varphi \prec \psi$ holds. We can then form

$$\overline{\gamma} := \bigsqcup{}^{\uparrow}\{\varphi \sqcap \psi \mid \varphi \in \bar{\mathsf{I}}\gamma, \psi \in \overline{\mathsf{F}}\gamma\}$$

which is directed by Lemma 7.8. As in the proof of 7.10 we know that $\overline{\gamma} = \langle \bigsqcup^{\uparrow}_{\varphi \in \bar{\mathsf{I}}\gamma} \varphi_+, \bigsqcup^{\uparrow}_{\psi \in \overline{\mathsf{F}}\gamma} \psi_- \rangle$.

As an aside, this provides an alternative proof for the fact that negation on a regular partial frame is unique if it exists.

In general, there is not much that we can say about the operation $\gamma \mapsto \overline{\gamma}$ but with a few more assumptions it moves closer to being a proper negation:

Lemma 9.11 *Let $\mathcal{P}$ be a partial frame satisfying the infinitary Gentzen cut rules* (G-CUT$_l$) *and* (G-CUT$_r$) *(cf. Table 2). Then for all $\gamma \in \mathcal{P}$, $\overline{\overline{\gamma}} \sqsubseteq \gamma$.*

Proof. Suppose $\psi \in \overline{\mathsf{F}}\overline{\gamma}$, that is, $tt \prec \psi \vee \overline{\gamma}$. By definition of $\overline{\gamma}$ as a directed supremum of meets plus $(\prec - \sqsubseteq)$, $tt \prec \psi \vee (\bigsqcup^{\uparrow} \bar{\mathsf{I}}\gamma)$ holds. Also by definition, for each $\varphi \in \bar{\mathsf{I}}\gamma$, $\varphi \wedge \gamma \prec ff$ holds. So (G-Cut$_r$) implies $\gamma \prec \psi$. Thus $\overline{\mathsf{F}}\overline{\gamma} \subseteq \mathsf{F}\gamma$. Similarly, $\bar{\mathsf{I}}\overline{\gamma} \subseteq \mathsf{I}\gamma$ by (G-Cut$_l$). Hence, $\overline{\overline{\gamma}} \sqsubseteq \gamma_\circ \sqsubseteq \gamma$. ∎

Next we observe that in the presence of interpolativity of $\prec$ and negation, Gentzen's cut rule can be *inverted*: Whenever $\gamma \wedge \gamma' \prec \delta \vee \delta'$ then this can be rewritten as $\gamma \wedge \neg\delta \prec \neg\gamma' \vee \delta'$; by interpolation there is some φ such that $\gamma \wedge \neg\delta \prec \varphi \prec \neg\gamma' \vee \delta'$, and another application of $(\neg - \prec)$ yields $\gamma \prec \delta \vee \varphi$ and $\varphi \wedge \gamma' \prec \delta'$. This motivates our main result on the matter:

Theorem 9.12 *Let $\mathcal{P}$ be a regular partial frame. The following are equivalent:*

(i) The infinitary Gentzen cut rules (G-CUT$_l$) *and* (G-CUT$_r$) *hold, and Gentzen's finitary cut rule can be inverted.*

(ii) The operation $\gamma \mapsto \overline{\gamma}$ is a negation on $\mathcal{P}$, and $\prec \subseteq \prec\circ\prec$.

Proof. (i) $\Rightarrow$ (ii): We already know $\overline{\overline{\gamma}} \sqsubseteq \gamma$ by the preceding lemma. For the converse, consider $\varphi \in \mathsf{I}\gamma$, which is equivalent to $tt \wedge \varphi \prec \gamma \vee ff$. By an application of (G$-$cut)$^{-1}$ we obtain an element ψ such that $tt \prec \gamma \vee \psi$ and $\varphi \wedge \psi \prec ff$. By regularity, $\alpha \prec ff$ holds if and only if $\alpha = ff$. Because we have $\varphi \wedge \psi \prec ff$ and $\varphi \wedge ff \prec ff$, it follows that $\varphi \wedge (\psi \sqcap ff) \prec ff$. But $\psi \in \overline{\mathsf{F}}\gamma$ and $ff \in \bar{\mathsf{I}}\gamma$, so $(\prec - \sqsubseteq)$ implies $\varphi \wedge \overline{\gamma} \prec ff$. Thus $I\gamma \subseteq \bar{\mathsf{I}}\overline{\gamma}$, and similarly $F\gamma \subseteq \overline{\mathsf{F}}\overline{\gamma}$. So by regularity, $\gamma = \gamma_\circ \sqsubseteq \overline{\overline{\gamma}}$.

(ii) $\Rightarrow$ (i): The validity of all cut rules was stated already as Proposition 9.2. Invertibility of Gentzen's cut rule was presented above. ∎

Proposition 6.13 and Lemma 8.1 provide us with the following special case:

Corollary 9.13 *A compact regular partial frame carries a negation if and only if Gentzen's cut rule is invertible.*

Corollary 9.14 *The full subcategory of* **pFrm** *consisting of compact regular partial frames in which Gentzen's cut rule is invertible is dually equivalent to the category of compact Hausdorff spaces.*

9.2.3 Boolean algebras

We are ready to take the final step towards Boolean algebras.

Definition 9.15 *A partial frame is called* Boolean *if in addition to satisfying the conditions of a Stone partial frame, it admits a negation.*

Theorem 9.16 *The category of Boolean partial frames is equivalent to the category of Boolean algebras.*

Proof. We already know that the set of reflexive elements forms a sub-lattice; negation restricts to this because $\gamma \prec \gamma$ implies $\neg\gamma \prec \neg\gamma$. We showed in Proposition 9.6 that de Morgan's laws hold and this is sufficient to prove that the set of reflexive elements is a Boolean algebra. Everything else is a special case of Theorem 8.10. ∎

10 Discussion

In Figure 7 we have listed the main dualities discussed in the paper. It illustrates our central contention that the classical Stone dualities are best understood as special cases of the dual adjunction between bitopological spaces and d-frames. The accompanying Figure 8 displays the dualising objects as sub-structures of 2.2, with or without symmetry. It shows quite clearly that the dualiser $\mathbb{B} = \{\mathit{tt}, \mathit{ff}\}$ in the bottom row is different from the dualiser $2 = \{0, 1\}$ for frames, and thus that the duality of Boolean algebras is *not* a special case of that of frames.

In the traditional understanding of the situation, expounded for example in [Joh82], the connection between the finitary structures of Boolean algebras and distributive lattices is given by ideal completion of the *logical order*, and by selecting the sub-poset of compact elements, in the other. This works up to a point, but note that one must restrict frame homomorphisms to those that preserve $\ll$, a condition that is not really justified when considering a single topology.

Indeed, this methodology breaks down when one tries to apply it to the middle layer of strong proximity lattices in Figure 7. While (round) ideal completion

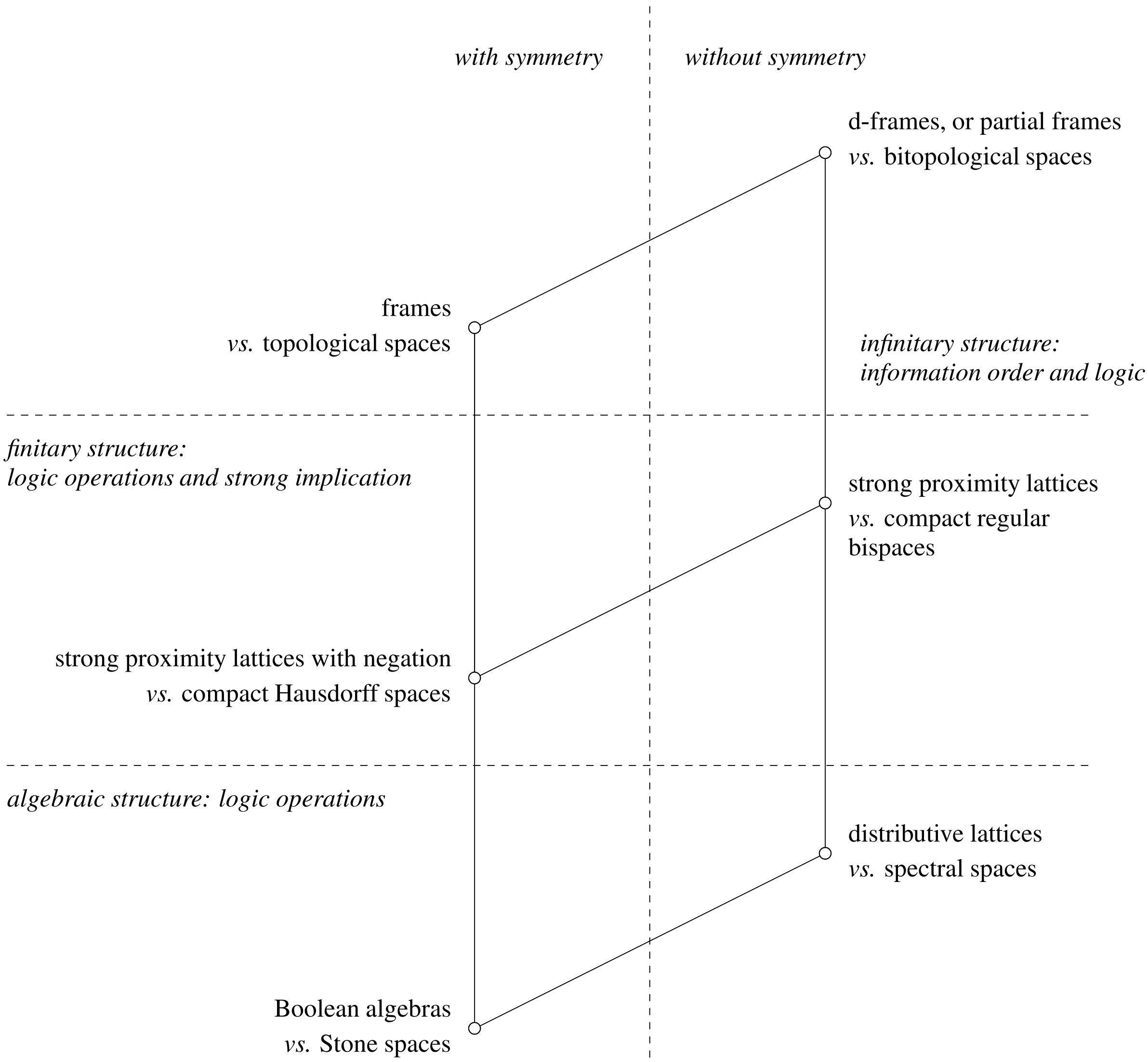


Figure 7: A hierarchy of Stone-type dualities.

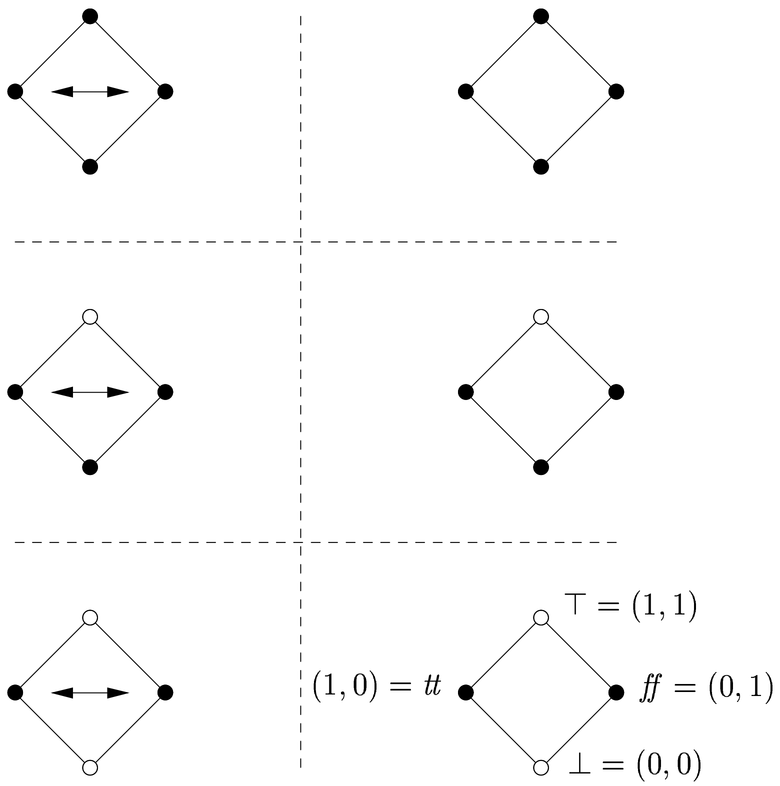


Figure 8: The dualising d-frames for the dualities of Figure 7. In the second row, solid lines and filled elements indicate the dualising compact regular partial frame. In the third row, filled elements indicate the dualising distributive lattice (in the "logical" order). Double arrows indicate negation.

can still be applied, there is no way to recover the finitary structure as a *substructure* of the associated stably continuous frame. We believe that it is the (rather amazing) fact that a compact regular d-frame is completely determined by its first component, Theorem 6.15, that has obscured the bitopological character of Stone duality for so long. The coincidence of the well-inside relation $\lhd$ with the way-below relation $\ll$ in a compact regular d-frame, Lemma 6.10, also pointed us in the wrong direction. As the bitopological treatment makes clear, the strong implication $\prec$ of partial frames in general captures the former and not the latter.

With this paper we have not answered all questions that naturally present themselves when one generalises from frames to d-frames. We consider the following as the most important open problems that remain.

Open Problem 1 *Does the Hofmann-Mislove Theorem 6.6 hold for general d-frames, or under more general assumptions than those made in 6.6?*

We have invested a considerable amount of energy on this question and now believe the answer to be "no."

Open Problem 2 *Is every reasonable d-frame that satisfies the infinitary cut rules derived from a biframe?*

We included some thoughts on this question in Section 5, after Proposition 5.8; our conjecture is that this is true.

Open Problem 3 *Develop a notion of "locally compact d-frame."*

Not just any notion will do; we would expect that with the right definition a locally compact d-frame would be spatial. If things work really well, then one could also hope for a link between local compactness and exponentiability in **dFrm**.

Open Problem 4 *Develop a "generators and relations" method of constructing d-frames.*

In the construction of a free biframe from a d-frame, we employed the technique of covers on a semi-lattice. This same construction yields the symmetrisation of a d-frame. But as noted in Section 5.2, the resulting elements of the constructed frame are characterised as sub-lattices with respect to $\leq$ and as being Scott closed with respect to $\sqsubseteq$ (along with other conditions particular to this construction). This suggests that objects may be constructed from generators and a combination of logical and informational relations. The separate roles that logic and information play in such a construction should help illuminate their relationships more generally.

Priestley duality considers the *join* of the two natural topologies on the spectrum of a distributive lattice. By also keeping the specialisation order as explicit data, it manages to faithfully capture the underlying bitopology. This works equally well for strong proximity lattices (Proposition 2.12) but one wonders whether there is a more general principle available. Ultimately, one will need biframes, which essentially add the two basic topologies explicitly. Since we now understand how free biframes arise from d-frames, a more finitistic construction may be possible. Again, logic and information are likely to play distinct roles in such a construction.

In this study, we have defined d-frames and shown that they constitute the Stone duals of bitopological spaces. As a result, we are able to see several known Stone-type duality results as special cases of this. D-frames also open up the study of bitopological spaces to the "point-free" techniques that have become so useful in topology. In particular, studies of d-sobriety for bitopological spaces and spatiality for d-frames show that these concepts are quite rich. Bitopological analogues of the Hofmann-Mislove Theorem are available and find very natural ties to Escardó's approach to quantification over subsets of a topological space. D-frames also yield many "choice-free" proofs of classical results and have helped to further illuminate the relationship between regularity and continuity.

Perhaps the most surprising feature of d-frames is the natural split they make between logic and information. This was only partly anticipated when we began this investigation nearly two years ago, and yet has turned out to be an organising theme throughout this paper. Many proofs are significantly simplified, particularly for reasonable d-frames, by maintaining logic and information as separate, but interacting, notions. Indeed, one indication that logical aspects of d-frames are "naturally occurring" is the repeated appearance of Gentzen's cut rules, and their reflection in d-frames, in unexpected places.

Partial frames provide a general topological theory of partial propositional logic that accounts for partiality in terms of information order where accumulation of information is the main operation, and logic where conjunction and disjunction are the main operations, and strong implication is distinct from $\leq$. In this regard, one can regard partial frames as the Lindenbaum algebras for a generalised form of Kleene's three-valued logic. This generalisation points out the importance of treating information as well as strong implication as distinct concepts.

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