# Uniform Approximation of Topological Spaces

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## Abstract

We sharpen the notion of a quasi-uniform space to spaces which carry with them *functional* means of approximating points, opens and compacts. Assuming nothing but sobriety, the requirement of uniform approximation ensures that such spaces are compact ordered (in the sense of Nachbin). We study uniformly approximated spaces with the means of topology, uniform topology, order theory and locale theory. In each case it turns out that one can give a succinct and meaningful characterization. This leads us to believe that uniform approximation is indeed a concept of central importance.

## 1  Introduction

In this paper we re-visit the time-honored subject of compact ordered spaces, first introduced by Leopoldo Nachbin in 1948 [Nac48]. These are compact Hausdorff spaces endowed with a partial order relation which is closed as a subset of the cartesian product of the space with itself. Both order and topology can be recovered from the collection of open upper sets which in itself forms a (typically non-Hausdorff) topology. The topological spaces

arising in this fashion can be characterized independently and show many similarities with compact Hausdorff spaces. Indeed, one may say that they are the $T_0$-analogue of compact Hausdorff spaces, and, going one step further, that they occupy an even more canonical and central position in topology than the former. Discussions of these and further connections may be found in [GHK+80, Chapter VII-1] and [Law88].

In [JS96] we showed how these *coherent spaces*, as we chose to call them, can be described via certain distributive lattices. The most striking feature of that work is that a faithful and satisfactory algebraic representation can be obtained by considering both open and compact upper sets.

Topology provides us with a notion of nearness and convergence but not by itself with means of *approximation*. We will explore below a particular suggestion of what it means that a coherent space is *uniformly approximated*. Our definition is very simple and natural: we require that the canonical quasi-uniformity be generated by continuous functions. We will show that this stipulation leads to ordered structures which have arisen in the denotational semantics of programming languages, the so-called *FS-domains* [Jun90, AJ94]. This is our first main result. It places these structures, which were developed with rather different motivations, in the context of classical topology. We then go on to extend the framework of [JS96] to deal with these quantitative aspects of coherent spaces. Rather pleasingly, the localic treatment is also very smooth and elegant.

We have tried to keep this paper self-contained as much as possible but since it draws together concepts from a number of different areas, the reader may at times wish to get more background information. We recommend [AJ94, Chapter 7] and [Joh82] for Stone-duality, [DP90] for order theory, [AJ94, SHLG94] for domain theory, and [FL82] for quasi-uniform spaces.

## 2 Topology and order

Most topological spaces arising in Mathematics satisfy the Hausdorff Separation Axiom and subsets of the plane generally suffice to illustrate Hausdorff topological concepts. $T_0$-spaces, in contrast, appear to be anarchic and strange at first sight. Yet, there is a simple way to develop just as useful and general intuition about them as for their Hausdorff counterparts. The key to understanding $T_0$-spaces is provided by the *specialization order*, defined by

$$x \sqsubseteq_s y \quad \text{if} \quad x \in \mathsf{cl}\{y\} \ .$$

It is immediate that $\sqsubseteq_s$ is indeed a partial order and that open sets are always upper sets with respect to $\sqsubseteq_s$. Vice versa, sets of the form $X \setminus {\downarrow}x$ are always open, indeed ${\downarrow}x = \{y \in X \mid y \sqsubseteq_s x\}$ equals $\mathsf{cl}\{x\}$. Of course, there may be more open sets around. Thus we may visualize $T_0$-spaces as ordered sets together with a topology of certain upper sets. Since $X \setminus {\downarrow}x$ is always open it follows that every upper set is equal to the intersection of all its open neighborhoods.

Throughout this paper we will always assume that our spaces are *sober*. This can be interpreted as either a certain completeness of the space of points or as a certain richness in terms of open sets. Technically, a space is *sober*, if every closed set is the closure of a unique point or the union of two proper subsets. [GHK+80, Joh82, AJ94] give more details on this concept.

An ordered set, in which every directed subset has a supremum, is called a *directed complete partial order* or *dcpo*, for short. We write directed suprema as $\bigsqcup^{\uparrow} x_i$. It follows immediately from Stone-duality that a sober space is directed complete in its specialization order. On a dcpo one defines the *Scott-topology* whose closed sets are those which are closed under suprema of directed sets. It is again immediate from Stone-duality that every open set in a sober space is Scott-open with respect to the specialization order.

A map $f$ between dcpo's is called *Scott-continuous* if it is monotone and preserves least upper bounds of directed sets. One can show that the concept of Scott-continuity coincides with topological continuity in terms of the respective Scott-topologies. On the other hand, a continuous map between sober spaces is also Scott-continuous with respect to the specialization orders even though the topology of a sober space may be weaker than the associated Scott-topology. (The proof is the same as in [AJ94, Proposition 2.3.4].)

Besides open sets we will make crucial use of *compact upper sets*. Simple examples are sets of the form ${\uparrow}x$. We denote the collection of all compact upper sets of a topological space $X$ by $\mathsf{K}_X$. When we think of this collection as an ordered set in its own right then it is advantageous to use $\supseteq$ rather than $\subseteq$ as the order relation.

The single most important property of sober spaces is the so-called *Hofmann-Mislove Theorem* [HM81, KP94, AJ94] which states that $(K_X, \supseteq)$ is isomorphic to the set of Scott-open filters on $X$. We will access it through two of its consequences:

**Proposition 1** *Let $(X, \mathcal{T})$ be a sober space. Then*

   *1. $(\mathsf{K}_X, \supseteq)$ is a dcpo;*

2. *if the filtered intersection of compact upper sets $K_i$ is contained in an open set $O$, then some $K_i$ is contained in $O$.*

**Lemma 2** *Let $f\colon X \to Y$ be a continuous function where both $X$ and $Y$ are sober. Define a function $f_{\mathsf{K}}\colon \mathsf{K}_X \to \mathsf{K}_Y$ by $f_{\mathsf{K}}(K) := \mathord{\uparrow} f(K)$. Then $f_{\mathsf{K}}$ is Scott-continuous.*

**Proof.** Assume that $(K_i)_{i \in I}$ is a filtered family of compact upper subsets of $X$. Let $O$ be an open set in $Y$. We argue as follows to show that it is a neighborhood of $\mathord{\uparrow} f(\bigcap K_i)$ if and only if $\bigcap \mathord{\uparrow} f(K_i)$ is contained in $O$:

$$
\begin{aligned}
\mathord{\uparrow} f(\textstyle\bigcap K_i) \subseteq O \quad &\Longleftrightarrow \quad f(\textstyle\bigcap K_i) \subseteq O \\
&\Longleftrightarrow \quad \textstyle\bigcap K_i \subseteq f^{-1}(O) \\
(\text{Proposition 1}) \quad &\Longleftrightarrow \quad \exists i.\ K_i \subseteq f^{-1}(O) \\
&\Longleftrightarrow \quad \exists i.\ f(K_i) \subseteq O \\
&\Longleftrightarrow \quad \exists i.\ \mathord{\uparrow} f(K_i) \subseteq O \\
(\text{Proposition 1}) \quad &\Longleftrightarrow \quad \textstyle\bigcap \mathord{\uparrow} f(K_i) \subseteq O
\end{aligned}
$$

This concludes the proof because every upper set equals the intersection of its open neighborhoods. ∎

In a dcpo we say that $x$ *approximates* $y$ (or $x$ is *way-below $y$*), and we write $x \ll y$, if every directed set $A$ whose supremum is above $y$ contains an element above $x$. This concept arose in the theory of continuous lattices [GHK+80] but it is also present in many arguments from topology and analysis, though not always fully explicit.

**Lemma 3** *Let $(X, \mathcal{T})$ be a sober space. Then for $O, O' \in \mathcal{T}$, $K, K' \in \mathsf{K}_X$ we have*

$$
\begin{aligned}
O \subseteq K \subseteq O' \quad &\text{implies} \quad O \ll O' \quad \text{in } (\mathcal{T}, \subseteq); \\
K \subseteq O \subseteq K' \quad &\text{implies} \quad K' \ll K \quad \text{in } (\mathsf{K}_X, \supseteq).
\end{aligned}
$$

*If, in addition, $X$ is locally compact then the converses also hold.*

# 3 Coherent spaces and uniform approximation

We cite from [Nac65] the following definition.

**Definition 4** *A* compact ordered space *is given by a set $X$ together with a compact Hausdorff topology and a partial order, which is closed in the product topology.*

It was observed by the authors of [GHK$^+$80] that compact ordered spaces can be characterized purely topologically as follows.

**Definition 5** *A topological space $X$ which is sober, compact and locally compact and in which $K \cap K'$ is compact for all $K, K' \in \mathsf{K}_X$, is called* coherent.

Note that since we do not require the $T_2$ Separation Axiom we have to require local compactness and the intersection property explicitly. For a Hausdorff space these properties follow from compactness.

**Definition 6** *If $(X, \mathcal{T})$ is a topological space then the* cocompact topology $\mathcal{T}_c$ *is generated by the collection of all sets of the form $X \setminus K$, where $K \in \mathsf{K}_X$. The* patch topology $\mathcal{T}_p$ *is the common refinement of $\mathcal{T}$ and $\mathcal{T}_c$.*

It is an easy consequence of Proposition 1 that for coherent spaces all open sets in the cocompact topology have the form $X \setminus K$, $K \in \mathsf{K}_X$.

We are now in a position to make the connection between coherent and compact ordered spaces precise.

**Theorem 7**   *1. Let $(X, \mathcal{T}, \leq)$ be a compact ordered space. The collection of all open upper sets forms a topology $\mathcal{T}^{\uparrow}$ on $X$ and $(X, \mathcal{T}^{\uparrow})$ is coherent.*

   *2. Let $(X, \mathcal{T})$ be a coherent space. Then $X$ together with the patch topology and the specialization order is compact ordered.*

   *3. The translations in (1) and (2) are inverses of each other.*

The proof of (1) consists of showing that there are sufficiently many open upper sets, while the main hurdle in (2) is to show that every compact upper set is also compact with respect to the patch topology. See [GHK$^+$80, VII.1 Exercises] for details.

We now come to our main topic, the problem of approximating a topological space. The necessary quantitative information is traditionally captured by uniformities, and, in the non-Hausdorff setting, by quasi-uniformities. (For general information see [FL82].) For coherent spaces there is already a smooth theory at hand, going back to Nachbin and later developed by Künzi and Brümmer [KB87].

**Theorem 8** *Every coherent space $(X, \mathcal{T})$ carries a unique quasi-uniformity $\mathcal{U}$ such that $\mathcal{T}(\mathcal{U}) = \mathcal{T}$ and $\mathcal{T}(\mathcal{U}^{-1}) = \mathcal{T}_c$. It consists of all $\mathcal{T}_p \times \mathcal{T}_p$-neighborhoods of $\sqsubseteq_s$ in $X \times X$. This structure is also unique with regard to the properties $\mathcal{T}(\mathcal{U}^*) = \mathcal{T}_p$ and $\leq_{\mathcal{U}} = \sqsubseteq_s$. ([KB87])*

We propose to go a step further and to require that the space is equipped with functions which yield uniform approximations to points, opens, and compacts. We formalize this idea as follows.

**Definition 9** *Let $(X, \mathcal{T})$ be a topological space. A continuous function $f \colon X \to X$ is said to be* uniformly approximating *if for all $O \in \mathcal{T}$ there exists $K \in \mathsf{K}_X$ such that $f^{-1}(O) \subseteq K \subseteq O$ and if for all $K \in \mathsf{K}_X$ there exists $O \in \mathcal{T}$ such that $K \subseteq O \subseteq \uparrow f(K)$.*

Note that this is a purely topological definition. It stipulates that $f$ provides "neighborhoods" for opens and compacts alike. Since it is a function, it also yields approximations to points. (Note that, by $\uparrow x \in \mathsf{K}_X$, we will always have $f(x) \sqsubseteq_s x$.) Uniform approximation seems to be a desirable property — if you can get it! Indeed, non-trivial connected $T_1$ spaces will never allow such maps; there the specialization order is trivial and the condition $K \subseteq O \subseteq \uparrow f(K)$ implies that $f$ is constant and every point an open set. If we give up on $T_1$ separation then examples abound. For example, on the unit interval with the upper topology $\{(a, 1] \mid 0 \leq a \leq 1\} \cup \{[0, 1]\}$ the functions $f_\epsilon(x) = \max\{0, x - \epsilon\}$ are uniformly approximating for $\epsilon > 0$.

The recent work of Abbas Edalat [Eda94, Eda95] illustrates quite clearly the need to have an effective notion of approximation if one wants to perform actual calculations (for example, integration) over a space. For his purposes, Edalat replaces the unit interval by the space of all subintervals of the unit interval. His work provided part of the motivation for the investigations reported in the present paper.

We give two alternative formulations of uniform approximation, one in the spirit of topology and one in the spirit of domain theory.

**Definition 10** *For a function $f \colon X \to Y$ between topological spaces we define the* hypergraph *of $f$ by $U_f := \{(x, y) \mid f(x) \sqsubseteq_s y\}$.*

**Lemma 11**  *1. Let $X$ be a topological space and $f \colon X \to X$ be uniformly approximating. Then the hypergraph of $f \circ f$ is a $\mathcal{T}_c \times \mathcal{T}$-neighborhood of $\sqsubseteq_s$ in $X \times X$.*

2. *Let $X$ be coherent and $f: X \to X$ be a continuous function whose hypergraph is a $\mathcal{T}_c \times \mathcal{T}$-neighborhood of $\sqsubseteq_s$ in $X \times X$. Then $f$ is uniformly approximating.*

**Proof.** (1) Assume $x \sqsubseteq_s y$. Since $f$ is monotone with respect to the specialization order we have $\uparrow f(\uparrow x) = \uparrow f(x)$ for the compact upper set $\uparrow x$. By assumption, there is an open set $W$ with $x \in W \subseteq \uparrow f(x)$. This will be the $y$-part of the $\mathcal{T}_c \times \mathcal{T}$-neighborhood we are searching for. For the $x$-part, observe that $O := f^{-1}(X \setminus \downarrow f(x))$ does not contain $x$ and again by assumption there is a compact upper set $K$ such that $f^{-1}(O) \subseteq K \subseteq O$. We let $V := X \setminus K$. Now each element of $W$ is above $f(x)$ and every element of $V$ is mapped below $f(x)$ by $f^2$. Hence $V \times W$ is contained in $U_{f \circ f}$.

(2) For a single element $x \in X$ we have $(x, x) \in \sqsubseteq_s$ and so there is a $\mathcal{T}_c \times \mathcal{T}$ basic open $S := (X \setminus K) \times O$ which contains $(x, x)$ and is contained in $U_f$. This means $f(a) \sqsubseteq_s b$ whenever $(a, b) \in S$. In particular, every element of $O$ is above $f(x)$. Hence we have $x \in O \subseteq \uparrow f(x)$. The extension to arbitrary compact upper sets is straightforward.

Next let $W$ be an open set in $X$. For every element $x \notin W$ we let $S = (X \setminus K) \times O$ be a $\mathcal{T}_c \times \mathcal{T}$ basic open neighborhood of $(x, x)$ contained in $U_f$ as before. Now we may conclude that for every element $a$ in $X \setminus K$ we have $f(a) \sqsubseteq_s x$ and hence $f(a) \notin W$. Dually, every element of $f^{-1}(W)$ is contained in $K$. The intersection of all such compact upper sets is contained in $W$. It is compact because of coherence. ∎

Now we employ order theory to describe uniform approximation. It is clear that we can reformulate uniform approximation as $f^{-1}(O) \ll O$ in $(\mathcal{T}, \subseteq)$ and $\uparrow f(K) \ll K$ in $(\mathsf{K}_X, \supseteq)$ because Lemma 3 applies. This requires more concepts but once the order theoretic language is accepted, it also amplifies the simplicity of our definition.

More in the spirit of order theory is the following concept:

**Definition 12** *A Scott-continuous function $f$ on a dcpo $D$ is said to be finitely separated from $\mathsf{id}_D$, if there exists a finite subset $M$ of $D$ such that for all $x \in D$ there exists $m \in M$ with $f(x) \leq m \leq x$. The same concept applies to continuous functions between sober spaces because these are Scott-continuous with respect to the specialization order. specialization orders.*

**Lemma 13** *If $D$ is a dcpo and $f: D \to D$ is finitely separated from $\mathsf{id}_D$ then for all $x \in D$ we have $f(x) \ll x$.*

**Proof.** Let $x \in D$ and $A \subseteq D$ be directed such that $x \le \bigsqcup^\uparrow A$. Further let $M \subseteq_{\text{fin}} D$ be the finite separating set for $f$. For each $m \in M$ let $A_m$ be the set of those elements $a$ of $A$ for which $f(a) \le m \le a$. By assumption, $A = \bigcup_{m \in M} A_m$ and since $A$ is directed, some $A_{m_0}$ is cofinal in $A$. Scott-continuity of $f$ yields $f(x) = f(\bigsqcup^\uparrow A) = f(\bigsqcup^\uparrow A_{m_0}) = \bigsqcup^\uparrow_{a \in A_{m_0}} f(a) \le m_0 \le a \in A_{m_0} \subseteq A$. ∎

It appears that finite separation is slightly stronger than uniform approximation:

**Lemma 14**     *1. If $(X, \mathcal{T})$ is sober and locally compact and $f \colon X \to X$ is finitely separated from $\mathsf{id}_X$ then $f$ is uniformly approximating.*

    *2. If $(X, \mathcal{T})$ is coherent and $f \colon X \to X$ is uniformly approximating then $f \circ f$ is finitely separated from $\mathsf{id}_X$.*

**Proof.**     (1) Let $M \subseteq_{\text{fin}} X$ be the separating subset for $f$. If $O$ is open then one sees immediately that $f^{-1}(O) \subseteq {\uparrow}(M \cap O) \subseteq O$ and ${\uparrow}(M \cap O)$ is the interpolating compact upper set. For the corresponding property for compact upper sets we use the fact that $f_\mathsf{K} \colon \mathsf{K}_X \to \mathsf{K}_X$ is Scott-continuous and that $\{{\uparrow}N \mid N \subseteq M\}$ is a finite separating set for $f_\mathsf{K}$. By the previous lemma (since $\supseteq$ is the order on $\mathsf{K}_X$), we infer that $f_\mathsf{K}(K) \ll K$ in $\mathsf{K}_X$ which implies $K \subseteq O \subseteq {\uparrow}f(K) = f_\mathsf{K}(K)$ for some $O \in \mathcal{T}$ by Lemma 3.

(2) For $x \in X$, ${\uparrow}x$ is compact and ${\uparrow}f({\uparrow}x) = {\uparrow}f(x)$ because $f$ is monotone with respect to the specialization order. By assumption there is an open set $U$ with $x \in U \subseteq {\uparrow}f(x)$. Furthermore, $O := f^{-1}(X \setminus {\downarrow}f(x))$ does not contain $x$ and again by assumption there is a compact upper set $K$ such that $f^{-1}(O) \subseteq K \subseteq O$. The set $U \setminus K$ is open in the patch topology. It contains $x$ and for each of its elements $y$, $f(x)$ separates between $f^2(y)$ and $y$: The inequality $f(f(y)) \sqsubseteq_s f(x)$ holds because $y \notin K \supseteq f^{-1}(f^{-1}(X \setminus {\downarrow}f(x)))$; $f(x) \sqsubseteq_s y$ is true because $y \in U \subseteq {\uparrow}f(x)$. Because $X$ is coherent it is compact Hausdorff in the patch topology and finitely many such open sets cover the whole space. The corresponding elements "$f(x)$" constitute a finite separating set. ∎

**Definition 15** *A sober space $X$ is said to be* uniformly approximated *if there exists a directed family of uniformly approximating functions $(f_i)_{i \in I}$ whose pointwise supremum (with respect to the specialization order) equals the identity on $X$.*

Note that if $\bigsqcup^{\uparrow}_{i \in I} f_i = \mathsf{id}_X$ then also $\bigsqcup^{\uparrow}_{i \in I} f_i^2 = \mathsf{id}_X$ because composition of functions is Scott-continuous. The slight differences showing up in Lemmas 11 and 14 are therefore of no importance when it comes to uniformly approximated spaces.

The unit interval with the functions $f_\epsilon$ from above is an example of a uniformly approximated space. A more systematic way to construct examples of such spaces is described in the following theorem.

**Theorem 16** *If $X$ is a compact Hausdorff space then $\mathsf{K}_X \setminus \{\emptyset\}$ together with the Scott-topology derived from $\supseteq$ is a uniformly approximated space. The function $\eta: X \to \mathsf{K}_X$, $x \mapsto \{x\}$, is a topological embedding.*

This is a direct consequence of standard results in domain theory which ensure that $\mathsf{K}_X \setminus \{\emptyset\}$ is a so-called *bc-domain* [AJ94, Definition 4.1.1(2)], which in turn are FS-domains [AJ94, Proposition 4.2.12] and hence uniformly approximated by Theorem 21 below.

It should also be mentioned that the Scott-topology on $(\mathsf{K}_X \setminus \{\emptyset\}, \supseteq)$ is nothing but the "upper topology" of Vietoris [Vie21], generated by sets of the form $\Box O := \{K \in \mathsf{K}_X \setminus \{\emptyset\} \mid K \subseteq O\}$, $O \in \mathcal{T}$. The claim about $\eta$ being an embedding is then obvious.

It is presently unknown which Hausdorff spaces allow an embedding into a uniformly approximated space (as the set of maximal points), though progress in that direction has recently been made by Lawson [Law95] and Edalat and Heckmann [EH95].

**Lemma 17** *Let $(X, \mathcal{T})$ be uniformly approximated by the functions $f_i$, $i \in I$. Then*
$$\forall O \in \mathcal{T}. \ O = \bigcup_{i \in I} f_i^{-1}(O)$$
$$\forall K \in \mathsf{K}_X. \ K = \bigcap_{i \in I} \uparrow f_i(K)$$

**Proof.** Every open set $O$ in a sober space is Scott-open with respect to the specialization order. If $x \in O$ then by assumption $x = \bigsqcup^{\uparrow}_{i \in I} f_i(x)$ and from the definition of the Scott-topology it follows that some $f_{i_0}(x)$ belongs to $O$. Hence $x \in f_{i_0}^{-1}(O)$.

Next let $O$ be an open neighborhood of some $K \in \mathsf{K}_X$. For every $x \in K$ there is $i_x \in I$ with $f_{i_x}(x) \in O$ as we have just seen. Hence $K \subseteq \bigcup_{i \in I} f_i^{-1}(O)$ and by compactness it follows that $K$ is contained in some $f_{i_0}^{-1}(O)$. This is tantamount to $f_{i_0}(K) \subseteq O$ and since $K$ equals the intersection of its open neighborhoods it also equals $\bigcap_{i \in I} \uparrow f_i(K)$. ∎

**Theorem 18** *A coherent space is uniformly approximated if and only if its canonical quasi-uniformity has a base consisting of hypergraphs of continuous functions.*

**Proof.** First of all, observe that $\bigsqcup^{\uparrow}_{i \in I} f_i = \mathrm{id}_X$ is equivalent to $\bigcap_{i \in I} U_{f_i} = \sqsubseteq_s$. Now assume that a base for the canonical quasi-uniformity consisting of hypergraphs of continuous functions is given for the coherent space $X$. It follows from Theorem 8 that for every $\mathcal{T}_p \times \mathcal{T}_p$-neighborhood $U$ of $\sqsubseteq_s$, $U \circ U \circ U$ is a $\mathcal{T}_c \times \mathcal{T}$-neighborhood. The claim now follows from Lemma 11(2).

Conversely, Lemma 11(1) shows that hypergraphs of uniformly approximating functions are entourages of the canonical quasi-uniformity. It remains to prove that they constitute a base. Now for any continuous function $f$, its hypergraph equals $(f \times \mathrm{id}_X)^{-1}(\sqsubseteq_s)$. Thus the hypergraphs are certainly $\mathcal{T} \times \mathcal{T}_c$-closed in $X \times X$ and hence $\mathcal{T}_p \times \mathcal{T}_p$-compact. We conclude, employing Proposition 1, that every neighborhood of $\sqsubseteq_s$ contains some $U_{f_i}$. ∎

**Lemma 19** *A uniformly approximated space is coherent and its topology coincides with the Scott-topology.*

**Proof.** Let $O$ be a Scott-open set in $(X, \sqsubseteq_s)$ and $x \in O$. There must be an index $i_x$ such that $f_{i_x}(x) \in O$ because $x = \bigsqcup^{\uparrow}_{i \in I} f_i(x)$. Since $f_{i_x}$ is uniformly approximating, $\uparrow f_{i_x}(x)$ is a compact neighborhood of $\uparrow x$. This shows that every Scott-open set belongs to $\mathcal{T}$ and that the space is locally compact. Compactness is trivial since there is at least one uniformly approximating function on $X$.

For the intersection property assume $K, K' \in \mathsf{K}_X$. The set $K \cap K'$ equals the intersection of all $O \cap O'$ where $O$ is a neighborhood of $K$ and $O'$ is a neighborhood of $K'$. Let $i \in I$ be such that both $f_i(K) \subseteq O$ and $f_i(K') \subseteq O'$. Such an index must exist by Lemma 17. It follows that $K \cap K' \subseteq f_i^{-1}(O) \cap f_i^{-1}(O') = f_i^{-1}(O \cap O') \ll O \cap O'$ and we see that $K \cap K'$ equals the filtered intersection of its compact neighborhoods. By Proposition 1(1) it is itself compact. ∎

**Definition 20** *A dcpo $D$ is called an* FS-domain *if there exists a directed family of Scott-continuous functions on $D$, each of which is finitely separated from $\mathrm{id}_D$ and whose pointwise supremum equals $\mathrm{id}_D$.*

These domains were introduced in [Jun90]. They have a property which is rather rare in topology, namely, they form a cartesian closed category. In

[Jun90] it was shown that it is a maximal cartesian closed category among certain dcpo's (the so-called *continuous domains*).

**Theorem 21**     *1. If $(X, \mathcal{T})$ is uniformly approximated, then $(X, \sqsubseteq_s)$ is an FS-domain.*

    *2. If $(D, \sqsubseteq)$ is an FS-domain, then $(D, \Sigma)$ is uniformly approximated (where $\Sigma$ is the Scott-topology on $D$).*

    *3. The translations in (1) and (2) are inverses of each other.*

**Proof.**    Continuous functions on a sober space are Scott-continuous, so (1) follows from the Lemma 14(2). The second part follows from the fact that FS-domains are always coherent [AJ94, Theorem 4.2.18] and hence Lemma 14(1) applies. The specialization order derived from the Scott-topology always coincides with the original order. For uniformly approximated spaces we have shown in Lemma 19 that the topology comprises all Scott-open sets. ∎

Summing up the results of this section we may say that uniformly approximated spaces and FS-domains are one and the same concept, one formulated in the language of topology and the other in the language of order theory.

# 4    Quantitative proximity lattices

In [JS96] we showed how to represent coherent spaces through certain *proximity lattices*. We extend that theory to also deal with the quantitative aspects. We start by recalling the main results from [JS96].

**Definition 22** *A* strong proximity lattice *is a distributive bounded lattice $(B; \vee, \wedge, 0, 1)$ together with a binary relation $\prec$ on $B$ satisfying $\prec^2 = \prec$. The two structures are connected through the following four axioms:*

($\vee$-$\prec$)   $\forall a \in B \; \forall M \subseteq_{fin} B. \; M \prec a \iff \bigvee M \prec a$ ;

($\prec$-$\wedge$)   $\forall a \in B \; \forall M \subseteq_{fin} B. \; a \prec M \iff a \prec \bigwedge M$ ;

($\prec$-$\vee$)   $\forall a, x, y \in B. \; a \prec x \vee y \implies \exists x', y' \in B. \; x' \prec x, \; y' \prec y \; \& \; a \prec x' \vee y'.$

($\wedge$-$\prec$)   $\forall a, x, y \in B. \; x \wedge y \prec a \implies \exists x', y' \in B. \; x \prec x', \; y \prec y' \; \& \; x' \wedge y' \prec a.$

Here $M \prec a$ stands for $\forall m \in M.\ m \prec a$ and similarly for $a \prec M$. Moreover, we use the notation $\uparrow A$ as before to denote the upwards closure but now it refers to $\prec$ rather than the specialization order. Since $\prec$ is not necessarily reflexive, $\uparrow A$ need not contain $A$. But $\uparrow$ is still idempotent because of $\prec^2 = \prec$.

**Definition 23** *Suppose* $(B; \vee, \wedge, 0, 1; \prec)$ *is a strong proximity lattice. We define the set of all* ideals *on* $B$:

$$\mathsf{Idl}(B) = \{I \subseteq B \mid I = {\downarrow}I,\ M \subseteq_{fin} I \Longrightarrow \bigvee M \in I\}\ ;$$

*the set of all* filters *on* $B$:

$$\mathsf{filt}(B) = \{F \subseteq B \mid F = {\uparrow}F,\ M \subseteq_{fin} F \Longrightarrow \bigwedge M \in F\}\ ;$$

*the* spectrum *of* $B$, *which comprises all* prime filters *of* $B$:

$$\mathsf{spec}(B) = \{F \in \mathsf{filt}(B) \mid (M \subseteq_{fin} B\ \&\ \bigvee M \in F) \Longrightarrow M \cap F \neq \emptyset\}\ ;$$

*and for* $x \in B$ *the* basic open set

$$O_x = \{F \in \mathsf{spec}(B) \mid x \in F\}\ .$$

*Finally, let* $\mathcal{T}_B$ *denote the topology on* $\mathsf{spec}(B)$ *generated by the sets* $O_x$, $x \in B$. *We refer to it as the* canonical topology.

**Theorem 24 ([JS96])** *Let* $(B; \vee, \wedge, 0, 1; \prec)$ *be a strong proximity lattice. Then* $X = \mathsf{spec}(B)$ *with the canonical topology is a coherent space, the topology on* $X$ *is isomorphic to* $(\mathsf{Idl}(B), \subseteq)$, *and* $(\mathsf{K}_X, \supseteq)$ *is isomorphic to* $(\mathsf{filt}(B), \subseteq)$. *Furthermore, the map* ${\downarrow}\colon B \to \mathsf{Idl}(B)$ *is a lattice homomorphism and* ${\uparrow}\colon B \to \mathsf{filt}(B)$ *is an anti-homomorphism.*

We refine Definition 22 to deal with the quantitative aspect of coherent spaces as follows.

**Definition 25** *A* quantitative proximity lattice *is given by a distributive bounded lattice* $(B; \vee, \wedge, 0, 1)$ *together with a directed (wrt $\subseteq$) family of transitive relations* $(\prec_i)_{i \in I}$ *which satisfy the interpolation axiom*

**(INT)** $\forall i \in I\ \exists j \in I.\ \prec_i\ \subseteq\ \prec_j \circ \prec_j.$

($\prec$-$\vee$) and ($\wedge$-$\prec$) are satisfied by $\prec = \bigcup_{i \in I} \prec_i$. In place of the other two axioms in Definition 23, we require their adaptations to the quantitative setting:

($\vee$-$\prec_i$) $\forall i \in I \, \forall a \in B \, \forall M \subseteq_{fin} B. \ M \prec_i a \iff \bigvee M \prec_i a;$

($\prec_i$-$\wedge$) $\forall i \in I \, \forall a \in B \, \forall M \subseteq_{fin} B. \ a \prec_i M \iff a \prec_i \bigwedge M;$

Finally, all relations $\prec_i$ have to satisfy the following condition of finiteness:

**(FIN)** $\forall i \in I \, \exists M \subseteq_{fin} B \, \forall a, b \in B. \ a \prec_i b \Longrightarrow \exists m \in M. \ a \prec m \prec b.$

**Remark.** The reader might have noticed the resemblence between quantitative proximity lattices and *syntopologies*. These are systems of strong inclusions on powersets introduced by Á. Császár in [Csá63] as a foundation for general topology. In fact, Császár suggests that his work could be a starting point for pointless topology; one might understand the present paper to move in this direction.

**Proposition 26** *For every quantitative proximity lattice* $(B; \vee, \wedge, 0, 1; (\prec_i)_{i \in I})$, *we have that* $(B; \vee, \wedge, 0, 1; \prec)$ *is a strong proximity lattice.*

**Proof.** Transitivity of the $\prec_i$ together with directedness implies $\prec^2 \subseteq \prec$, Axiom (INT) implies $\prec \subseteq \prec^2$. The axioms ($\vee$-$\prec$) and ($\prec$-$\wedge$) are immediate consequences of their quantitative counterparts. ∎

**Definition 27** *Suppose* $(B; \vee, \wedge, 0, 1; (\prec_i)_{i \in I})$ *is a quantitative proximity lattice and* $i \in I$. *We define a binary relation* $U_i$ *on* $\mathsf{spec}(B)$ *by*

$$F \, U_i \, G \iff \uparrow_i F \subseteq G,$$

*where* $\uparrow_i F = \{x \in B \mid \exists a \in F. \ a \prec_i x\}$. *The filter on* $\mathsf{spec}(B) \times \mathsf{spec}(B)$ *generated by this collection is denoted by* $\mathcal{U}$.

**Proposition 28** *The relations* $U_i$ *form a base for* $\mathcal{U}$. *Furthermore,* $\mathcal{U}$ *is a quasi-uniformity on* $\mathsf{spec}(B)$.

**Proof.** Clearly, $F \, U_i \, F$ holds for all $i \in I$ and all $F \in \mathsf{spec}(B)$. Moreover, $\prec_i \subseteq \prec_j$ implies $U_i \supseteq U_j$. Thus the $U_i$ form indeed a filterbase and axiom (INT) gives us for any $i \in I$ an index $j \in I$ with $U_j^2 \subseteq U_i$. ∎

**Theorem 29** *Let $(B; \vee, \wedge, 0, 1; (\prec_i)_{i \in I})$ be a quantitative proximity lattice and $\mathcal{U}$ as in Definition 27. Then $\mathcal{U}$ is the unique quasi-uniformity with the property that $\mathcal{T}(\mathcal{U})$ is the canonical topology and $\mathcal{T}(\mathcal{U}^*)$ is the patch topology on the coherent space $\mathsf{spec}(B)$.*

**Proof.** To see that $\mathcal{T}(\mathcal{U})$ refines the canonical topology on $\mathsf{spec}(B)$, suppose $F \in O_a$, i.e. $a \in F$. Since $F = \uparrow F$, there is $b \in F$ with $b \prec a$. This means that there is some $i \in I$ with $b \prec_i a \in F$ implying $a \in \uparrow_i F$. Hence $G \in O_a$ whenever $F \, U_i \, G$. This shows $[F]U_i \subseteq O_a$. Conversely, fix $i \in I$. By (FIN), $(\wedge\text{-}\prec)$, and the property of $F$ being a filter, there is some $a \in F$ satisfying $\uparrow_i F \subseteq \uparrow a$. Now for all $G \in \mathsf{spec}(B)$, surely $a \in G$ implies $\uparrow a \subseteq G$, hence $F \, U_i \, G$. Therefore, $F \in O_a \subseteq [F]U_i$.

It remains to prove that the $\mathcal{T}(\mathcal{U}^{-1})$-open sets on $\mathsf{spec}(B)$ are exactly the complements of compact upper sets. The latter sets correspond via $\mathsf{K} \mapsto \bigcap \mathsf{K}$ and $K \mapsto \{F \in \mathsf{spec}(B) \mid K \subseteq F\}$ to $\mathsf{filt}(B)$, the set of all filters on $B$ [JS96, Theorem 27]. Suppose $F \in \mathsf{spec}(B) \setminus \mathsf{K}$ for a compact upper set $\mathsf{K}$, i.e. $K \not\subseteq F$ for some $K \in \mathsf{filt}(B)$. Then there is a point $a \in K$ with $a \notin F$. Since $K = \uparrow K$, there is some $i \in I$ and some $b \in K$ with $b \prec_i a$. Then $U_i[F] \subseteq \mathsf{spec}(B) \setminus \mathsf{K}$: If $\uparrow_i G \subseteq F$, then $b \in G$ implies $a \in F$, contradicting the construction. Hence $G \in U_i[F]$ implies $b \notin G$ which implies $K \not\subseteq G$ and this means $G \in \mathsf{spec}(B) \setminus \mathsf{K}$.

To verify the reverse inclusion of topologies, the goal is, given an index $i \in I$ and some $F \in \mathsf{spec}(B)$, to find $K \in \mathsf{filt}(B)$ such that

$$F \in \{G \in \mathsf{spec}(B) \mid K \not\subseteq G\} \subseteq U_i[F] \, .$$

To this end let $M$ be the finite interpolating set associated with $\prec_i$ whose existence is guaranteed by (FIN) and define $N := \{m \in M \mid \exists a \in B \setminus F. \, m \prec a\}$ and $n := \bigvee N$. Since $N$ is finite and $F$ is a prime filter, the supremum $n$ is not contained in $F$. Furthermore, $N$ is contained in $\downarrow(B \setminus F)$ and so is $n$. We let $K := \uparrow n$. From what we just said, it follows that $K$ is not contained in $F$. Next let $G$ be any prime filter which does not belong to $U_i[F]$. Then $\uparrow_i G \not\subseteq F$ and hence there are elements $a \prec_i b$ such that $a \in G$ and $b \notin F$. Because of (INT) some element of $M$ interpolates between $a$ and $b$. It belongs to $N$ and therefore $n \in G$. ∎

As mentioned above (Theorem 8), every coherent space carries a canonical quasi-uniformity which may be constructed from the topology. On the localic side of the world, this construction is even more transparent:

**Theorem 30** *Suppose $(B; \vee, \wedge, 0, 1; \prec)$ is a strong proximity lattice. Denote the set of all finite 0-1-sublattices of $(B; \vee, \wedge, 0, 1)$ by $\mathbb{F}$. For every $F \in \mathbb{F}$, we define the relation $\prec_F$ on $B$ by*

$$x \prec_F y \iff \exists a, b \in F. \, x \prec a \prec b \prec y.$$

*Then $(B; \vee, \wedge, 0, 1; (\prec_F)_{F \in \mathbb{F}})$ is a quantitative proximity lattice with $\bigcup_{F \in \mathbb{F}} \prec_F = \prec$.*

**Proof.** By distributivity, finitely generated sublattices are finite, hence the set of all $\prec_F$ is directed. If $x \prec y$, then there are $a, b \in B$ with $x \prec a \prec b \prec y$, hence $x \prec_F y$ for $F = \langle a, b \rangle$. Therefore, $\bigcup \prec_F = \prec$.

By the nature of our construction, the relations $\prec_F$ are clearly transitive and satisfy (FIN). To prove (INT), suppose a sublattice $F$ is given. Whenever $a, b \in F$ with $a \prec b$, interpolate twice to get $a', b' \in B$ such that $a \prec a' \prec b' \prec b$. The sublattice $G$ generated by $F$ together with all these new elements is finite. To see that $\prec_F \, \subseteq \, \prec_G \circ \prec_G$, suppose $x \prec_F y$. Then there are $a, b \in F$ with $x \prec a \prec b \prec y$, by construction there are $a', b' \in G$ such that $a \prec a' \prec b' \prec b$. Interpolating between $a'$ and $b'$, we get $z \in B$ with $a' \prec z \prec b'$. Then $x \prec_G z \prec_G y$, thus (INT) holds.

Now for $(\vee\text{-}\prec_F)$. If $M \prec_F x$, then for all $m \in M$, there are $a_m, b_m \in F$ with $m \prec a_m \prec b_m \prec x$. With $a = \bigvee_{m \in M} a_m \in F$ and $b = \bigvee_{m \in M} b_m \in F$, we have $\bigvee M \prec a \prec b \prec x$, thus $\bigvee M \prec_F x$. The reverse implication is trivial. This is ensured by the assumption $0 \in F$. Axiom $(\prec_F\text{-}\wedge)$ follows by symmetry. ∎

In order to achieve uniform approximation we have to change the axioms as follows.

**Definition 31** *A* finitary proximity lattice *$(B; \vee, \wedge, 0, 1; (\prec_i)_{i \in I})$ is a quantitative proximity lattice which satisfies the following quantitative version of $(\prec\text{-}\vee)$:*

$(\prec_i\text{-}\vee)$ *$\forall i \in I \, \forall a, x, y \in B. \, a \prec_i x \vee y \implies \exists x', y' \in B. \, x' \prec_i x, \, y' \prec_i y \, \& \, a \prec x' \vee y'.$*

*Moreover, transitivity of the $\prec_i$ is strengthened to*

$(\prec_i\text{-}\prec)$ *$\forall i \in I. \, \prec_i \circ \prec \, = \, \prec_i.$*

$(\prec\text{-}\prec_i)$ *$\forall i \in I. \, \prec \circ \prec_i \, = \, \prec_i.$*

While we could have required ($\prec_i$-$\prec$) and ($\prec$-$\prec_i$) for quantitative proximity lattices already without losing any generality (that is to say, these two axioms are satisfied by the relations $\prec_F$ constructed in Theorem 30), the change from ($\prec$-$\vee$) to its quantitative version ($\prec_i$-$\vee$) is crucial. Its effect is that each approximating relation defines a *function* on the spectrum which yields the desired uniform approximations to points, opens, and compacts. For each $i \in I$ we can define the corresponding function explicitly as follows

$$f_i \colon \mathsf{spec}(B) \to \mathsf{spec}(B), \ F \mapsto \uparrow_i F = \{a \in B \mid \exists b \in F.\ b \prec_i a\} \ .$$

It is straightforward to check that this map is well-defined and that it is continuous with respect to the canonical topology. We may point out, however, that it is precisely the new axiom ($\prec_i$-$\vee$) which allows us to conclude that $\uparrow_i F$ is prime.

We have cited in Theorem 24 the fact that $\mathsf{Idl}(B)$ is isomorphic to the canonical topology and that $\mathsf{filt}(B)$ is isomorphic to $\mathsf{K}_{\mathsf{spec}(B)}$. Using these isomorphisms one calculates without difficulty the following concrete pendants to $f_i^{-1}$ and $f_{i\mathsf{K}}$:

$$f_{iO} \colon \mathsf{Idl}(B) \to \mathsf{Idl}(B), \ I \mapsto \downarrow_i I$$
$$f_{i\mathsf{K}} \colon \mathsf{filt}(B) \to \mathsf{filt}(B), \ F \mapsto \uparrow_i F$$

The way-below relation on $\mathsf{Idl}(B)$ (and similarly on $\mathsf{filt}(B)$) is characterized by

$$I \ll I' \quad \text{if and only if} \quad \exists a \in I'.\ I \subseteq \downarrow a \ .$$

Because of the condition of finiteness in Definition 25 we see immediately that $f_{iO}(I) \ll O$ and $f_{i\mathsf{K}}(F) \ll F$. Combining this observation with the results in the previous section we conclude:

**Theorem 32** *The spectrum of a finitary proximity lattice is a uniformly approximated space.*

It remains to see that every uniformly approximated space arises as the spectrum of some finitary proximity lattice. The construction is very similar to the construction of representing strong proximity lattices for arbitrary coherent spaces which we gave in [JS96]. Suppose $(X, \mathcal{T})$ is a uniformly approximated space. Let $(f_i)_{i \in I}$ be a directed family of continuous functions on $X$, each finitely separated from $\mathsf{id}_X$ such that $\bigvee^\uparrow_I f_i = \mathsf{id}_X$. (The order-theoretic point of view makes the following proof easier.) We define:

- $B := \{(O, K) \in \mathcal{T} \times \mathsf{K}_X \mid O \subseteq K\}$

- $(O, K) \vee (O', K') := (O \cup O', K \cup K')$

- $(O, K) \wedge (O', K') := (O \cap O', K \cap K')$

- $0 := (\emptyset, \emptyset); \; 1 := (X, X)$

- $(O, K) \prec_i (O', K') :\iff f_i^2(K) \subseteq O' \iff K \subseteq f_i^{-2}(O')$

where we work with $f_i \circ f_i$ rather than $f_i$ as we have done before on several occasions.

**Theorem 33** *If $(X, \mathcal{T})$ is a uniformly approximated space, then the above defined structure is a finitary proximity lattice with $\mathsf{spec}(B) \cong X$.*

**Proof.**    Clearly, the strong proximity lattice constructed in this fashion is the same as the one constructed in Section 6 of [JS96]. Hence we get a strong proximity lattice $B$ with $\mathsf{spec}(B) \cong D$. It is a trivial observation that $(\vee\text{-}\prec_i)$ and $(\prec_i\text{-}\wedge)$ hold. We also know $(\prec\text{-}\vee)$ to hold and from this we deduce $(\prec_i\text{-}\vee)$ by the following trick: If $(O, K) \prec_i (O_1, K_1) \vee (O_2, K_2)$, then certainly $(O, K) \prec (f_i^{-2}(O_1), K_1) \vee (f_i^{-2}(O_2), K_2)$. Now we apply $(\prec\text{-}\vee)$ to get interpolating tokens $x_1$ and $x_2$ with $(O, K) \prec x_1 \vee x_2$ and $x_{1/2} \prec (f_i^{-2}(O_{1/2}), K_{1/2})$. The latter is equivalent to $x_{1/2} \prec_i (O_{1/2}, K_{1/2})$.

Directedness, $(\prec_i\text{-}\prec)$, and $(\prec\text{-}\prec_i)$ are trivial. To see (INT), pick for a given $i \in I$ an index $j \in I$ such that $f_j^4(m) \geq f_i(m)$ for all $m \in M_i$ where $M_i$ is the separating set for $f_i$. Such an $f_j$ must exist because of Lemma 13. It is then obvious that $(f_j \circ f_j)^2 \geq f_i \circ f_i$ and that the associated relations obey $\prec_i \; \subseteq \; \prec_j \circ \prec_j$.

Finally, (FIN) is seen as follows. For each $N \subseteq M_i$ we have a compact upper set $\uparrow N$ and a Scott-open set $\Uparrow N := \{x \in D \mid \exists n \in N. \, n \ll x\}$. Now, if $f_i^2(K) \subseteq O'$, then for $N := M_i \cap O'$ we have $\uparrow N \subseteq O'$. Moreover, for every $x \in K$ there is some $n \in M_i$ with $f_i^2(x) \leq n \leq f_i(x)$. Then $n \in N$ and $n \ll x$ by Lemma 13. Hence $K \subseteq \Uparrow N$. So the token $(\Uparrow N, \uparrow N)$ interpolates between $(O, K)$ and $(O', K')$. Since $M_i$ is finite there are only finitely many of these tokens. ∎

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# References

[AJ94]     S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3, pages 1–168. Clarendon Press, 1994.

[Csá63]    Á. Császár. *Foundations of General Topology*. Pergamon Press, 1963.

[DP90]     B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, Cambridge, 1990.

[Eda94]    A. Edalat. Domain theory and integration. In *Ninth Annual IEEE Symposium on Logic in Computer Science*, pages 115–124. IEEE Computer Society Press, 1994.

[Eda95]    A. Edalat. Dynamical systems, measures and fractals via domain theory. *Information and Computation*, 120(1):32–48, 1995.

[EH95]     A. Edalat and R. Heckmann. Computational models for metric spaces, October 1995. Manuscript.

[FL82]     P. Fletcher and W. F. Lindgren. *Quasi-Uniform Spaces*. Marcel-Dekker, 1982.

[GHK⁺80]  G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *A Compendium of Continuous Lattices*. Springer Verlag, 1980.

[HM81]     K. H. Hofmann and M. Mislove. Local compactness and continuous lattices. In B. Banaschewski and R.-E. Hoffmann, editors, *Continuous Lattices, Proceedings Bremen 1979*, volume 871 of *Lecture Notes in Mathematics*, pages 209–248. Springer Verlag, 1981.

[Joh82]    P. T. Johnstone. *Stone Spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1982.

[JS96]      A. Jung and Ph. Sünderhauf. On the duality of compact vs. open. In S. Andima, R. C. Flagg, G. Itzkowitz, P. Misra, Y. Kong, and R. Kopperman, editors, *Papers on General Topology and Applications: Eleventh Summer Conference at University of Southern Maine*, Annals of the New York Academy of Sciences, 1996. To appear.

[Jun90]     A. Jung. The classification of continuous domains. In *Logic in Computer Science*, pages 35–40. IEEE Computer Society Press, 1990.

[KB87]      H. P. A. Künzi and G. C. L. Brümmer. Sobrification and bicompletion of totally bounded quasi-uniform spaces. *Mathematical Proceedings of the Cambridge Philosophical Society*, 101:237–246, 1987.

[KP94]      K. Keimel and J. Paseka. A direct proof of the Hofmann-Mislove theorem. *Proceedings of the AMS*, 120:301–303, 1994.

[Law88]     J. D. Lawson. The versatile continuous order. In M. Main, A. Melton, M. Mislove, and D. Schmidt, editors, *Mathematical Foundations of Programming Language Semantics*, volume 298 of *Lecture Notes in Computer Science*, pages 134–160. Springer Verlag, 1988.

[Law95]     J. D. Lawson. Spaces of maximal points, October 1995. Manuscript.

[Nac48]     L. Nachbin. Sur les espaces uniformes ordonnés. *C.R.Acad.Sci*, MR 9, 455:774–775, 1948. English translation in [Nac65].

[Nac65]     L. Nachbin. *Topology and Order*. Von Nostrand, Princeton, N.J., 1965. Reprinted by Robert E. Kreiger Publishing Co., Huntington, NY, 1967.

[SHLG94]    V. Stoltenberg-Hansen, I. Lindström, and E. R. Griffor. *Mathematical Theory of Domains*, volume 22 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1994.

[Vie21]     L. Vietoris. Stetige Mengen. *Monatshefte für Mathematik und Physik*, 31:173–204, 1921.

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