The Troublesome Probabilistic Powerdomain

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Abstract

In [12] it is shown that the probabilistic powerdomain of a continuous domain is again continuous. The category of continuous domains, however, is not cartesian closed, and one has to look at subcategories such as **RB**, the retracts of bifinite domains. [8] offers a proof that the probabilistic powerdomain construction can be restricted to **RB**.

In this paper, we give a counterexample to Graham's proof and describe our own attempts at proving a closure result for the probabilistic powerdomain construction. We have positive results for finite trees and finite reversed trees. These illustrate the difficulties we face, rather than being a satisfying answer to the question of whether the probabilistic powerdomain and function spaces can be reconciled.

We are more successful with coherent or Lawson-compact domains. These form a category with many pleasing properties but they fall short of supporting function spaces.

Along the way, we give a new proof of Jones' Splitting Lemma.

1 Introduction

This paper attempts to highlight one of the unresolved issues in the theory of the probabilistic powerdomain. Briefly, the question is whether the probabilistic powerdomain construction can be defined on a universe of semantic domains which is closed under the usual constructions. What we find, in particular, is that the probabilistic powerdomain construction is in conflict with function spaces.

The probabilistic powerdomain was first defined by Saheb-Djahromi in 1980, [26]. It has since been studied extensively by Plotkin, Graham, Jones, Kirch, Heckmann and the second author, [25, 8, 13, 12, 21, 9, 10, 28]. Originally, the probabilistic powerdomain was introduced as a tool in denotational semantics but, more recently, Edalat demonstrated its usefulness in more mainstream mathematics, most notably in the theory of integration, [2, 4, 3].

From a structural point of view, the probabilistic powerdomain construction leads to domains with complex internal structure. Topologically, it produces continuous rather than algebraic domains because of its connection with real numbers. Order-theoretically, it seems to destroy all lattice-like structure (see Example 14 below). The first phenomenon is not particularly worrying because continuous domains have been studied alongside algebraic ones since the very beginning of domain theory, [27, 7, 1]; the second is not new either as the Plotkin powerdomain construction, [24], has a similar effect.

Considering the use of domains in semantics, one would hope for a "universe" of domains which allows one to perform all kinds of constructions easily and without restrictions. Furthermore, one would like these constructions to have good (i.e. meaningful) categorical properties and simple concrete definitions. One way to go about creating such a semantic universe is to concentrate on the categorical properties of the constructions. This is the route taken by axiomatic domain theory, [6, 5]. The more traditional way is to define constructions concretely and then prove the categorical properties. The latter approach frequently requires additional assumptions about the spaces employed.

Let us illustrate these two alternatives with the probabilistic powerdomain construction. While it is true that the probabilistic powerdomain can be defined for arbitrary dcpo's (all topological spaces, in fact) and will always yield another dcpo, it has been shown to satisfy the axioms for a commutative monad only on the much smaller category **CONT** of continuous domains. If one wants to insist on the categorical properties for all dcpo's one must work with an abstract definition of a probabilistic powerdomain (for example, as a free probabilistic algebra), and one loses useful tools and intuitions from measure and integration theory.

On the other hand, the concrete approach is not without difficulties either. In fact, the work reported in this paper leads us to believe that some problems may be insurmountable. These difficulties stem from the fact that **CONT** as a whole is not closed under the function space construction. In order to accommodate function spaces it is necessary to confine oneself to one of the closed subcategories of **CONT**. These have been completely classified, [15, 1], and the candidate categories in the present setting are **RB** (also known as **RSFP**) and **FS**. The more lattice-like categories such as continuous Scott-domains are unsuitable because the probabilistic powerdomain, like the Plotkin powerdomain, destroys existing suprema. It was claimed in [25, 8] that the probabilistic powerdomain construction applied to an RB-domain yields another RB-domain. The proof offered is not valid, unfortunately, and whether the statement holds or not is an open question. We explore this problem in some detail in Section 3, concentrating on the probabilistic powerdomain of *finite* posets. Even in this very restricted setting the problem seems extremely difficult. Our positive results concern trees and reversed trees but there does not seem to be an easy way to generalize the methods to all finite posets.

In the last section we explore a more lenient notion of "closed" category, encouraged by our work on *relational* rather than *functional* semantics, [16]. We are able to establish that the probabilistic powerdomain construction behaves well on Lawson-compact domains. The proof is a bit technical but not too difficult. A more structural proof, preferably applicable to all coherent spaces, would be desirable.

2 Background

We assume familiarity with the theory of continuous domains as laid out in [1] or [23].

The definition of the probabilistic powerdomain employs the unit interval $I = [0,1] \subseteq \mathbb{R}$. We will frequently refer to the order of approximation $\ll_I$ on I, which is characterized simply by $a \ll_I b$ iff $a = 0$ or $a < b$.

Definition 1 *Let (X, τ) be a topological space. A* valuation *on X is a function $\mu\colon \tau \to [0,1] \subseteq \mathbb{R}$ with the properties*

1. $\mu(\emptyset) = 0$;
2. $\mu(O) + \mu(U) = \mu(O \cup U) + \mu(O \cap U)$, $O, U \in \tau$;
3. $O \subseteq U \implies \mu(O) \leq \mu(U)$.

In deviation from general practice we will also require a valuation to be Scott-continuous with respect to the Scott-topologies on I and the complete lattice $(\tau, \subseteq)$.

The set of all (continuous) valuations on (X, τ) is called the probabilistic powerdomain *of X. We denote it by $\mathsf{P}X$. On $\mathsf{P}X$ one considers the pointwise ordering between valuations*

$$\mu \leq \mu' \quad \text{if} \quad \mu(O) \leq \mu'(O) \text{ for all } O \in \tau \ .$$

Valuations have a long history in measure and lattice theory, see [22] and the references given there. As a construction in denotational semantics, the probabilistic powerdomain was first defined by Saheb-Djahromi in [26], with the additional restriction $\mu(X) = 1$. The definition we have chosen is the one of [12]. It was later shown by Kirch, [21], that one can extend the range of valuations to $\mathbb{R}_{0,+}$ or even $\mathbb{R}_{0,+} \cup \{\infty\}$, retaining the core properties. This extension has the advantage that we can freely add valuations

$$(\mu + \mu')(O) = \mu(O) + \mu'(O)$$

and multiply by non-negative scalars

$$(r \cdot \mu)(O) = r \cdot \mu(O) \ .$$

$\mathsf{P}X$ then becomes an ordered cone. For more details see [28]. For technical reasons we will stick with Jones' definition, i.e. we will limit the range of a valuation to the unit interval.

As every dcpo D is also a topological space when equipped with the Scott-topology $\sigma(D)$, we can define the probabilistic powerdomain on all dcpo's. Because addition is a Scott-continuous operation on $(\mathbb{R}, \leq)$ it follows that $\mathsf{P}X$ is again a dcpo if $X = (D, \sigma(D))$. Furthermore, if $f\colon D \to E$ is a Scott-continuous map between dcpo's then so is $\mathsf{P}f\colon \mathsf{P}D \to \mathsf{P}E$, where

$$\mathsf{P}f(\mu)(O) = \mu(f^{-1}(O)), \quad O \in \sigma(E) \ .$$

It follows that P is indeed a functor on the category **DCPO**.

Very little is known about the properties of this functor in general. However, if we restrict its domain of definition to **CONT**, the category of continuous domains, then the situation is much better. That is because for continuous domains we can make use of so-called *simple valuations*. In the remainder of this section we develop the theory of simple valuations in as far as it is relevant for the purposes of this paper.

Definition 2 *A valuation is called* simple *if it takes on only finitely many different values.*

Alternatively, simple valuations can be characterised with the help of point valuations, as follows.

Proposition 3 *Let (X, τ) be a topological space and $x \in X$. Then the following defines a valuation η_x on X*

$$\eta_x(O) = \begin{cases} 1, & \textit{if } x \in O; \\ 0, & \textit{otherwise.} \end{cases}$$

We call η_x the *point valuation* centered at x.

Proposition 4 ([28]) *On a sober space X, a valuation μ is simple if and only if it is expressible as a linear combination of point evaluations $\mu = \sum_{m \in M} r_m \eta_m$, where M is a finite subset of X.*

For a simple valuation $\mu = \sum_{m \in M} r_m \eta_m$ the measure of an open set O is just $\sum_{m \in O} r_m$. On a finite poset D equipped with it Scott-topology $\sigma(D)$, every valuation is simple since there are only finitely many open sets. If we allow zero weights then we can write every valuation as a linear combination of point evaluations

$$\mu = \sum_{x \in D} r_x \eta_x$$

with the index set being all points of D. In this case, we can give the following simple formula for the weights

$$r_x = \mu(\uparrow x) - \mu(\uparrow x \setminus \{x\}) \; . \tag{1}$$

The key result for studying the probabilistic powerdomain construction on continuous domains is the so-called *Splitting Lemma*:

Lemma 5 ([12]) *For two simple valuations $\mu = \sum_{m \in M} r_m \eta_m$ and $\nu = \sum_{n \in N} r_n \eta_n$ on a continuous domain the following are equivalent*

1. $\mu \leq \nu$.
2. *There exist non-negative real numbers $(t_{m,n})_{m \in M, n \in N}$ such that*
   - $\forall m \in M. \; \sum_{n \in N} t_{m,n} = r_m$,
   - $\forall n \in N. \; \sum_{m \in M} t_{m,n} \leq r_n$,
   - $\forall m \in M, n \in N.$ *If $t_{m,n} \neq 0$ then $m \leq n$.*

Throughout this paper, we will call the $t_{m,n}$ appearing in this characterisation the *transport numbers*. Although the proof of the Splitting Lemma in [12] is very pretty one may wonder whether there exists a more direct argument. We therefore include the following.

Proof. We show (1) $\implies$ (2), which is the difficult direction. Let us call a family of non-negative real numbers $(t_{m,n})_{m \in M, n \in N}$ a *semi-splitting* if the following is true:

$$\begin{array}{l} \forall m \in M. \; \sum_{n \in N} t_{m,n} \leq r_m \\ \forall n \in N. \; \sum_{m \in M} t_{m,n} \leq r_n \\ \forall m \in M, n \in N. \; t_{m,n} \neq 0 \implies m \leq n. \end{array}$$

This, of course, is just a slight weakening of the conditions in the statement of the lemma.

The set of all semi-splittings is a subset of some $\mathbb{R}^k$ (where $k = |N| \cdot |M|$), which is non-empty (because the null vector belongs to it), closed (because it is defined through inequalities), and bounded (because each $t_{m,n}$ is less than or equal to r_n and r_m). Therefore this set is compact in the usual metric topology on $\mathbb{R}^k$.

For a given semi-splitting $(t_{m,n})_{m \in M, n \in N}$ we call $(\sum_{n \in N} t_{m,n})_{m \in M}$ the *rs-vector* (as in *row summation*). The goal is to show that there exists a semi-splitting whose rs-vector equals $(r_m)_{m\in M}$. Such a semi-splitting would satisfy the conditions in (2).

Since the set of all semi-splittings is compact, and since the passage from semi-splittings to their rs-vectors is continuous, there exists a semi-splitting with *maximal* rs-vector, where "maximal" is taken with respect to the coordinatewise order on $\mathbb{R}^k$. We will show that such a semi-splitting is indeed a splitting as required.

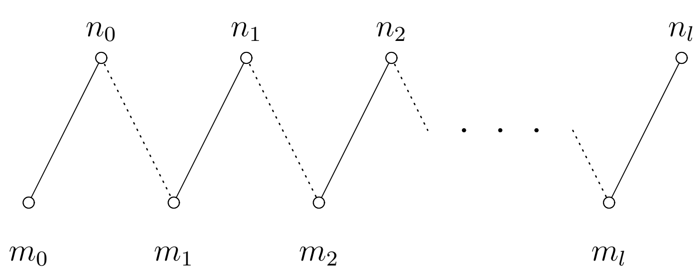
Assume, for the sake of contradiction, that $(t_{m,n})_{m\in M,n\in N}$ is a semi-splitting with maximal rs-vector and that there exists $m_0 \in M$ with $\sum_{n\in N} t_{m_0,n} < r_{m_0}$. Define subsets $M' \subseteq M$, $N' \subseteq N$ inductively by:

1. $m_0 \in M'$
2. $m \in M', m \leq n \in N \implies n \in N'$
3. $n \in N', n \geq m \in M, t_{m,n} > 0 \implies m \in M'$

Since M and N are finite sets, these subsets are well-defined. Further, let $M'' = {\uparrow}M' \cap M$ and O be an open set containing ${\uparrow}M' = {\uparrow}M''$, which does not contain any element from either $M \setminus M''$ or $N \setminus N'$. We calculate:

$$\begin{array}{rcll}
\sum_{n\in N'}\sum_{m\in M} t_{m,n} & = & \sum_{n\in N'}\sum_{m\in M'} t_{m,n} & \text{Rule 3} \\
 & = & \sum_{n\in N'}\sum_{m\in M''} t_{m,n} & M' \subseteq M'' \subseteq M \\
 & = & \sum_{m\in M''}\sum_{n\in N'} t_{m,n} & \\
 & < & \sum_{m\in M''} r_m & \text{since } m_0 \in M'' \\
 & = & (\sum_{m\in M} r_m\eta_m)(O) & \text{since } O \cap M = M'' \\
 & \leq & (\sum_{n\in N} r_n\eta_n)(O) & \text{Assumption} \\
 & = & \sum_{n\in N'} r_n & \text{since } O \cap N = N'
\end{array}$$

Comparing the first and the last term in this chain of inequalities, we observe that there must exist some $\hat{n} \in N'$ with $\sum_{m\in M} t_{m,\hat{n}} < r_{\hat{n}}$. Since $\hat{n}$ belongs to the inductively defined set N', there is a finite chain $m_0 \leq n_o \geq m_1 \leq n_1 \geq \ldots \leq n_l = \hat{n}$, pictorially:



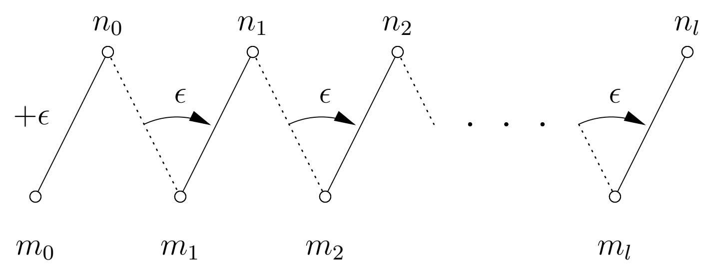
On all dotted edges, the transport number t_{m_{i+1},n_i} is strictly positive. Therefore, the following number is also strictly positive:

$$\epsilon = \min\,\{t_{m_{i+1},n_i} \mid i = 0,\ldots,l-1\} \cup \{r_{\hat{n}} - \sum_{m\in M} t_{m,\hat{n}}\} \cup \{r_{m_0} - \sum_{n\in N} t_{m_0,n}\}$$

We define a new semi-splitting by setting

$$\tilde{t}_{m,n} = \begin{cases} t_{m,n} + \epsilon & (m,n) = (m_i, n_i), i = 0, \ldots, l; \\ t_{m,n} - \epsilon & (m,n) = (m_{i+1}, n_i), i = 0, \ldots, l-1; \\ t_{m,n} & \text{otherwise} \end{cases}$$

Pictorially:



This adjustment does not change the value of the row summations $\sum_{m \in M} t_{m,n_i}$ for $i = 0, \ldots, l-1$, and the column summations $\sum_{n \in N} t_{m_i,n}$ for $i = 1, \ldots, l$. The values for $\sum_{m \in M} t_{m,n_l}$ and $\sum_{n \in N} t_{m_0,n}$ increase by ϵ each but this is all right because of the second and the third term in the definition of ϵ. Now observe that we have created a semi-splitting $(\tilde{t}_{m,n})$ whose rs-vector is strictly larger than that of $(t_{m,n})$. This contradicts the assumed maximality of the rs-vector for $(t_{m,n})$ and the lemma is proven. ■

Later on we will be concerned with valuations on a fixed finite domain. For this case we can formulate the Splitting Lemma even more nicely. Recall that every valuation on a finite domain can be written in the form $\sum_{x \in D} r_x \eta_x$ with $r_x \in [0,1]$. The following is then obvious:

Lemma 6 (Elementary Steps) *Let D be a finite domain. Consider the following relations between valuations on D.*

1. $\sum_{x \in D} r_x \eta_x \leq_1 \sum_{x \in D} s_x \eta_x$ *if there exists $x_0 \in D$ with $r_{x_0} \leq s_{x_0}$, and for all $x \in D \setminus \{x_0\}$, $r_x = s_x$.*

2. $\sum_{x \in D} r_x \eta_x \leq_2 \sum_{x \in D} s_x \eta_x$ *if there exist $x_0 < x_1$, where x_1 is an upper neighbour of x_0, such that $r_{x_1} \leq s_{x_1}$ and $r_{x_0} + r_{x_1} = s_{x_0} + s_{x_1}$, and for all $x \in D \setminus \{x_0, x_1\}$, $r_x = s_x$.*

The order between valuations on D is the transitive hull of $\leq_1 \cup \leq_2$.

In other words, a step of type 1 consists of increasing the mass at some point of D, and a step of type 2 consists of shifting some mass from a point to one of its upper neighbours. The lemma states that any two valuations on D, which are comparable, can be connected by a finite sequence of elementary steps.

Carefully exploiting the information contained in the Splitting Lemma, one can also give a characterisation of the order of approximation:

Lemma 7 ([12]) *For two simple valuations $\mu = \sum_{m \in M} r_m \eta_m$ and $\nu = \sum_{n \in N} r_n \eta_n$ on a continuous domain the following are equivalent*

1. $\mu \ll \nu$.
2. *There exist non-negative real numbers $(t_{m,n})_{m \in M, n \in N}$ such that*
   - $\forall m \in M.\ \sum_{n \in N} t_{m,n} = r_m$,
   - $\forall n \in N.\ \sum_{m \in M} t_{m,n} \ll_I r_n$,
   - $\forall m \in M, n \in N.$ *If $t_{m,n} \neq 0$ then $m \ll n$.*

This is instrumental in proving the following:

Theorem 8 ([12]) *The probabilistic powerdomain of a continuous domain is again continuous. A basis is given by the set of simple valuations.*

3 The probabilistic powerdomain on cartesian closed categories

The interpretation of functional types requires a function space construction in the semantic universe. Since **CONT** is well pointed there is no choice in the definition of a function space; it has to be the set of all continuous functions ordered pointwise. Unfortunately, this dcpo is not continuous in general, see [1, Chapter 4] for a full discussion of this phenomenon. The way out is to consider continuous domains with additional properties and, indeed, there are a number of possible definitions. Broadly, these fall into two categories, the lattice-like domains, where one assumes the existence of certain least upper bounds, and the compact domains, which are defined with reference to finite posets. Claire Jones demonstrated that lattice-like structure is destroyed by the probabilistic powerdomain construction in even the simplest cases, so we concentrate attention on the second kind.

3.1 Trees and RB-domains

Let us first look at retracts of bifinite domains, or RB-domains. We will mostly work with the following simple internal characterisation of these spaces, [14, Theorem 4.1].

Definition 9 *A dcpo D is called an* RB-domain *if there exists a directed family $(f_i)_{i \in I}$ of Scott-continuous functions from D to itself with the following properties.*

1. $\bigvee^{\uparrow}_{i\in I} f_i = \mathsf{id}_D$;

2. *The image of each f_i is finite.*

Functions with these properties are called deflations. *The full subcategory of* **CONT** *consisting of RB-domains is denoted by* **RB**.

Example 10 *Consider the unit interval $I = [0,1]$ with its usual order. We can define functions $f_n\colon I \to I$ with the desired properties by setting*

$$f_n(x) = \frac{\max\{m \in \mathbb{N} \mid m \ll_I n \cdot x\}}{n} ,$$

in other words, $f_n(x)$ is the largest multiple of $\frac{1}{n}$ approximating x. (Recall that $r \ll_I s$ if and only if $r = 0$ or $r < s$.)

The function space of two RB-domains is again an object in **RB** and there are a number of other pleasing closure properties of this category. The question then is whether the probabilistic powerdomain construction can be restricted (or, rather, co-restricted) to **RB**. This was claimed in [25, 12]. [8] offers a proof but unfortunately it is not valid, and the question in fact remains open.

The problem can easily be reduced to finite posets as follows.

Lemma 11 *If it is true that the probabilistic powerdomain of every finite poset is an RB-domain then* **RB** *is closed under the probabilistic powerdomain construction.*

Proof. Let D be a retract of the bifinite domain $E = \mathsf{bilim}(E_i)$ where all E_i are finite posets. The probabilistic powerdomain functor is locally continuous, hence $\mathsf{P}D$ is a retract of $\mathsf{P}E = \mathsf{bilim}(\mathsf{P}E_i)$. By assumption, all $\mathsf{P}E_i$ are RB-domains. It was shown in [14, Theorem 4.6] that **RB** is closed under the formation of bilimits (non-trivial) and retracts (trivial), hence $\mathsf{P}D$ belongs to **RB**. ∎

In order to get the desired closure result one might first try to post-compose valuations directly with the deflations $f_n\colon I \to I$ from Example 10. This, however, would destroy modularity.

Since by the previous lemma we can restrict ourselves to finite posets, we can exploit the fact that every valuation on a finite poset D is of the form $\sum_{d\in D} r_d\eta_d$, with $r_d \in I$, η_d the point valuation centered at $d \in D$, and $\sum_{d\in D} r_d \leq 1$. In a second attempt we can then apply the deflations f_n to weights rather than valuations:

$$G_n\colon \mathsf{P}D \to \mathsf{P}D, \quad G_n(\sum_{d\in D} r_d\eta_d) = \sum_{d\in D} f_n(r_d)\eta_d \,. \tag{2}$$

This idea is the starting point for the "proof" contained in [8]. While it is true that each G_n is below $\mathsf{id}_{\mathsf{P}D}$ and has finite image, and also that $\bigvee^{\uparrow}_{n\in\mathbb{N}} G_n = \mathsf{id}_{\mathsf{P}D}$, it is unfortunately not the case that they are monotone:

Example 12 *Consider the two-element chain $\bot \leq \top$ and let $0 < \epsilon < \frac{1}{n} < 2\epsilon$. Then $2\epsilon\eta_\bot < \epsilon\eta_\bot + \epsilon\eta_\top$ but the images are in reversed order, $\frac{1}{n}\eta_\bot > 0$.*

What is happening here is that the G_n deal with elementary steps of type 1 but not with those of type 2. Our positive result regarding **RB** concerns finite trees only; we have not been able to extend it to more general posets.

Theorem 13 *The probabilistic powerdomain of a finite tree belongs to* **RB**.

Proof. Every open set in a finite tree is a unique disjoint union of principal filters (i.e. sets of the form $\uparrow x$). We define a deflation on valuations by describing its action on the values for principal filters:

$$F_n \colon \mathsf{P}D \to \mathsf{P}D, \quad F_n(\mu)(\uparrow x) = f_n(\mu(\uparrow x)) ,$$

where the f_n are the deflations on I from Example 10. For an arbitrary open set $O = \uparrow y_1 \cup \ldots \cup \uparrow y_k$ set $F_n(\mu)(O) = \sum_{i=1}^{k} f_n(\mu(\uparrow y_i))$. We need to show that the resulting function F_n is a deflation on $\mathsf{P}D$. To this end, we first need to establish that $F_n(\mu)$ is again a valuation on D. Consider first whether $F_n(\mu)$ is compatible with the relationships between principal filters: if $\uparrow x \subseteq \uparrow y$ then $\mu(\uparrow x) \leq \mu(\uparrow y)$ and we have $F_n(\mu)(\uparrow x) \leq F_n(\mu)(\uparrow y)$ because the f_n are monotone.

Next let O be an arbitrary open subset contained in $\uparrow x$. Because we are working on a finite tree, O can be written as a disjoint union $\uparrow y_1 \cup \ldots \cup \uparrow y_k$. We need $F_n(\mu)(O) \leq F_n(\mu)(\uparrow x)$ which is equivalent to $\sum_{i=1}^{k} f_n(\mu(\uparrow y_i)) \leq f_n(\mu(\uparrow x))$. This relation is a consequence of the super-additivity of the deflations f_n:

$$f_n(x) + f_n(y) \leq f_n(x + y)$$

and the fact that μ itself is a valuation. Now we can apply 1 and recover the point masses. This shows that $F_n(\mu)$ is indeed a valuation.

So we see that the weight at each node x, calculated as $F_n(\mu)(\uparrow x) - F_n(\mu)(\uparrow x \setminus \{x\})$, is non-negative. Hence $F_n(\mu)$ is a valuation.

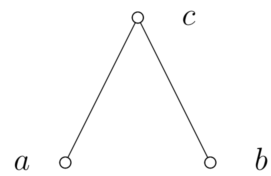
Monotonicity and continuity of F_n as an operation from $\mathsf{P}D$ to $\mathsf{P}D$ follow from the monotonicity and continuity of f_n.

Clearly, F_n has finite image because only multiples of $\frac{1}{n}$ occur as values for $F_n(\mu)(O)$. ∎

Applying F_n to the valuation $(\frac{1}{n} - \epsilon)\eta_\bot + 2\epsilon\eta_\top$ on the two-element chain (where $\epsilon < \frac{1}{2n}$) we get $\frac{1}{n}\eta_\bot$. From this we see that the mass assigned to individual points can be increased by the deflation.

An alternative method of proof for the preceding theorem is to establish that the probabilistic powerdomain of a finite tree is a continuous bounded-complete domain [1, Definition 4.1.1], and to rely on the fact that bc-domains belong to **RB**. We will not pursue this any further, because the situation for trees is indeed very special in this respect. Already the simplest poset, which is not a tree, has a probabilistic powerdomain which is not bounded-complete:

Example 14 ([12]) *Consider the poset*



and the valuations $\frac{1}{2}\eta_a$ *and* $\frac{1}{2}\eta_b$. *They have two distinct minimal upper bounds:* $\frac{1}{2}\eta_a + \frac{1}{2}\eta_b$ *and* $\frac{1}{2}\eta_c$.

3.2 Reversed trees and FS-domains

A category related to **RB** was introduced in [15]. It is also cartesian closed and in fact maximal with this property among the full subcategories of **CONT**.

Definition 15 *A dcpo D is called an* FS-domain *if there exists a directed family $(f_i)_{i \in I}$ of Scott-continuous functions from D to itself with the following properties.*

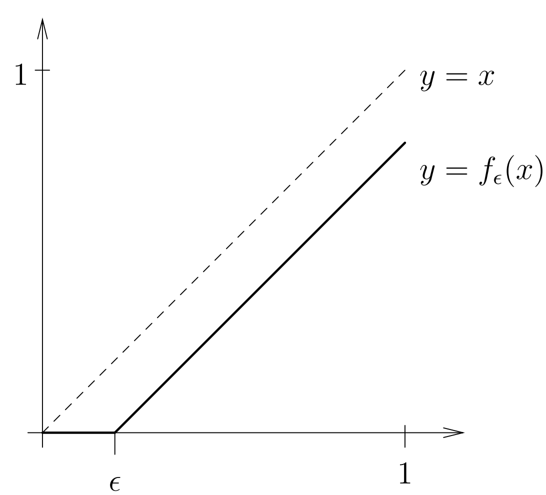
1. $\bigvee^{\uparrow}_{i \in I} f_i = \mathsf{id}_D$;
2. *Every f_i is* finitely separated *from* id_D *in the sense that there exists a finite set $M_i \subseteq D$ with the property $\forall x \in D \exists m \in M_i.\ f_i(x) \leq m \leq x$.*

The full subcategory of **CONT** *consisting of FS-domains is denoted by* **FS**.

It is immediate from the definition that every RB-domain belongs to **FS** as well. Whether this inclusion is proper is not known.

We will need the extra freedom that finitely separated functions over those with finite image allow. To this end we consider the following maps on the unit interval (rather than the deflations f_n from Example 10):

$$f_\epsilon \colon I \to I, \quad f_\epsilon(x) = \max\{0, x - \epsilon\}, \quad \epsilon > 0 \ .$$



We shall need some special properties of these functions in our calculations below.

Lemma 16 *The functions f_ϵ are monotone, Scott-continuous, Hausdorff-continuous, and convex. They satisfy the following laws:*

$$a \leq b \quad \Longrightarrow \quad a - f_\epsilon(a) \leq b - f_\epsilon(b) \tag{3}$$

$$a \leq b, \delta \geq 0 \quad \Longrightarrow \quad f_\epsilon(b) - f_\epsilon(a) \leq f_\epsilon(b+\delta) - f_\epsilon(a+\delta) \tag{4}$$

Each f_ϵ is finitely separated from the identity on I. Their supremum equals id_I.

Proof. Convexity of functions is equivalent to convexity of the hyper-graph. This is obvious in this case, as are monotonicity and continuity. The two inequalities are more interesting:

The first law is proven by a case analysis. If $b \geq a \geq \epsilon$ then $f_\epsilon(a) = a - \epsilon$ and $f_\epsilon(b) = b - \epsilon$ and both sides of the inequality equal ϵ. If $a < \epsilon \leq b$ then $f_\epsilon(a) = 0$ and $f_\epsilon(b) = b - \epsilon$. We get $a - f_\epsilon(a) = a < \epsilon = b - f_\epsilon(b)$. Finally, if $a \leq b \leq \epsilon$ then both $f_\epsilon(a)$ and $f_\epsilon(b)$ equal zero.

The second law is a consequence of convexity. By the convexity property we have

$$\begin{array}{rcccl} f_\epsilon(a+\delta) & = & f_\epsilon(\frac{b-a}{b+\delta-a} \cdot a + \frac{\delta}{b+\delta-a} \cdot (b+\delta)) & \leq & \frac{b-a}{b+\delta-a} \cdot f_\epsilon(a) + \frac{\delta}{b+\delta-a} \cdot f_\epsilon(b+\delta) \\ f_\epsilon(b) & = & f_\epsilon(\frac{\delta}{b+\delta-a} \cdot a + \frac{b-a}{b+\delta-a} \cdot (b+\delta)) & \leq & \frac{\delta}{b+\delta-a} \cdot f_\epsilon(a) + \frac{b-a}{b+\delta-a} \cdot f_\epsilon(b+\delta)) \end{array}$$

Adding the two inequalities we get

$$f_\epsilon(a+\delta) + f_\epsilon(b) \leq f_\epsilon(a) + f_\epsilon(b+\delta)$$

from which the desired inequality follows by re-arrangement.

As a separating subset for f_ϵ one can choose all multiples of ϵ in I. It is clear that in the limit we get back id_I. ∎

No deflation on I except the constant zero function is convex because convexity implies Hausdorff-continuity.

At this point we must confess that we do not have an analogue of Lemma 11 for FS-domains. Whether the following results about finite posets will ever be useful is therefore not at all clear. They do, however, illustrate the technical difficulties one faces when trying to establish a general result about the structure of the probabilistic powerdomain construction.

Theorem 17 *The probabilistic powerdomain of a finite reversed tree is an FS-domain.*

Proof. Since we are working with a reversed tree, every element, except the top element, has a unique ancestor. Denote the ancestor of x by px. We will need this (partial) function only to denote open sets of the form $\uparrow px$. Setting $\uparrow px = \emptyset$ for $x = \top$, the top element, allows us to use a more uniform formalism below.

In particular, the translation from values for principal filters to point evaluations takes the following form. Assume that $\mu = \sum_{x\in D} r_x \eta_x$ and $O \in \sigma(D)$. Then

$$\mu(O) = \sum_{x\in O} r_x = \sum_{x\in O} \Big(\mu(\uparrow x) - \mu(\uparrow px)\Big) . \tag{5}$$

Using the functions $f_\epsilon \colon I \to I$ from above we define a mapping F_ϵ on $\mathsf{P}D$ as follows. For principal filters we set

$$F_\epsilon(\mu)(\uparrow x) = f_\epsilon(\mu(\uparrow x)), \quad x \in D . \tag{6}$$

For general open sets O we use the translation from measures for filters to weights on points (Equation 1):

$$\begin{aligned} F_\epsilon(\mu)(O) &= \sum_{x\in O} \Big(F_\epsilon(\mu)(\uparrow x) - F_\epsilon(\mu)(\uparrow px)\Big) \\ &= \sum_{x\in O} \Big(f_\epsilon(\mu(\uparrow x)) - f_\epsilon(\mu(\uparrow px))\Big) . \end{aligned} \tag{7}$$

The resulting function $F_\epsilon(\mu)$ is again a valuation, because f_ϵ is monotone and so with $\mu(\uparrow x) \geq \mu(\uparrow px) = \mu(\uparrow x \setminus \{x\})$ we also have $F_\epsilon(\mu)(\uparrow x) = f_\epsilon(\mu(\uparrow x)) \geq f_\epsilon(\mu(\uparrow px)) = F_\epsilon(\mu)(\uparrow px)$, that is, the resulting weights are all non-negative.

The crucial part of the proof is in showing that the F_ϵ are monotone. For this we need to employ Lemma 6. Assume, therefore, that μ is related to μ' by an elementary step of type 1, that is, the point mass at some $x_0 \in D$ is smaller for μ than it is for μ' but all other weights are the same. We need to show that $F_\epsilon(\mu)(O) \leq F_\epsilon(\mu')(O)$ for all open set O. We will use the definition of $F_\epsilon(\mu)(O)$ as given in equation 7. To this end we distinguish three kinds of points in D:

Class I consists of those $x \in D$ for which $x \not\leq x_0$. Here we have no change in the measure of principal filters:

$$f_\epsilon(\mu(\uparrow x)) - f_\epsilon(\mu(\uparrow px)) = f_\epsilon(\mu'(\uparrow x)) - f_\epsilon(\mu'(\uparrow px))$$

Class II consists of just x_0. Here we have $\mu(\uparrow x_0) \leq \mu'(\uparrow x_0)$ and $\mu(\uparrow px_0) = \mu'(\uparrow px_0)$. Hence

$$f_\epsilon(\mu(\uparrow x_0)) - f_\epsilon(\mu(\uparrow px_0)) \leq f_\epsilon(\mu'(\uparrow x_0)) - f_\epsilon(\mu'(\uparrow px_0))$$

Class III contains all elements strictly below x_0. This is the trickiest part because both $\uparrow x$ and $\uparrow px$ are affected by the change at x_0. It is here that we make use of the convexity of f_ϵ through rule 4, instantiated as

$$a := \mu(\uparrow px) \quad b := \mu(\uparrow x) \quad \delta := \mu'(\uparrow x) - \mu(\uparrow x) = \mu'(\uparrow px) - \mu(\uparrow px)$$

We get

$$f_\epsilon(\mu(\uparrow x)) - f_\epsilon(\mu(\uparrow px)) \le f_\epsilon(\mu'(\uparrow x)) - f_\epsilon(\mu'(\uparrow px))$$

which is the inequality we need. Summing up gives monotonicity for F_ϵ because every $x \in O$ belongs to precisely one of the three classes.

Assume now that μ and μ' are related by an elementary step of type 2, that is, there exists $x_0 \in D$ such that some mass has been shifted from x_0 to px_0 in the passage from μ to μ'. In order to evaluate equation 7 we again distinguish a number of cases.

$$\begin{aligned}
\mathrm{I} &:= \{x \in D \mid x \not\le px_0\} \\
\mathrm{II} &:= \{x \in D \mid x \not\le x_0, x < px_0\} \\
\mathrm{III} &:= \{x_0, px_0\} \\
\mathrm{IV} &:= \{x \in D \mid x < x_0\}
\end{aligned}$$

There is no change in passing from μ to μ' for elements of class I and IV. The effect for elements of class II is the same as that for those of class III in the previous paragraph. The two elements in class III need to be considered together.

$$\begin{aligned}
&f_\epsilon(\mu(\uparrow x_0)) - f_\epsilon(\mu(\uparrow px_0)) + f_\epsilon(\mu(\uparrow px_0)) - f_\epsilon(\mu(\uparrow ppx_0)) = \\
&\qquad = f_\epsilon(\mu(\uparrow x_0)) - f_\epsilon(\mu(\uparrow ppx_0)) \\
&\qquad = f_\epsilon(\mu'(\uparrow x_0)) - f_\epsilon(\mu'(\uparrow ppx_0)) \\
&\qquad = f_\epsilon(\mu'(\uparrow x_0)) - f_\epsilon(\mu'(\uparrow px_0)) + f_\epsilon(\mu'(\uparrow px_0)) - f_\epsilon(\mu'(\uparrow ppx_0))
\end{aligned}$$

Summing up over all x gives the desired inequality.

Scott-continuity of the F_ϵ follows from the Scott-continuity of the f_ϵ.

We next show that $F_\epsilon(\mu) \le \mu$ holds. To this end we show that the weight at each point of D is decreased. We use equation 3 from Lemma 16, instantiated with $b = \mu(\uparrow x)$ and $a = \mu(\uparrow px)$:

$$f_\epsilon(\mu(\uparrow x)) - f_\epsilon(\mu(\uparrow px)) \le \mu(\uparrow x) - \mu(\uparrow px)$$

We also need to check that F_ϵ is finitely separated from the identity on $\mathsf{P}D$. For this we use Graham's non-monotone (!) functions G_n (Equation 2) with $n \in \mathbb{N}$ chosen so that $\frac{1}{n} < \frac{\epsilon}{|D|}$ holds. We prove that for every $\mu \in \mathsf{P}D$, $F_\epsilon(\mu) \le G_\epsilon(\mu) \le \mu$ holds. Since G_n produces only finitely many different valuations, this will show finite separation for F_ϵ.

For $F_\epsilon(\mu) \le G_\epsilon(\mu)$ let O be an open subset of D. We distinguish two cases: either there exists a principal filter $\uparrow X \subseteq O$ with $\mu(\uparrow x) \ge \epsilon$ or not. In the first case, $F_\epsilon(\mu)(O) = f_\epsilon(\mu(\uparrow x)) + \sum_{y \in O, y \not\ge x} f_\epsilon(\mu(\uparrow x)) - f_\epsilon(\mu(\uparrow px)) \le \mu(\uparrow x) - \epsilon + \sum_{y \in O, y \not\ge x} \mu(\uparrow x) - \mu(\uparrow px)$ where we have used the definition of f_ϵ and the fact that F_ϵ reduces the weight at every point. Since G_n can reduce the weight at each point by at most $\frac{1}{n} < \frac{\epsilon}{|D|}$, we have $F_\epsilon(\mu) \le G_\epsilon(\mu)$.

In the second case, $F_\epsilon(\mu)(O) = 0$ and the desired relationship also holds.

The inequality $G_n \leq \mathsf{id}_{\mathsf{P}D}$ is trivial.

Finally, we want $\bigvee^{\uparrow}_{\epsilon>0} F_\epsilon = \mathsf{id}_{\mathsf{P}D}$. This is obvious from the way the F_ϵ are constructed. ■

In the proof we have pointed out why it is necessary to have *convex* approximating functions f_ϵ on I. There are no convex *deflations* on I except for the constant zero map and, indeed, we do not know whether the probabilistic powerdomain of a finite reversed tree belongs to **RB** or not. These spaces, therefore, provide a whole family of domains who may serve as examples that **FS** is strictly larger than **RB**. The only other example is due to Jimmie Lawson; it is described in [1, p. 60].

4 A positive result for compact domains

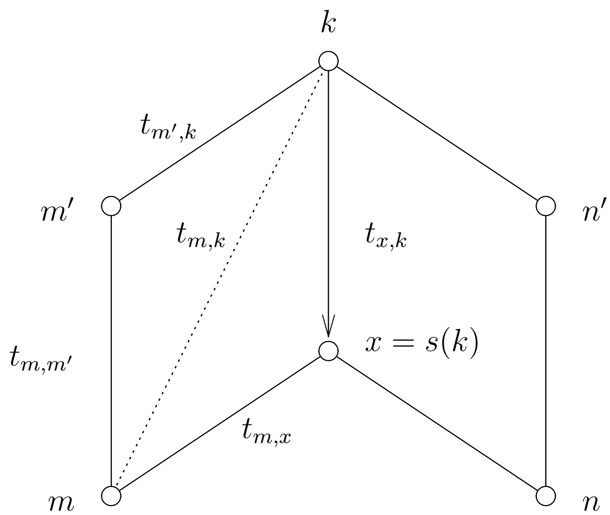
If we relax the requirement for function spaces in our universe of semantic domains then we get new possibilities. Foremost, there is Jones' result that the probabilistic powerdomain construction maps continuous domains to continuous domains. Topologically, continuous domains are still quite general spaces and it makes sense to impose further conditions. One of the best known in this context is *coherence*, introduced in [11]. See [1, Section 7.2.4] for an introduction and [23, 18, 19] for some of the many pleasing properties of coherent domains. Recently, it was also shown that these spaces arise quite naturally in a logical approach to denotational semantics, [16, 17].

In combination with a continuous dcpo structure, coherence can be characterized by Lawson-compactness. We will work with the following criterion for Lawson-compactness whose prove is similar to that of Lemma 4.18 in [14]:

Lemma 18 *A continuous domain D with bottom element is Lawson-compact, if and only if, for every situation $x \ll x'$, $y \ll y'$ there exist finitely many points $a_1, \ldots, a_n$ in $\mathsf{ub}\{x, y\}$ such that $\mathsf{ub}\{x', y'\} \subseteq \uparrow\{a_1, \ldots, a_n\}$.*

Theorem 19 *Let D be a Lawson-compact, continuous domain with bottom element. Then the probabilistic powerdomain is also Lawson-compact.*

Proof. We use the characterisation given in Lemma 18 in a slightly sharpened form by assuming that the two strongly related points are actually taken from a basis. So let $\mu \ll \mu'$ and $\nu \ll \nu'$ be simple valuations with support M, M', N, and N', respectively. We look for finitely many (simple) valuations χ above μ and ν, such that every valuation κ above μ' and ν' is above some χ. We may, without loss of generality, assume that κ, too, is simple. In the calculations to follow it may be helpful to refer to the following picture:



All χ will have the same support X, which we now define. For each pair of subsets $A' \subseteq M'$, $B' \subseteq N'$ let $X_{A',B'}$ be a finite set of upper bounds for $\mathord{\downarrow\!\!\downarrow} A' \cap M$, $\mathord{\downarrow\!\!\downarrow} B' \cap N$ which covers $\mathsf{ub}(A' \cup B')$. The existence of these sets is guaranteed by the Lawson-compactness of D (Lemma 18). Let X be the union of all $X_{A',B'}$.

In a first step we will, for a given simple valuation κ above μ' and ν', define a simple valuation χ which is below κ, above μ and ν, and which has its support in X.

Let such a κ be given. We denote its support by K. The transport numbers, whose existence is guaranteed by the Splitting Lemma, are denoted by $t_{m',k}$ etc. From the construction of X it then follows that there exists a (not necessarily injective) mapping s from K to X with the properties

1. $s(k) \leq k$,
2. $m \ll m' \leq k \implies m \leq s(k)$,
3. $n \ll n' \leq k \implies n \leq s(k)$.

Our definition of χ and the corresponding transport numbers are derived from particular transport numbers $t_{m,k}$ and $t_{n,k}$. We calculate these from the transport numbers corresponding to $\mu \ll \mu' \leq \kappa$ as follows:

$$t_{m,k} := \sum_{m' \in M'} \frac{t_{m',k}}{r_{m'}} t_{m,m'} \quad \text{and} \quad t_{n,k} := \sum_{n' \in N'} \frac{t_{n',k}}{r_{n'}} t_{n,n'} \ .$$

These are valid transport numbers for $\mu \leq \kappa$ and $\nu \leq \kappa$, respectively, since

$$\begin{aligned}
\sum_{k\in K} t_{m,k} &= \sum_{k\in K}\sum_{m'\in M'} \frac{t_{m',k}}{r_{m'}} t_{m,m'} \\
&= \sum_{m'\in M'}\sum_{k\in K} \frac{t_{m',k}}{r_{m'}} t_{m,m'} \\
&= \sum_{m'\in M'} \frac{t_{m,m'}}{r_{m'}} \sum_{k\in K} t_{m',k} \\
&= \sum_{m'\in M'} \frac{t_{m,m'}}{r_{m'}} r_{m'} \\
&= \sum_{m'\in M'} t_{m,m'} = r_m
\end{aligned}$$

and

$$\begin{aligned}
\sum_{m\in M} t_{m,k} &= \sum_{m\in M}\sum_{m'\in M'} \frac{t_{m',k}}{r_{m'}} t_{m,m'} \\
&= \sum_{m'\in M'}\sum_{m\in M} \frac{t_{m',k}}{r_{m'}} t_{m,m'} \\
&\leq \sum_{m'\in M'} \frac{t_{m',k}}{r_{m'}} r_{m'} \leq r_k
\end{aligned}$$

Now we set

$$t_{m,x} := \sum_{\substack{k\in K \\ s(k)=x}} t_{m,k} \quad \text{and} \quad t_{n,x} := \sum_{\substack{k\in K \\ s(k)=x}} t_{n,k} \; .$$

This definition is necessary because s might not be injective. But s is still a function, so we retain the properties

$$r_m = \sum_{x\in X} t_{m,x} \quad \text{and} \quad r_n = \sum_{x\in X} t_{n,x} \; .$$

Next we set

$$t_{s(k),k} := \mathsf{max}\left\{ \sum_{m\in M} t_{m,k}, \sum_{n\in N} t_{n,k} \right\}$$

and for all other $x \in X$ we let $t_{x,k} := 0$. Finally, we can define the weights for χ:

$$r_x := \sum_{k\in K} t_{x,k} \; .$$

Let us now check that χ is indeed above μ and ν and below κ. For this we employ the Splitting Lemma in the reverse direction. We begin with $\mu \leq \chi$: We have already noted that $r_m = \sum_{x \in X} t_{m,x}$. For the inequality we calculate

$$\begin{aligned}\sum_{m \in M} t_{m,x} &= \sum_{m \in M} \sum_{\substack{k \in K \\ s(k)=x}} t_{m,k} \\ &= \sum_{\substack{k \in K \\ s(k)=x}} \sum_{m \in M} t_{m,k} \\ &\leq \sum_{\substack{k \in K \\ s(k)=x}} t_{x,k} = r_x\end{aligned}$$

The third condition also holds because if $t_{m,x}$ is non-vanishing, then by definition at least one $t_{m,k}$ with $x = s(k)$ is non-zero. Since we defined $t_{m,k}$ as the sum $\sum_{m' \in M'} \frac{t_{m',k}}{r_{m'}} t_{m,m'}$, at least one term $\frac{t_{m',k}}{r_{m'}} t_{m,m'}$ is different from zero. This implies that $m \ll m' \leq k$ holds for this point $m' \in M'$ and then (2) above yields $m \leq x$.

Let us now go through the same three steps to show $\chi \leq \kappa$. The first condition holds by definition of the weights of χ. For the second we calculate

$$\begin{aligned}\sum_{x \in X} t_{x,k} &= t_{s(k),k} \\ &= \sum_{n \in N} t_{n,k} \quad (\text{or } \textstyle\sum_{m \in M} t_{m,k}) \\ &\leq r_k\end{aligned}$$

The third condition was explicitly enforced.

So far, so good. But we get too many valuations χ this way, depending on how the weight is distributed in the κ's. We will now show that it is in fact possible to restrict the weights for the valuations χ.

From the relations $\mu \ll \mu'$ and $\nu \ll \nu'$ we know that $\sum_{m \in M} t_{m,m'} < r_{m'}$ and $\sum_{n \in N} t_{n,n'} < r_{n'}$, respectively. As there are just $|M'| + |N'|$-many of these differences we may take their minimum ε_1 and set

$$\varepsilon := \frac{\varepsilon_1}{\mathsf{max}\{|M|, |N|\} + 1} \ .$$

Consider new valuations $\bar{\mu}$, $\bar{\nu}$ with weights

$$\bar{r}_m := r_m + \varepsilon \quad \text{and} \quad \bar{r}_n := r_n + \varepsilon \ .$$

We define transport numbers from μ' to $\bar{\mu}$ and from ν' to $\bar{\nu}$ by setting

$$\bar{t}_{m,m'} := \frac{t_{m,m'}}{r_m} \bar{r}_m \quad \text{and} \quad \bar{t}_{n,n'} := \frac{t_{n,n'}}{r_n} \bar{r}_n \ .$$

Then $\bar{\mu}$ is still way-below μ' (and also $\bar{\nu} \ll \nu'$):

$$\sum_{m' \in M'} \bar{t}_{m,m'} = \sum_{m' \in M'} \frac{t_{m,m'}}{r_m} \bar{r}_m = \frac{r_m}{r_m} \bar{r}_m = \bar{r}_m$$

and

$$\begin{aligned}
\sum_{m\in M} \bar{t}_{m,m'} &= \sum_{m\in M} \frac{t_{m,m'}}{r_m}\bar{r}_m \\
&= \sum_{m\in M} \frac{t_{m,m'}}{r_m}(r_m+\varepsilon) \\
&= \sum_{m\in M} t_{m,m'} + \varepsilon \sum_{m\in M} \frac{t_{m,m'}}{r_m} \\
&\leq \sum_{m\in M} t_{m,m'} + \varepsilon|M| \\
&< \sum_{m\in M} t_{m,m'} + \varepsilon_1 \leq r_{m'}
\end{aligned}$$

For an upper bound κ of μ' and ν' we perform the construction as before, but in the end we let the weight at each $x \in X$ be $\tilde{r}_x := \lfloor r_x \rfloor_{\tilde{\varepsilon}}$, where $\tilde{\varepsilon} := \frac{\varepsilon}{|X|}$ and $\lfloor r \rfloor_{\tilde{\varepsilon}}$ is the largest multiple of $\tilde{\varepsilon}$ below or equal to r. Because of this alteration, the valuation $\tilde{\chi}$ may no longer be above $\bar{\mu}$ or $\bar{\nu}$, but it will still be above μ and ν. For this we argue from the definition. Let O be a Scott-open set in D which contains at least one element of M (otherwise $\mu(O) = 0$). Then

$$\mu(O) \leq \bar{\mu}(O) - \varepsilon \leq \chi(O) - \varepsilon \leq \tilde{\chi}(O) \ .$$

There are only finitely many $\tilde{\chi}$ if we restrict the maximal weight at each $x \in X$ to be less than or equal to $\mathsf{max}\{\mu(D), \nu(D)\} + \varepsilon_1$. This completes the proof. ■

As a concluding remark we observe that this proof is valid independent of whether the total mass of valuations is restricted to be 1, to be less than or equal to 1, or whether it is allowed to be any number from the positive extended reals.

5 Open problems

It is annoying and almost embarrassing that we still don't know whether function spaces and the probabilistic powerdomain can be reconciled in a category of continuous domains. The question has the irritating feature that it is easier to come up with a "natural proof" than it is to find the right counterexample. We have gone through this iteration a number of times ourselves and our insight into the problem has not improved much. Theorems 13 and 17 demonstrate that even for well-structured posets the formal argument is quite involved.

If we were to suggest further work on the problem then we would probably recommend to start with parallel-serial posets. This, however, cannot be the whole story because we have a proof (not included in this paper) that every poset of height 2 leads to a probabilistic powerdomain which is FS, and not every such poset is parallel-serial.

More interesting than a further partial result for finite posets would be an analogue of Lemma 11 for FS-domains, that is, to show that if the probabilistic powerdomain of every finite poset is FS then every FS-domain has an FS probabilistic powerdomain. Such a proof would almost certainly shed light on other unresolved issues regarding the category **FS**.

As indicated at the end of Section 3, the results in this paper provide new examples of domains which are demonstrably FS but which are not known to be in **RB**. It would be very nice if we could make further progress on the question whether these two categories are different or not.

With respect to the last section it would be quite interesting to see whether a closure result holds for all coherent spaces, not just the coherent domains. A proof would have to work quite differently (for example, the topology on $\mathsf{P}X$ would not be the Scott-topology in general) and would most likely be more structural than the one offered here.

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