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FOUNDATIONS

A duality for two-sorted lattices

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Abstract

A series of representation theorems (some of which discovered very recently) present an alternative view of many classes of algebras related to non-classical logics (e.g. bilattices, semi-De Morgan, Nelson and quasi-Nelson algebras) as two-sorted algebras in the sense of many-sorted universal algebra. In all the above-mentioned examples, we are in fact dealing with a pair of lattices related by two meet-preserving maps. We use this insight to develop a Priestley-style duality for such structures, mainly building on the duality for meet-semilattices of G. Bezhanishvili and R. Jansana. Our approach simplifies all the existing dualities for these algebras and is applicable more generally; in particular, we show how it specialises to the class of quasi-Nelson algebras, which has not yet been studied from a duality point of view.

Keywords Priestley duality · Bilattice · Twist-structure · Semi-De Morgan algebra · Nelson algebra

1 Introduction

Many classes of algebras related to non-classical logics—e.g. bilattices, Nelson algebras and the more recently introduced quasi-Nelson algebras (see the next section for the relevant definitions and references)—can be conveniently represented through a product-like construction, known, among other names, as *twist-structure* (used already by, for example, Vakarelov 1977; but similar constructions have been studied since at least the 1950s: see, for example, Davey 2013). In this representation, one embeds an arbitrary algebra belonging to one of the above-mentioned classes (say, a Nelson algebra **A**) into a special power of some other (say, Heyting) algebra **H**, which is usually built using some extra structure on **H** (typically, a filter on **H**); one says that **A** is (isomorphic to) *a twist-structure over* **H**.

The main interest and usefulness of the twist-structure representation lie, of course, in the possibility of reducing, to a certain extent, the study of a somewhat esoteric class of algebras to more tame and well-known ones; as shown in the papers (Jansana and Rivieccio 2014; Jung and Rivieccio 2013; Rivieccio et al. 2017), such a strategy has already given its fruits in the field of duality, and indeed, the present paper may be seen as a continuation of the line of research of Jansana and Rivieccio (2014), Jung and Rivieccio (2013) and Rivieccio et al. (2017). In most cases, the twist-structure representation yields in fact a (covariant) categorical equivalence between the algebraic category of, for example, Nelson algebras and an algebraic category based on, for example, Heyting algebras with extra structure (Jansana and Rivieccio 2014, Section 2); in some cases, one only obtains an adjunction (Rivieccio 2014).

In all the above-mentioned examples, once such an equivalence of categories has been established, it is possible (and we will indeed do so, from a certain point on in this paper) to disregard the details of the concrete representation (i.e. *how* a Nelson algebra is constructed from the corresponding Heyting algebra, and vice versa) to focus only on the bijective correspondence between (say) Nelson algebras on the one side and (enriched) Heyting algebras on the other. This more abstract perspective will allow us to include yet another class of algebras in the framework we shall propose, namely Sankappanavar's semi-De Morgan algebras; for this, we will rely on the recent paper (Greco et al. 2017), which

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shows that semi-De Morgan algebras are representable via a construction that (while not being product-like) has strong analogies with twist-structures.

The essential element common to both representations (or rather categorical equivalences) is that they deal, on the one side, with (single-sorted) abstract algebras (Nelson algebras, semi-De Morgan algebras, etc.) and, on the other, with tuples of type $\langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$ where $\mathbf{L}_+$ and $\mathbf{L}_-$ are (usually, distributive) lattices, perhaps endowed with additional operations (e.g. implications, modalities) and structure, and $n\colon L_+ \to L_-$ and $p\colon L_- \to L_+$ are meet-preserving maps, in each case satisfying different additional requirements.[1] In fact, each such tuple $\langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$ can be viewed as one *many-sorted* abstract algebra in the standard sense of many-sorted universal algebra (see, for example, Birkhoff and Lipson 1970): the maps n and p are then viewed as unary many-sorted algebraic operations, whereas the lattice operations of $\mathbf{L}_+$ and $\mathbf{L}_-$ (as well as the additional ones such as implications or modalities) act within a single sort.

We will call tuples of type $\langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$, in their most general form, *two-sorted lattices*. Imposing specific restrictions on the structure of the two lattices and/or the maps, we shall then obtain (tuples corresponding to) the various classes of the above-mentioned algebras.

The main idea inspiring the present paper (as well as its relatives (Jung and Rivieccio 2013; Jansana and Rivieccio 2014; Rivieccio et al. 2017) mentioned earlier) is to exploit the many-sorted analysis of non-classical algebras sketched above in order to obtain a topological duality. This strategy has two main advantages: (1) it provides a uniform treatment (independent of algebraic language) for a number of non-classical algebras, namely (involutive and non-involutive) bilattices, Nelson and quasi-Nelson algebras, N4-lattices and semi-De Morgan algebras and (2) the dualities thus obtained and the topological spaces involved are in all known cases (in our opinion) simpler than the existing (uni-sorted) ones available for the same classes of algebras (see, for example, Cignoli 1986; Hobby 1996; Celani 1999; Odintsov 2010); in the cases of non-involutive bilattices and quasi-Nelson algebras, they are indeed the *only* dualities currently available. Furthermore, as new representation results are established for other classes of algebras [ongoing research of ours is on Kleene algebras (Rivieccio 2020b) and Kleene algebras with a weak pseudo-complement (Rivieccio 2020a; Rivieccio et al. 2020)], we will be able to apply the same strategy, with no essential modifications, to these as well.

From a technical point of view, the main novelty and challenge of the present paper (if compared, for example, to Jansana and Rivieccio 2014; Jung and Rivieccio 2013; Rivieccio et al. 2017) stem from the observation that in the topological part of our study, we need to go beyond standard Priestley duality for distributive lattices. This is because the maps n and p appearing in a general two-sorted lattice $\langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$ need not be lattice homomorphisms (although $\mathbf{L}_+$ and $\mathbf{L}_-$ are indeed bounded distributive lattices); thus, n and p do not exist as morphisms in the standard category of distributive lattices and would not be representable as Priestley functions in the dual category. From this observation arises the insight that we should instead be taking the category of meet-semilattices with meet-preserving maps as our "base category", and accordingly, we should look for a topological duality for meet-semilattices as one of the main ingredients in our approach.

Among the various dualities for meet-semilattices available in the literature (see the references given in Bezhanishvili and Jansana 2011), we have chosen to use the one(s) introduced by Bezhanishvili and Jansana (2011) and Bezhanishvili and Jansana (2013). Indeed, their approach turns out to be particularly convenient for us, because, on the one hand, we can build on their duality for meet-semilattices (Bezhanishvili and Jansana 2011) to deal with the implication-less algebras we are interested in (bilattices, semi-De Morgan algebras), and on the other, we may use the subsequent paper (Bezhanishvili and Jansana 2013) which extends their results to implicative semilattices in order to obtain dualities for bilattices with implication and quasi-Nelson algebras. Besides these papers (as we will indicate in the next sections), our results also rely on Cornish and Fowler's duality for De Morgan algebras (Cornish and Fowler 1977) and Esakia's duality (Esakia 1974) for Heyting algebras (both of them in turn obviously relying on Priestley's work on distributive lattices). The duality for lattices with a negation operator due to Celani (1999), though not often cited in the next sections, has also been very useful to us during the preparation of the paper.

The rest of the paper is organised as follows. Section 2 introduces and presents the principal results on the relevant classes of algebras: (non-involutive) bilattices (Sect. 2.1), semi-De Morgan (2.2) and quasi-Nelson algebras (2.3). We focus, in particular, on the representation theorems that allow us to view each member of these classes of algebras as a two-sorted lattice. In Sect. 3, we recall the basics of Priestley duality for distributive meet-semilattices as introduced by Bezhanishvili and Jansana, which is the main framework into which our dualities are placed. In Sect. 4, we formally introduce two-sorted lattices in their most general form and explain how the Bezhanishvili–Jansana duality can be adapted to obtain a duality for them. Finally, Sect. 5 shows how the general duality specialises to the different subclasses

[1] The notation $\langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$ originates in the literature on bilattices and bitopology (Jakl et al. 2016), in which $\mathbf{L}_+$ and $\mathbf{L}_-$ are viewed as truth value spaces corresponding to (resp.) positive and negative information concerning, say, a proposition. Thus, the map n (which has the lattice L_+ as source and L_- as target) allows one to "translate" positive information into negative, and likewise, p provides a translation the other way round.

of two-sorted lattices corresponding to the classes of algebras introduced in Sect. 2.

2 Bilattices, semi-De Morgan and quasi-Nelson algebras

In this section, we introduce the classes of algebras we are interested in. For each class of algebras, we give an abstract (equational) presentation and an equivalent one by means of tuples; we will rely on the latter for our duality.

2.1 Non-involutive bilattices and their representation

For further details and proofs of results about bilattices, we refer the reader to Bou et al. (2011) and Rivieccio et al. (2020).

Definition 2.1 (Interlaced pre-bilattice) An *interlaced pre-bilattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup \rangle$ of type $\langle 2, 2, 2, 2 \rangle$ such that $\langle B, \wedge, \vee \rangle$ and $\langle B, \sqcap, \sqcup \rangle$ are lattices, and each one of the four operations $\{\vee, \wedge, \sqcup, \sqcap\}$ is monotonic with respect to both lattice orders. $\mathbf{B}$ is *distributive* if $\langle B, \wedge, \vee \rangle$ and $\langle B, \sqcap, \sqcup \rangle$ are both distributive lattices.

We denote the lattice orderings of $\langle B, \wedge, \vee \rangle$ and $\langle B, \sqcap, \sqcup \rangle$ by $\leq$ and $\sqsubseteq$, respectively. If present, the lattice bounds of $\langle B, \wedge, \vee \rangle$ are denoted by f (minimum) and t (maximum). Likewise, we denote by $\perp$ and $\top$ the minimum and maximum (if present) of $\langle B, \sqcap, \sqcup \rangle$.

From now on, we write $x \equiv_+ y$ as a shorthand for the identity $x \wedge y = x \sqcup y$, and $x \equiv_- y$ as a shorthand for the identity $x \wedge y = x \sqcap y$. We shall be interested in the two (equivalence) relations determined by the preceding identities on a (pre-)bilattice; our notation is meant to suggest that the "positive information lattice" of the desired twist-representation (see, for example, Definition 2.3) is obtained as a (partial) quotient via the relation determined by $\equiv_+$, and likewise, the relation associated with $\equiv_-$ allows us to recover the "negative information lattice".

Following a terminology introduced by M. Fitting (and followed by O. Arieli, A. Avron, B. Davey, H. Priestley, etc.), we speak of *pre-bilattices* when the negation operator is not present, reserving the term *bilattice* to algebras with negation.

Definition 2.2 (Non-involutive bilattice) A *non-involutive bilattice* is an interlaced pre-bilattice $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \neg \rangle$ endowed with a unary *negation* operation $\neg$ satisfying the following identities:

(i) $\neg(x \sqcap y) = \neg x \sqcap \neg y$,
(ii) $\neg\perp = \perp \quad \neg\top = \top \quad \neg\mathsf{t} = \mathsf{f} \quad \neg\mathsf{f} = \mathsf{t}$ (if bounds are present),
(iii) $\neg\neg x \sqsubseteq x$.
(iv) $\neg(x \wedge y) \equiv_+ \neg(x \sqcup y) \quad \neg(x \wedge y) \equiv_- \neg(x \sqcap y)$.

$\mathbf{B}$ is *(bounded) distributive* if $\langle B, \wedge, \vee \rangle$ and $\langle B, \sqcap, \sqcup \rangle$ are both (bounded) distributive lattices.

A standard (involutive) bilattice (Bou et al. 2011, Definition 2.2) additionally satisfies $x \sqsubseteq \neg\neg x$ and $\neg(x \sqcup y) = \neg x \sqcup \neg y$ (in which case the usual De Morgan identities $\neg(x \wedge y) = \neg x \vee \neg y$ and $\neg(x \vee y) = \neg x \wedge \neg y$ also hold).

Our presentation of (non-involutive) bilattices as two-sorted lattices relies on the following constructions.

Definition 2.3 (Product pre-bilattice) Let $\mathbf{L}_+ = \langle L_+, \wedge_+, \vee_+ \rangle$ and $\mathbf{L}_- = \langle L_-, \wedge_-, \vee_- \rangle$ be lattices. The *product pre-bilattice* $\langle L_+ \times L_-, \wedge, \vee, \sqcap, \sqcup \rangle$ is defined as follows. For all $\langle a_+, a_- \rangle, \langle b_+, b_- \rangle \in L_+ \times L_-$,

$$\langle a_+, a_- \rangle \wedge \langle b_+, b_- \rangle = \langle a_+ \wedge_+ b_+, a_- \vee_- b_- \rangle$$
$$\langle a_+, a_- \rangle \vee \langle b_+, b_- \rangle = \langle a_+ \vee_+ b_+, a_- \wedge_- b_- \rangle$$
$$\langle a_+, a_- \rangle \sqcap \langle b_+, b_- \rangle = \langle a_+ \wedge_+ b_+, a_- \wedge_- b_- \rangle$$
$$\langle a_+, a_- \rangle \sqcup \langle b_+, b_- \rangle = \langle a_+ \vee_+ b_+, a_- \vee_- b_- \rangle.$$

Thus, the lattice reduct $\langle L_+ \times L_-, \sqcap, \sqcup \rangle$ is just the standard direct product $\mathbf{L}_+ \times \mathbf{L}_-$, while the reduct $\langle L_+ \times L_-, \wedge, \vee \rangle$ is the direct product $\mathbf{L}_+ \times (\mathbf{L}_-)^{op}$, with $(\mathbf{L}_-)^{op}$ denoting the order-theoretic dual of the lattice $\langle L_-, \vee_-, \wedge_- \rangle$.

Definition 2.4 (Non-involutive product bilattice) Let $\mathbf{L}_+ = \langle L_+, \wedge_+, \vee_+ \rangle$ and $\mathbf{L}_- = \langle L_-, \wedge_-, \vee_- \rangle$ be lattices, and let $n \colon L_+ \to L_-$ and $p \colon L_- \to L_+$ be maps satisfying the following properties:

(i) n, p are both meet-semilattice homomorphisms (preserve finite meets);
(ii) n, p preserve all lattice bounds of $\mathbf{L}_+$ and $\mathbf{L}_-$ (if present);
(iii) $p \circ n \leq_+ Id_{L_+}$ and $n \circ p \leq_- Id_{L_-}$.

The *non-involutive product bilattice* is the algebra $\mathbf{L}_+ \bowtie \mathbf{L}_- = \langle L_+ \times L_-, \wedge, \vee, \sqcap, \sqcup, \neg \rangle$ where $\langle L_+ \times L_-, \wedge, \vee, \sqcap, \sqcup \rangle$ is the product pre-bilattice as per Definition 2.3 and the *negation* is given, for all $\langle a_+, a_- \rangle \in L_+ \times L_-$, by

$$\neg\langle a_+, a_- \rangle = \langle p(a_-), n(a_+) \rangle.$$

The standard (involutive) product bilattice construction of, for example, Bou et al. (2011) corresponds to the case where n and p are mutually inverse lattice isomorphisms.

Let $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \neg \rangle$ be a (bounded, distributive) non-involutive bilattice. Consistently with the notation introduced earlier, we let:

$$\equiv_+ = \{\langle a, b\rangle \in B \times B : a \wedge b = a \sqcup b\}$$
$$\equiv_- = \{\langle a, b\rangle \in B \times B : a \wedge b = a \sqcap b\}.$$

Then the quotients $\mathbf{B}_+ = \langle B/{\equiv_+}, \wedge, \vee\rangle$ and $\mathbf{B}_- = \langle B/{\equiv_-}, \vee, \wedge\rangle$ are both (bounded, distributive) lattices. Moreover, we can define maps $n \colon B/{\equiv_+} \to B/{\equiv_-}$ and $p \colon B/{\equiv_-} \to B/{\equiv_+}$ satisfying the conditions of Definition 2.4 by letting $n([a]_+) = [\neg a]_-$ and $p([a]_-) = [\neg a]_+$. In this way, we obtain the representation result for non-involutive bilattices first proved in Rivieccio et al. (2020, Theorem 3.5).

Theorem 2.5 *Every non-involutive bilattice* $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \neg\rangle$ *is isomorphic to the non-involutive product bilattice* $\mathbf{B}_+ \bowtie \mathbf{B}_-$ *constructed according to Definitions 2.3 and 2.4, with negation defined by* $\neg\langle [a]_+, [a]_-\rangle = \langle p([a]_-), n([a]_+)\rangle$ *for all* $a \in B$*. The isomorphism is given by the map* $\iota \colon B \to B/{\equiv_+} \times B/{\equiv_-}$ *defined as* $\iota(a) = \langle [a]_+, [a]_-\rangle$ *for all* $a \in B$*.*

Theorem 2.5 tells us that we can view a non-involutive bilattice $\mathbf{B}$ as a tuple $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ satisfying the conditions of Definition 2.4. As mentioned in Rivieccio et al. (2020, Section 3), this correspondence extends to a (covariant) categorical equivalence between naturally associated categories. On the one side, we have non-involutive bilattices with algebraic homomorphisms among them; on the other, we have tuples $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ together with the notion of two-sorted morphism, which we will apply uniformly to all two-sorted lattices (see Definition 4.5).

In Rivieccio et al. (2020), we also considered the case where the lattices $\mathbf{L}_+$ and $\mathbf{L}_-$ possess an intuitionistic implication. We recall the relevant definitions below.

An *implicative lattice* (also known in the literature as a *Brouwerian lattice* or *relatively pseudo-complemented lattice*) is a lattice $\mathbf{L} = \langle L, \wedge, \vee, \to\rangle$ expanded with an extra binary operation $\to$ (called *implication*) which satisfies the following *residuation* property: $a \wedge b \leq c$ if and only if $b \leq a \to c$, for all $a, b, c \in L$. Implicative lattices are the algebraic counterpart of the negation-free fragment of intuitionistic logic and correspond precisely to the bottom-free subreducts of Heyting algebras. This implies, in particular, that any implicative lattice is distributive and has a top element (denoted by 1). For our purposes, it is also useful to keep in mind that implicative lattices form an equational class.

Definition 2.6 (Non-involutive implicative product bilattice) Let $\mathbf{L}_+ = \langle L_+, \wedge_+, \vee_+, \to_+, 1_+\rangle$ and $\mathbf{L}_- = \langle L_-, \wedge_-, \vee_-, \to_-, 1_-\rangle$ be implicative lattices, and let $n \colon L_+ \to L_-$ and $p \colon L_- \to L_+$ be maps satisfying properties (i)–(iii) of Definition 2.4. The *non-involutive implicative product bilattice* is the algebra $\mathbf{L}_+ \bowtie \mathbf{L}_- = \langle L_+ \times L_-, \wedge, \vee, \sqcap, \sqcup, \to, \not\leftarrow, \neg\rangle$, whose $\{\to, \not\leftarrow\}$-free reduct is the product bilattice of Definition 2.4, and where the two binary *implication* operations are given, for all $\langle a_+, a_-\rangle, \langle b_+, b_-\rangle \in L_+ \times L_-$, by

$$\langle a_+, a_-\rangle \to \langle b_+, b_-\rangle = \langle a_+ \to_+ b_+,\ n(a_+) \wedge_- b_-\rangle$$
$$\langle a_+, a_-\rangle \not\leftarrow \langle b_+, b_-\rangle = \langle p(a_-) \wedge_+ b_+,\ a_- \to_- b_-\rangle.$$

Definition 2.6 generalises both the construction for the *Brouwerian bilattices* of Bou et al. (2011) and that for the *nd-frames* of Jakl et al. (2016, Definition 3.1). In fact, any Brouwerian bilattice can be seen as a non-involutive implicative product bilattice $\mathbf{L} \bowtie \mathbf{L}$ where the maps n, p are both the identity on $\mathbf{L}$ and $\not\leftarrow$ is definable as $\neg(\neg x \to \neg y)$.[2] The operation $\not\leftarrow$, though not considered in Jakl et al. (2016), structurally exists on any *nd*-frame, for both underlying frames of an *nd*-frame are complete Heyting algebras (in which the implications $\to_+$ and $\to_-$ are the residua of the lattice meets).

As in the implication-less case, it is possible to give a presentation for a class of abstract algebras in the language $\langle \wedge, \vee, \sqcap, \sqcup, \to, \not\leftarrow, \neg\rangle$ which correspond to non-involutive implicative product bilattices. Let $\mathbf{B}$ be one such algebra and $a, b \in B$. Writing $\Phi(a)$ as an abbreviation for $a \to a$, define:

$a \preccurlyeq_+ b$ if and only if $a \to b = \Phi(a \to b)$

$a \preccurlyeq_- b$ if and only if $\neg(a \not\leftarrow b) = \Phi(\neg(a \not\leftarrow b))$.

Definition 2.7 A *non-involutive implicative bilattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \to, \not\leftarrow, \neg\rangle$ of type $\langle 2, 2, 2, 2, 2, 2, 1\rangle$ satisfying the following properties:

(i) the relations $\preccurlyeq_+$ and $\preccurlyeq_-$ are pre-orders (i.e. reflexive and transitive),
(ii) $\leq\ = \preccurlyeq_+ \cap (\preccurlyeq_-)^{-1}$,
(iii) the equivalence relation $\equiv_+$ induced by $\preccurlyeq_+$ is compatible with the operations $\wedge, \vee, \to$,
(iv) the equivalence relation $\equiv_-$ induced by $\preccurlyeq_-$ is compatible with the operations $\wedge, \vee, \not\leftarrow$,
(v) the quotients $\mathbf{B}_+ = \langle B/{\equiv_+}, \wedge, \vee, \to\rangle$ and $\mathbf{B}_- = \langle B/{\equiv_-}, \vee, \wedge, \not\leftarrow\rangle$ are implicative lattices[3],
(vi) $x \equiv_+ y$ entails $\neg x \equiv_- \neg y$ and $x \equiv_- y$ entails $\neg x \equiv_+ \neg y$,
(vii) $x \not\leftarrow y \equiv_+ \neg x \wedge y$ and $x \to y \equiv_- \neg x \vee y$,
(viii) $x \not\leftarrow x \equiv_- \neg(x \to x)$ and $x \to x \equiv_+ \neg(x \not\leftarrow x)$,
(ix) $\neg(x \vee y) \equiv_+ \neg x \wedge \neg y$ and $\neg(x \wedge y) \equiv_- \neg x \vee \neg y$,
(x) $\neg\neg x \preccurlyeq_+ x$ and $\neg\neg x \preccurlyeq_- x$.
(xi) $\neg\bot = \bot$ $\neg\top = \top$ $\neg\mathsf{t} = \mathsf{f}$ $\neg\mathsf{f} = \mathsf{t}$ (if the bounds mentioned are present),
(xii) $\langle B, \wedge, \vee, \sqcap, \sqcup\rangle$ is an interlaced pre-bilattice on which the relations $\equiv_+$ and $\equiv_-$ coincide with those of Definition 2.2.

[2] In classical propositional logic, $\neg(\neg x \to \neg y)$ is equivalent to $\neg(y \to x)$, hence our notation $x \not\leftarrow y$.

[3] Notice that $\mathbf{B}_-$ has $\vee$ as meet (whose residuum is $\not\leftarrow$) and $\wedge$ as join.

As expected, every non-involutive implicative bilattice has a non-involutive bilattice reduct, and every non-involutive implicative product bilattice is a non-involutive implicative bilattice (Rivieccio et al. 2020, Proposition 4.6). Moreover, the class of all non-involutive implicative bilattices is equationally definable (Rivieccio et al. 2020, Proposition 4.9). The representation result is Rivieccio et al. (2020, Theorem 4.7), which we repeat below.

Theorem 2.8 *Every non-involutive implicative bilattice* $\mathbf{B}$ *is isomorphic to the non-involutive implicative product bilattice* $\mathbf{B}_+ \bowtie \mathbf{B}_-$ *via the map* ι *defined in Theorem 2.5.*

As in the implication-less case, Theorem 2.8 allows us to identify non-involutive implicative bilattices with tuples $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ satisfying the conditions of Definition 2.6. This correspondence also extends to a (covariant) categorical equivalence between naturally associated categories. On the one side, we have non-involutive implicative bilattices with algebraic homomorphisms; on the other, we have tuples $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ together with two-sorted morphisms (Definition 4.5) which additionally preserve the two intuitionistic implications.

2.2 Semi-De Morgan algebras and their representation

In the literature on non-classical and algebraic logic, one finds a wide and constantly growing variety of generalisations of Boolean algebras. Some of these are algebras that have (at least) one implication operation (e.g. Heyting algebras, residuated lattices, (quasi-)Nelson algebras), which is obviously a weakening of the classical implication; others are algebras in a signature only comprising conjunction(s), disjunction(s) and negation(s). Well-known examples of the latter are De Morgan algebras (algebraic models of the four-valued Belnap–Dunn logic) and bilattices (models of the logics introduced by Ginsberg 1988 and by Arieli and Avron 1996); some of these algebras are well known and have been studied by logically oriented algebraists and topologists since several decades [e.g. Ockham lattices, Stone algebras, pseudo-complemented distributive lattices (Balbes and Dwinger 1974)].

The paper Sankappanavar (1987) proposed semi-De Morgan algebras as a common generalisation of many of the above-mentioned classes, encompassing in particular both De Morgan algebras and pseudo-complemented distributive lattices, that is, both the algebraic models of the Belnap–Dunn logic and the models of the implication-free fragment of intutionistic logic. Probably due to the technical difficulty of axiomatising implication-free logics, a logical calculus corresponding to semi-De Morgan algebras has been introduced only recently in Greco et al. (2017), together with an algebraic representation on which our approach will be based. For further details and proofs of results about semi-De Morgan algebras, we refer the reader to the above-mentioned papers Sankappanavar (1987) and Greco et al. (2017).

We begin with the abstract (one-sorted) definition, which is due to Sankappanavar.

Definition 2.9 (Semi-De Morgan algebra) A *semi-De Morgan algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg, \mathsf{f}, \mathsf{t}\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ satisfying the following:

(S1) The reduct $\langle A, \wedge, \vee, \mathsf{f}, \mathsf{t}\rangle$ is a bounded distributive lattice (with order $\leq$),
(S2) $\neg\mathsf{f} = \mathsf{t}$ and $\neg\mathsf{t} = \mathsf{f}$,
(S3) $\neg(x \vee y) = \neg x \wedge \neg y$,
(S4) $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y$,
(S5) $\neg x = \neg\neg\neg x$.

$\mathbf{A}$ is a *De Morgan algebra* if $x = \neg\neg x$.

The postulates (S2) to (S5) can be viewed as minimal requirements for a unary operation that interprets the negation in a logic; notice, in particular, that they all correspond to intuitionistically sound equational laws. As shown in Greco et al. (2017), semi-De Morgan algebras can be represented as tuples, but unlike the previously considered cases, the representation theorem is not based on a product construction.

Definition 2.10 (Heterogeneous Semi-De Morgan algebra) A *heterogeneous semi-De Morgan algebra* is a tuple $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ such that:

(H1) $\mathbf{L}_+ = \langle L_+, \leq_+, \wedge_+, \vee_+, 0_+, 1_+\rangle$ is a bounded distributive lattice,
(H2) $\mathbf{L}_- = \langle L_-, \leq_-, \wedge_-, \vee_-, \neg_-, 0_-, 1_-\rangle$ is a De Morgan algebra,
(H3) $p\colon L_- \to L_+$ is an (injective) bounded meet-semilattice homomorphism,
(H4) $n\colon L_+ \to L_-$ is a (surjective) bounded lattice homomorphism,
(H5) $Id_{L_-} = n \circ p$.

Given a heterogeneous semi-De Morgan algebra $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$, we can obtain a semi-De Morgan algebra $\langle L_+, \wedge_+, \vee_+, \neg_+, 0_+, 1_+\rangle$ by endowing $\mathbf{L}_+$ with a negation $\neg_+$ defined, for all $a_+ \in L_+$, by

$$\neg_+ a_+ = p(\neg_- n(a_+)).$$

Conversely, every semi-De Morgan algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg, \mathsf{f}, \mathsf{t}\rangle$ determines a heterogeneous semi-De Morgan algebra as follows. Consider the set $A_- = \{\neg a : a \in A\}$ and the operation defined on it by $a \vee_- b = \neg\neg(a \vee b)$ for all $a, b \in A_-$. Then $\langle A_-, \wedge, \vee_-, \neg, \mathsf{f}, \mathsf{t}\rangle$ is a De Morgan algebra, and we obtain a heterogeneous semi-De Morgan algebra $\langle \mathbf{A}_+, \mathbf{A}_-, n, p\rangle$ by defining:

(i) $\mathbf{A}_+ = \langle A, \wedge, \vee, \mathsf{f}, \mathsf{t}\rangle$,
(ii) $\mathbf{A}_- = \langle A_-, \wedge, \vee_-, \neg, \mathsf{f}, \mathsf{t}\rangle$,
(iii) $p(a) = a$ for all $a \in A_-$,
(iv) $n(b) = \neg\neg b$ for all $b \in A$.

As in the preceding cases, the two constructions yield an equivalence; that is, every semi-De Morgan algebra $\mathbf{A}$ is isomorphic to the one obtained from the heterogeneous semi-De Morgan algebra $\langle \mathbf{A}_+, \mathbf{A}_-, n, p\rangle$ (Greco et al. 2017, Proposition 4). The equivalence extends straightforwardly to morphisms, which are semi-De Morgan algebra homomorphisms, on the one side, and two-sorted morphisms (Definition 4.5) between heterogeneous semi-De Morgan algebras, on the other. Thus, in this case too we have a (covariant) categorical equivalence between semi-De Morgan algebras and heterogeneous semi-De Morgan algebras.

2.3 Quasi-Nelson algebras and their representation

As mentioned earlier, Nelson's logic (Nelson 1949) is a generalisation of Boolean logic. As in other well-known non-classical systems (e.g. linear logic) where the Boolean conjunction and disjunction are replaced by two pairs of connectives, so in Nelson's logic the classical implication is replaced by two connectives, the *weak implication* ($\to$) and the *strong implication* ($\Rightarrow$). Some classical properties (e.g. so-called *contraction* and the deduction–detachment theorem) are retained (only) by the former implication, while others (e.g. so-called *contraposition* and *residuation*) are enjoyed by the latter. Each of the two Nelson implications determines its own negation, and to each is associated a distinct conjunction. This richness of language makes Nelson's logic particularly interesting, both intrinsically and as a tool for applications, in particular in the area of philosophical logic where it was originally introduced.

The algebraic models of Nelson's logic (*Nelson algebras*) have been studied intensely since the 1970s (by such logicians as H. Rasiowa, D. Vakarelov, V. Goranko, M. Kracht) and are by now fairly well understood. However, our insight on one of the key features (perhaps *the* key feature) of Nelson's logic, namely the *Nelson axiom* relating strong and weak implication (see Spinks et al. 2019), remains to this day somewhat unsatisfactory. The papers Spinks et al. (2019) and the subsequent Rivieccio and Spinks (2019) and Rivieccio and Spinks were aimed at clarifying the meaning and implications of the Nelson axiom in the context of residuated lattices (i.e. models of the Full Lambek Calculus: see Galatos et al. (2007)). The class of *quasi-Nelson algebras* arose as a natural object of interest in such a setting, corresponding to the subclass of (commutative, integral, bounded) residuated lattices that satisfy the Nelson axiom, but not necessarily the classical law of double negation.

As argued in Rivieccio and Spinks (2019) and Rivieccio and Spinks, the interest in quasi-Nelson algebras has different sources. From a logical point of view, quasi-Nelson algebras may be viewed as (the algebraic models of) a common generalisation of intuitionistic and Nelson's logic which retains a great deal of the characteristic features of the latter. From the point of view that concerns us here, quasi-Nelson algebras are particularly interesting in that they provide an example of non-involutive algebras which can be represented via twist-structures.

The algebraic language of quasi-Nelson algebras is a fragment of that of bounded non-involutive implicative bilattices, which allows us to adopt similar notational conventions. In particular, we define $a \preccurlyeq_+ b$ if and only if $a \to b = \mathsf{t}$, and $x \equiv_+ y$ for the conjunction of the two identities $x \to y = y \to x = \mathsf{t}$. (Because of the structural properties of quasi-Nelson algebras, it is not necessary to consider the "negative" relations $\preccurlyeq_-$ and $\equiv_-$.)

Definition 2.11 A *quasi-Nelson algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \to, \neg, \mathsf{f}, \mathsf{t}\rangle$ of type $\langle 2, 2, 2, 1, 0, 0\rangle$ satisfying the following properties:

(QN1) The reduct $\langle A, \wedge, \vee, \mathsf{f}, \mathsf{t}\rangle$ is a bounded distributive lattice (with order $\leq$).
(QN2) The relation $\preccurlyeq_+$ is a pre-order (i.e. reflexive and transitive).
(QN3) The equivalence relation $\equiv_+$ induced by $\preccurlyeq_+$ is compatible with the operations $\wedge, \vee, \to$, and the quotient algebra $\mathbf{A}_+ = \langle A, \wedge, \vee, \to, \mathsf{f}, \mathsf{t}\rangle/{\equiv_+}$ is a Heyting algebra.
(QN4) $\neg(x \to y) \equiv_+ \neg\neg(x \wedge \neg y)$.
(QN5) $x \leq y$ iff $x \preccurlyeq_+ y$ and $\neg y \preccurlyeq_+ \neg x$.
(QN6) The following identities hold:

(QN6.1) $\neg\neg(\neg x \to \neg y) \equiv_+ \neg x \to \neg y$
(QN6.2) $\neg(x \vee y) \equiv_+ \neg x \wedge \neg y$
(QN6.3) $\neg\neg x \wedge \neg\neg y \equiv_+ \neg\neg(x \wedge y)$
(QN6.4) $\neg x \equiv_+ \neg\neg\neg x$
(QN6.5) $x \preccurlyeq_+ \neg\neg x$
(QN6.6) $x \wedge \neg x \preccurlyeq_+ \mathsf{f}$.

$\mathbf{A}$ is a *Nelson algebra* if it satisfies the identity $\neg\neg x \leq x$ (or, equivalently, $\neg\neg x = x$).

The above definition is a generalisation of Rasiowa's presentation of Nelson algebras (Rasiowa 1974, Ch. V, p. 68) as well as Odintsov's definition of *N4-lattices* (Odintsov 2003, Definition 5.1). It follows from Rivieccio and Spinks (Rivieccio and Spinks 2019, Theorem 4.4) that quasi-Nelson algebras form an equational class. We observe that the $\langle \wedge, \vee, \neg, \mathsf{f}, \mathsf{t}\rangle$-reduct of every quasi-Nelson algebra is a semi-De Morgan algebra (Rivieccio and Spinks 2019, Proposition 2.7), in fact a special one, for certain identities are satisfied that are not valid on all semi-De Morgan algebras (e.g. $x \leq \neg\neg x$ which defines *lower quasi-De Morgan algebras* in Sankappanavar's terminology). On the other hand, the $\langle \wedge, \vee, \rightarrow, \neg, \mathsf{f}, \mathsf{t}\rangle$-reduct of a non-involutive implicative bilattice (or even of a Brouwerian bilattice in the sense of Bou et al. 2011) need not be a quasi-Nelson algebra, because, for example, item (QN6.4) or (QN6.6) may fail.

One of the main results of Rivieccio and Spinks (2019) is that quasi-Nelson algebras can also be represented through a product construction similar to the ones introduced earlier. (For further details and proofs of results, we refer the reader to Rivieccio and Spinks 2019.)

Definition 2.12 Let $\mathbf{H}_+ = \langle H_+, \wedge_+, \vee_+, \rightarrow_+, 0_+, 1_+\rangle$ and $\mathbf{H}_- = \langle H_-, \wedge_-, \vee_-, \rightarrow_-, 0_-, 1_-\rangle$ be Heyting algebras and $n\colon H_+ \to H_-$ and $p\colon H_- \to H_+$ be maps satisfying the following properties:

(i) n is a bounded lattice homomorphism,
(ii) p preserves meets and both lattice bounds,[4]
(iii) $n \circ p = Id_{H_-}$ and $Id_{H_+} \leq_+ p \circ n$.

The algebra $\mathbf{H}_+ \bowtie \mathbf{H}_- = \langle H_+ \times H_-, \wedge, \vee, \rightarrow, \neg, 0, 1\rangle$ is defined as follows. For all $\langle a_+, a_-\rangle, \langle b_+, b_-\rangle \in H_+ \times H_-$,

$$1 = \langle 1_+, 0_-\rangle$$
$$0 = \langle 0_+, 1_-\rangle$$
$$\neg\langle a_+, a_-\rangle = \langle p(a_-), n(a_+)\rangle$$
$$\langle a_+, a_-\rangle \wedge \langle b_+, b_-\rangle = \langle a_+ \wedge_+ b_+, a_- \vee_- b_-\rangle$$
$$\langle a_+, a_-\rangle \vee \langle b_+, b_-\rangle = \langle a_+ \vee_+ b_+, a_- \wedge_- b_-\rangle$$
$$\langle a_+, a_-\rangle \rightarrow \langle b_+, b_-\rangle = \langle a_+ \rightarrow_+ b_+, n(a_+) \wedge_- b_-\rangle.$$

A *twist-algebra* $\mathbf{A}$ over $\mathbf{H}_+ \bowtie \mathbf{H}_-$ is a $\{\wedge, \vee, \rightarrow, \neg, 0, 1\}$-subalgebra of $\mathbf{H}_+ \bowtie \mathbf{H}_-$ with carrier set A satisfying $\pi_1(A) = H_+$ and $a_+ \wedge_+ p(a_-) = 0_+$ for all $\langle a_+, a_-\rangle \in A$.

Remark 2.13 The set $B := \{\langle a_+, a_-\rangle \in H_+ \times H_-\colon a_+ \wedge_+ p(a_-) = 0_+\}$ is the universe of the largest twist-algebra $\mathbf{B}$ over $\mathbf{H}_+ \bowtie \mathbf{H}_-$, and all others are precisely those subalgebras of $\mathbf{B}$ that satisfy $\pi_1(A) = H_+$. Observe, moreover, that the condition $a_+ \wedge_+ p(a_-) = 0_+$ entails $n(a_+) \wedge_- a_- = 0_-$ for all $\langle a_+, a_-\rangle \in A$. Likewise, $\pi_2[A] = H_-$ follows from $\pi_1[A] = H_+$.

Every twist-algebra is a quasi-Nelson algebra (Rivieccio and Spinks 2019, Corollary 3.4). Moreover, every quasi-Nelson algebra $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, \neg, \mathsf{f}, \mathsf{t}\rangle$ can be viewed as a twist-algebra in the following way. By (QN3), we have a Heyting algebra quotient $\mathbf{A}_+ = \langle A_+, \wedge_+, \vee_+, \rightarrow_+, 0_+, 1_+\rangle$. The second Heyting algebra $\mathbf{A}_-$ is defined as follows. Denoting by $[b]$ the equivalence class of each $b \in A$ modulo $\equiv_+$, we consider the set $A_- = \{[\neg a] : a \in A\} \subseteq A_+$. We endow A_- with Heyting algebra operations as follows. For all $a, b \in A$, let

$[\neg a] \wedge_- [\neg b] = [\neg(a \vee b)]$
$(= [\neg a \wedge \neg b] = [\neg a] \wedge_+ [\neg b]$, by Def. 2.11 (QN6.2))
$[\neg a] \vee_- [\neg b] = [\neg(a \wedge b)]$
$[\neg a] \rightarrow_- [\neg b] = [\neg\neg(\neg a \rightarrow \neg b)]$ $(= [\neg a \rightarrow \neg b] = [\neg a] \rightarrow_+ [\neg b])$, by Def. 2.11 (QN6.1))
$0_- = [\neg \mathsf{t}]$ $(= [\mathsf{f}] = 0_+)$
$1_- = [\neg \mathsf{f}]$ $(= [\mathsf{t}] = 1_+)$.

It is easy to show that the above operations are well defined. (In particular, $[a] = [c]$ and $[b] = [d]$ imply $[\neg(a \wedge b)] = [\neg(c \wedge d)]$.) The set A_- is the universe of a $\langle \wedge_+, \rightarrow_+, 0_+, 1_+\rangle$-subalgebra of A_+. Note that $\vee_-$, though not coinciding with the restriction of $\vee_+$ to A_-, is nevertheless a join for $\wedge_-$ in $\mathbf{A}_-$, so that $\mathbf{A}_-$ forms a Heyting algebra in its own right, though not a Heyting subalgebra of $\mathbf{A}_+$. The maps $p\colon A_- \to A_+$ and $n\colon A_+ \to A_-$ are defined as follows: p is the identity map on A_- and $n[a] = [\neg\neg a]$ for all $a \in A$. These maps satisfy the conditions of Definition 2.12, which gives us the following result (Rivieccio and Spinks 2019, Proposition 4.2).

Theorem 2.14 *Every quasi-Nelson algebra* $\mathbf{A}$ *is isomorphic to a twist-algebra over* $\mathbf{A}_+ \bowtie \mathbf{A}_-$ *by the map* $\iota(a) = \langle [a], [\neg a]\rangle$.

Unlike the preceding cases, Theorem 2.14 alone would not allow us to identify each quasi-Nelson algebra with a tuple $\langle \mathbf{H}_+, \mathbf{H}_-, n, p\rangle$, because distinct quasi-Nelson algebras may be isomorphic to twist-algebras over the same algebra $\mathbf{H}_+ \bowtie \mathbf{H}_-$. To recover a one-to-one correspondence, we need to improve the representation result of Rivieccio and Spinks (2019) by adding one more ingredient to our tuples.

Recall that, given a Heyting algebra $\mathbf{H} = \langle H, \wedge, \vee, \rightarrow, 0, 1\rangle$, the set of *dense elements* of $\mathbf{H}$ (which indeed forms a lattice filter) is defined as

$$D(\mathbf{H}) := \{a \in H : a \rightarrow 0 = 0\}.$$

It is shown in Rivieccio and Spinks (2020) that every quasi-Nelson algebra can be identified with a tuple $\langle \mathbf{H}_+, \mathbf{H}_-, n, p, \nabla_+\rangle$, where ∇_+ is a lattice filter of $\mathbf{H}_+$ which contains $D(\mathbf{H})$. The following result is (Rivieccio and Spinks (2020), Proposition 9).

Proposition 2.15 *Let* $\mathbf{H}_+$ *and* $\mathbf{H}_-$ *be Heyting algebras and* $n\colon H_+ \to H_-$ *and* $p\colon H_- \to H_+$ *maps satisfying the conditions of Definition 2.12. Let* $\nabla \subseteq H_+$ *be a lattice filter of*

[4] As observed in Rivieccio and Spinks (2019) and Rivieccio and Spinks (2020), conditions (i)–(iii) entail that p also preserves the Heyting implication.

$\mathbf{H}_+$ *containing* $D(\mathbf{H}_+)$. *Then the set*

$$A := \{\langle a_+, a_-\rangle \in \mathbf{H}_+ \times \mathbf{H}_- : a_+ \vee_+ p(a_-) \in \nabla,\ a_+ \wedge_+ p(a_-) = 0_+\}$$

is the universe of a twist-algebra over $\mathbf{H}_+ \bowtie \mathbf{H}_-$.

By Proposition 2.15, each tuple $\langle \mathbf{H}_+, \mathbf{H}_-, n, p, \nabla\rangle$ determines a twist-algebra over $\mathbf{H}_+ \bowtie \mathbf{H}_-$. Following the notation of Odintsov (2004), we denote this twist-algebra by $Tw\langle \mathbf{H}_+, \mathbf{H}_-, n, p, \nabla\rangle$. The next result (Rivieccio and Spinks (2020), Proposition 10) entails that every quasi-Nelson algebra is isomorphic to $Tw\langle \mathbf{H}_+, \mathbf{H}_-, n, p, \nabla\rangle$ for a suitable choice of ∇.

Proposition 2.16 *Let* $\mathbf{A}$ *be a twist-algebra over* $\mathbf{H}_+ \bowtie \mathbf{H}_-$, *and define* $I(\mathbf{A}) := \{a \in A : \neg a \leq a\}$. *Then:*

(i) $I(\mathbf{A})$ *is a lattice filter of* $\mathbf{A}$, *and one has* $I(\mathbf{A}) = \{\langle a_+, 0_-\rangle : \langle a_+, 0_-\rangle \in A\}$.
(ii) $\nabla_{\mathbf{A}} := \pi_1[I(\mathbf{A})]$ *is a lattice filter of* $\mathbf{H}_+$.
(iii) $D(\mathbf{H}_+) \subseteq \nabla_{\mathbf{A}}$.
(iv) $\mathbf{A} = Tw\langle \mathbf{H}_+, \mathbf{H}_-, n, p, \nabla_{\mathbf{A}}\rangle$.

Corollary 2.17 *Every quasi-Nelson algebra* $\mathbf{A}$ *is isomorphic to* $Tw\langle \mathbf{A}_+, \mathbf{A}_-, n, p, \nabla_{\mathbf{A}}\rangle$, *where* $\nabla_{\mathbf{A}} = I(\mathbf{A})/{\equiv_+}$.

3 Priestley dualities

In this section, we briefly recall the main notions from Priestley duality (for bounded distributive lattices and for meet semi-lattices) that we shall need in the sequel of the paper.

The original Priestley duality concerns the category D of bounded distributive lattices and bounded lattice homomorphisms. To every bounded distributive lattice $\mathbf{L}$, one associates the set $X(\mathbf{L})$ of its prime filters. On $X(\mathbf{L})$, one has the *Priestley topology* τ, generated by the sets $\Phi(a) := \{x \in X(\mathbf{L}) : a \in x\}$ and $\Phi'(a) := \{x \in X(\mathbf{L}) : a \notin x\}$, and the inclusion relation between prime filters as an order. The resulting ordered topological spaces are called *Priestley spaces*[5]. A homomorphism h between bounded distributive lattices $\mathbf{L}$ and $\mathbf{L}'$ gives rise to a function $X(h) : X(\mathbf{L}') \to X(\mathbf{L})$, defined by $X(h)(x') = h^{-1}[x']$, that is continuous and order preserving. Taking functions with these properties, called *Priestley functions*, as morphisms between Priestley spaces one obtains the category PrSp, and X is now readily recognised as a contravariant functor from D to PrSp.

For a functor in the opposite direction, one associates with every Priestley space $X = \langle X, \tau, \leq\rangle$ the set $L(X)$ of clopen up-sets. This is a bounded distributive lattice with respect to the set-theoretic operations $\cap$, $\cup$, $\emptyset$ and X. To a Priestley map $f : X \to X'$, one associates the function $L(f)$, given by $L(f)(U') = f^{-1}[U']$, which is a bounded lattice homomorphism from $L(X')$ to $L(X)$. Together, then, L constitutes a contravariant functor from PrSp to D.

The two functors are adjoint to each other with the units given by:

$$\Phi_{\mathbf{L}} : \mathbf{L} \to L(X(\mathbf{L})) \quad \Phi_{\mathbf{L}}(a) = \{x \in X(\mathbf{L}) : a \in x\}$$
$$\Psi_X : X \to X(L(X)) \quad \Psi_X(x) = \{U \in L(X) : x \in U.\}$$

One shows that these are the components of a natural transformation from the identity functor on D to $L \cdot X$ and from the identity functor on PrSp to $X \cdot L$, respectively, satisfying the required diagrams for an adjunction. In particular, they are morphisms in their respective categories. Furthermore, they are *isomorphisms*, and thus, the central result of Priestley duality is obtained: *the categories* D *and* PrSp *are dually equivalent.*

All dualities in the rest of this paper concern bounded distributive lattices with additional structure; in each case, the functors X and L are defined as above and likewise for the units Φ and Ψ.

Section 2 indicates that our most general objects of interest are tuples $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ where $\mathbf{L}_+, \mathbf{L}_-$ are bounded distributive lattices and n, p are meet-preserving maps. This suggests that a suitable base category to work with will be one whose objects are bounded distributive (semi)-lattices and whose morphisms are meet-preserving maps; for in such a setting, we should be able to view our tuples $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ as diagrams in the base category. Following this intuition, we shall work within the framework of the Priestley-style duality for meet-semilattices introduced by Bezhanishvili and Jansana (2011). This approach is also appealing to us because in Bezhanishvili and Jansana (2013) the authors have extended their duality to meet-semilattices enriched with an intuitionistic implication, which means that we will be able to exploit their results when dealing with tuples $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ where $\mathbf{L}_+$ and $\mathbf{L}_-$ are Heyting algebras (i.e. the case of bilattices with implication and quasi-Nelson algebras).

Suppose $\mathbf{L}_+$ and $\mathbf{L}_-$ are bounded distributive lattices and $n : L_+ \to L_-$ is a map which is meet-preserving but not necessarily join-preserving. Then n does not give rise in any obvious way to a function between the corresponding Priestley spaces $X(\mathbf{L}_+)$ and $X(\mathbf{L}_-)$, but it can nevertheless be represented as a binary relation $X(n) \subseteq X(\mathbf{L}_-) \times X(\mathbf{L}_+)$ as follows:

$$X(n) = \{\langle x_-, x_+\rangle \in X(\mathbf{L}_-) \times X(\mathbf{L}_+) : n^{-1}[x_-] \subseteq x_+\}.$$

Any relation $X(n)$ defined in this particular way obviously satisfies non-trivial order-theoretic and topological properties, which we shall identify in the next section. Their lattice-

[5] Abstractly, a *Priestley space* is defined as a compact ordered topological space $\langle X, \tau, \leq\rangle$ such that, for all $x, y \in X$, if $x \not\leq y$, then there is a clopen up-set $U \subseteq X$ with $x \in U$ and $y \notin U$. It follows that $\langle X, \tau\rangle$ is a Stone space.

theoretic duals (corresponding to join-preserving maps), have been studied in Halmos (1962) for Boolean spaces, and generalised to the setting of Priestley spaces in Cignoli et al. (1991).

It may be interesting to notice that the maps n and p can be viewed as two-sorted analogues of the so-called necessity operator $\Box$ of modal logic. In fact, most notions from duality for modal algebras apply to our case as well: in particular, the morphisms between spaces that we shall obtain are a straightforward generalisation of the maps known in the modal logic literature as *bounded morphisms* (or *p-morphisms*; see, for example, Blackburn et al. (2001)).

Thus, in our approach the spatial dual to a tuple $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ will be the structure $\langle X(\mathbf{L}_+), X(\mathbf{L}_-), X(n), X(p)\rangle$ made of two Priestley spaces together with two binary relations between them. For the converse construction, a binary relation between Priestley spaces may give rise to a meet-preserving map on the corresponding lattices as follows. Given sets X, Y, a relation $R \subseteq X \times Y$ and a subset $X' \subseteq X$, define:

$$R[X'] = \{y \in Y : \text{there is } x \in X' \text{ s.t. } \langle x, y\rangle \in R\}.$$

In particular, for $X' = \{x\}$, we write $R[x]$ instead of $R[\{x\}]$. For $Y' \subseteq Y$, let:

$$\Box_R Y' = \{x \in X : R[x] \subseteq Y'\}.$$

Now, suppose $\langle X_+, X_-, R_n, R_p\rangle$ is a structure such that X_+ and X_- are Priestley spaces, and $R_n \subseteq X_- \times X_+$, $R_p \subseteq X_+ \times X_-$ are binary relations. Imposing suitable restrictions on the relations, one can show that the maps $\Box_{R_n} : L(X_+) \to L(X_-)$ and $\Box_{R_p} : L(X_-) \to L(X_+)$ given as above are well defined on the lattices $L(X_+)$ and $L(X_-)$ and preserve finite meets. Thus, one obtains a tuple $\langle L(X_+), L(X_-), \Box_{R_n}, \Box_{R_p}\rangle$ of the required type.

In order to obtain a general duality (to be later specialised to the various classes of algebras introduced in Sect. 2), it only remains to take care of morphisms. For tuples of type $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$, a natural notion of morphism arises in a straightforward way; one just needs to look at the algebraic homomorphisms of the corresponding class of abstractly presented algebras[6]. Having fixed these (see Definition 4.5 in the next section), the morphisms on the spatial side will be fixed as well, and we will only need to provide a convenient characterisation for them.

[6] This is nevertheless a choice, different, for example, from the one made in Jakl et al. (2016, Definition 3.3).

4 Two-sorted lattices and their duals

We now apply definitions and results (specialised to our particular setting) on the dualities for (implicative) meet-semilattices developed in Bezhanishvili and Jansana (2011) and Bezhanishvili and Jansana (2013); we refer the reader to these papers for further details and proofs. Keep in mind that, since we are dealing with distributive lattices rather than semilattices, we will be able to use most results and definitions from Bezhanishvili and Jansana (2011, 2013) in a simplified form (e.g. we will not have to deal with fine-grained notions such as the distributive envelope, optimal filter and Frink ideal; in particular, in our setting optimal and prime filters coincide).

Let us start with the official definition of our main objects of interest, that is, *two-sorted lattices*.

Definition 4.1 A *two-sorted lattice* is a structure $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ such that:

(i) $\mathbf{L}_+ = \langle L; \wedge_+, \vee_+, 0_+, 1_+; \leq_+\rangle$ and $\mathbf{L}_- = \langle L; \wedge_-, \vee_-, 0_-, 1_-; \leq_-\rangle$ are bounded distributive lattices;
(ii) $n : L_+ \to L_-$ and $p : L_- \to L_+$ are maps that preserve finite meets and all lattice bounds of $\mathbf{L}_+$ and $\mathbf{L}_-$.

Given a two-sorted lattice $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$, we define its dual space as the structure

$$X(\mathbb{L}) = \langle X(\mathbf{L}_+), X(\mathbf{L}_-), X(n), X(p)\rangle$$

where:

(i) $X(\mathbf{L}_+) = \langle X(\mathbf{L}_+), \tau_+, \subseteq\rangle$ is the Priestley space corresponding to $\mathbf{L}_+$;
(ii) $X(\mathbf{L}_-) = \langle X(\mathbf{L}_-), \tau_-, \subseteq\rangle$ is the Priestley space corresponding to $\mathbf{L}_-$;
(iii) $X(p) \subseteq X(\mathbf{L}_+) \times X(\mathbf{L}_-)$ and $X(n) \subseteq X(\mathbf{L}_-) \times X(\mathbf{L}_+)$ are relations defined as follows:

$$X(n) = \{\langle x_-, x_+\rangle \in X(\mathbf{L}_-) \times X(\mathbf{L}_+) : n^{-1}[x_-] \subseteq x_+\}$$
$$X(p) = \{\langle x_+, x_-\rangle \in X(\mathbf{L}_+) \times X(\mathbf{L}_-) : p^{-1}[x_+] \subseteq x_-\}.$$

Unless indicated otherwise, we will always write x_-, y_-, etc. for generic points (i.e. prime filters) in $X(\mathbf{L}_-)$ and x_+, y_+ for points (prime filters) in $X(\mathbf{L}_+)$. Every relation obtained in the above-defined way satisfies certain special properties, which we are now going to demonstrate.

Proposition 4.2 (Bezhanishvili and Jansana 2011, Prop. 6.1, Lemma 6.10) *Let* $X(\mathbb{L}) = \langle X(\mathbf{L}_+), X(\mathbf{L}_-), X(n), X(p)\rangle$ *be the space corresponding to a two-sorted lattice* $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$. *The following hold:*

(i) $X(n)[x_-] \subseteq X(\mathbf{L}_+)$ *and* $X(p)[x_+] \subseteq X(\mathbf{L}_-)$ *are non-empty closed up-sets;*
(ii) $\Box_{R_n} \circ \Phi_{\mathbf{L}_+} = \Phi_{\mathbf{L}_-} \circ n$ *and* $\Box_{R_p} \circ \Phi_{\mathbf{L}_-} = \Phi_{\mathbf{L}_+} \circ p$.

Taking the lead from Bezhanishvili and Jansana (2011), we include the properties of Proposition 4.2 in our official definition of a *two-sorted Priestley space*.

Definition 4.3 A *two-sorted Priestley space* is a structure $X = \langle X_+, X_-, R_n, R_p \rangle$ such that:

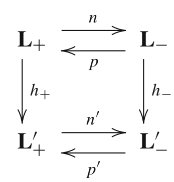
(i) $X_+ = \langle X, \tau_+, \leq_+ \rangle$ and $X_- = \langle X, \tau_-, \leq_- \rangle$ are Priestley spaces;
(ii) $R_n \subseteq X_- \times X_+$, $R_p \subseteq X_+ \times X_-$ are binary relations satisfying[7]:
  1. $R_n[x_-]$ and $R_p[x_+]$ are non-empty closed up-sets, for all x_- and x_+;
  2. for all $U_- \in \mathbf{L}(X_-)$, $U_+ \in \mathbf{L}(X_+)$, we have $\Box_{R_p} U_- \in \mathbf{L}(X_+)$ and $\Box_{R_n} U_+ \in \mathbf{L}(X_-)$.

By Proposition 4.2, the dual space $X(\mathbb{L}) = \langle X(\mathbf{L}_+), X(\mathbf{L}_-), X(n), X(p) \rangle$ of a two-sorted lattice $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$ is a two-sorted Priestley space. The next result entails that, conversely, to every two-sorted Priestley space we can associate a two-sorted lattice.

Proposition 4.4 (Bezhanishvili and Jansana 2011, Prop. 5.9, Lemma 6.5) *Let* $X = \langle X_+, X_-, R_n, R_p \rangle$ *be a two-sorted Priestley space. Then* $\mathbf{L}(X) = \langle \mathbf{L}(X_+), \mathbf{L}(X_-), \Box_{R_n}, \Box_{R_p} \rangle$ *is a two-sorted lattice.*

Next we check that every two-sorted lattice is isomorphic to its double dual. For this, we need to stipulate what a (homo)morphism of two-sorted lattices is. We shall adopt the notion of many-sorted homomorphism that is standard in many-sorted universal algebra (Meinke and Tucker 1992, Definition 3.4.1.); as mentioned earlier, such a choice is also consistent with the fact that, in the case of semi-De Morgan and quasi-Nelson algebras, many-sorted homomorphisms correspond precisely to algebraic homomorphisms.

Definition 4.5 Let $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$ and $\mathbb{L}' = \langle \mathbf{L}'_+, \mathbf{L}'_-, n', p' \rangle$ be two-sorted lattices. A *morphism* $h \colon \mathbb{L} \to \mathbb{L}'$ consists of a pair of bounded lattice homomorphisms $h = \langle h_+, h_- \rangle$ with $h_+ \colon L_+ \to L'_+$ and $h_- \colon L_- \to L'_-$ such that $h_+ \circ p = p' \circ h_-$ and $n' \circ h_+ = h_- \circ n$.



One easily checks that two-sorted lattices together with the above-defined morphisms form a category that we shall denote by 2Lat. Given a two-sorted lattice $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$, let $\Phi_{\mathbb{L}} = \langle \Phi_{\mathbf{L}_+}, \Phi_{\mathbf{L}_-} \rangle$. The next result follows from Priestley duality together with Proposition 4.2.(ii).

Proposition 4.6 *Let* $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$ *be a two-sorted lattice. Then the map* $\Phi_{\mathbb{L}} = \langle \Phi_{\mathbf{L}_+}, \Phi_{\mathbf{L}_-} \rangle$ *is an isomorphism in* 2Lat *between* $\mathbb{L}$ *and* $\mathbf{L}(X(\mathbb{L}))$.

Given two-sorted lattices $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$ and $\mathbb{L}' = \langle \mathbf{L}'_+, \mathbf{L}'_-, n', p' \rangle$ and a morphism $h = \langle h_+, h_- \rangle \colon \mathbb{L} \to \mathbb{L}'$, we let $X(h) = \langle X(h_+), X(h_-) \rangle$, where $X(h_+) \colon X(\mathbf{L}'_+) \to X(\mathbf{L}_+)$ $X(h_-) \colon X(\mathbf{L}'_-) \to X(\mathbf{L}_-)$ are the Priestley functions given by $X(h_+)(x'_+) = h_+^{-1}[x'_+]$ and $X(h_-)(x'_-) = h_-^{-1}[x'_-]$.

The spatial category dual to 2Lat, which we shall denote by 2PrSp, has as objects two-sorted Priestley spaces; we define the corresponding morphisms below.

Definition 4.7 Let $X = \langle X_+, X_-, R_n, R_p \rangle$ and $X' = \langle X'_+, X'_-, R'_n, R'_p \rangle$ be two-sorted Priestley spaces and let $f_+ \colon X_+ \to X'_+$ and $f_- \colon X_- \to X'_-$ be maps. The pair $f = \langle f_+, f_- \rangle$ is a 2PrSp-*morphism* if the following conditions hold:

(i) f_+ and f_- are Priestley functions.
(ii) f preserves R_p and R_n, that is, if $\langle x_+, x_- \rangle \in R_p$, then $\langle f_+(x_+), f_-(x_-) \rangle \in R'_p$, etc.
(iii) f_+ and f_- are bounded morphisms, in the sense that
  (a) if $\langle f_+(x_+), x'_- \rangle \in R'_p$, then there is $x_- \in X_-$ such that $f_-(x_-) \leq'_- x'_-$ and $\langle x_+, x_- \rangle \in R_p$,
  (b) if $\langle f_-(x_-), x'_+ \rangle \in R'_n$, then there is $x_+ \in X_+$ such that $f_+(x_+) \leq'_+ x'_+$ and $\langle x_-, x_+ \rangle \in R_n$.

In keeping with the view of n and p as necessity-like modalities, one can view 2PrSp-morphisms as a two-sorted version of so-called bounded morphisms or p-morphisms, i.e. the morphisms of modal spaces. It is straightforward to check that the composition map $f \circ g$ of two 2PrSp-morphisms f and g is a 2PrSp-morphism. We proceed to

[7] Our formulation of (ii).1 is different from the one in Bezhanishvili and Jansana (2011, Definition 6.2), but the two are easily seen to be equivalent in our context. Also, we always require R_n and R_p to be *total* (Bezhanishvili and Jansana 2011, Definition 6.11) because our maps n, p preserve all lattice bounds.

check that 2PrSp-morphisms are precisely the duals of 2Lat-morphisms.

Proposition 4.8 *Let $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ and $\mathbb{L}' = \langle \mathbf{L}'_+, \mathbf{L}'_-, n', p'\rangle$ be two-sorted lattices and $h = \langle h_+, h_-\rangle\colon \mathbb{L} \to \mathbb{L}'$ a morphism between them. The following hold:*

(i) *For all $x'_+ \in X(\mathbf{L}'_+)$ and $x'_- \in X(\mathbf{L}'_-)$, if $\langle x'_+, x'_-\rangle \in X(p')$, then $\langle X(h_+)(x'_+), X(h_-)(x'_-)\rangle \in X(p)$, and analogously for pairs $\langle x'_-, x'_+\rangle \in X(n')$.*
(ii) *For all $x'_+ \in X(\mathbf{L}'_+)$ and $x_- \in X(\mathbf{L}_+)$, if $\langle X(h_+)(x'_+), x_-\rangle \in X(p)$, then there is $x'_- \in X(\mathbf{L}'_-)$ such that $X(h_-)(x'_-) \subseteq x_-$ and $\langle x'_+, x'_-\rangle \in X(p')$; analogously for elements $x'_- \in X(\mathbf{L}'_-)$ and $x_+ \in X(\mathbf{L}_+)$.*

Hence, $X(h)$ is a 2PrSp-morphism.

Proof (i). Assume $\langle x'_+, x'_-\rangle \in X(p')$, i.e. $(p')^{-1}[x'_+] \subseteq x'_-$. Then $h_-^{-1}[(p')^{-1}[x'_+]] \subseteq h_-^{-1}[x'_-] = X(h_-)(x'_-)$. From the assumption that $h_+ \circ p = p' \circ h_-$, we have

$$(h_+ \circ p)^{-1} = p^{-1} \circ h_+^{-1} = h_-^{-1} \circ (p')^{-1} = (p' \circ h_-)^{-1}.$$

Hence,

$$\begin{aligned} h_-^{-1}[(p')^{-1}[x'_+]] &= p^{-1}[h_+^{-1}[x'_+]] \\ &= p^{-1}[X(h_+)(x'_+)] \subseteq X(h_-)(x'_-), \end{aligned}$$

which means that $\langle X(h_+)(x'_+), X(h_-)(x'_-)\rangle \in X(p)$ as required.

(ii). Assume $\langle X(h_+)(x'_+), x_-\rangle \in X(p)$, which means that $p^{-1}[h_+^{-1}[x'_+]] \subseteq x_-$. As seen in (i), we have $p^{-1} \circ h_+^{-1} = h_-^{-1} \circ (p')^{-1}$ and so we can rewrite our assumption as $h_-^{-1}[(p')^{-1}[x'_+]] \subseteq x_-$. We need to show that there is $x'_- \in X(\mathbf{L}'_-)$ such that $h_-^{-1}[x'_-] \subseteq x_-$ and $(p')^{-1}[x'_+] \subseteq x'_-$. We shall then need to extend $(p')^{-1}[x'_+]$ to a prime filter of $\mathbf{L}'_-$ with the required properties. In order to do so, notice that $(p')^{-1}[x'_+]$ is a filter and that $\downarrow h_-[x_-^c]$ is an ideal of $\mathbf{L}'_-$ such that $(p')^{-1}[x'_+] \cap \downarrow h_-[x_-^c] = \emptyset$. In fact, if there was some $a'_- \in (p')^{-1}[x'_+] \cap \downarrow h_-[x_-^c]$, then we would have, on the one hand, $p'(a'_-) \in x'_+$ and, on the other, $a'_- \leq'_- h_-(b_-)$ for some $b_- \notin x_-^c$. Then, by monotonicity of p', we would have $p'(a'_-) \leq'_- p'(h_-(b_-))$ and so $p'(h_-(b_-)) \in x'_+$ because x'_+ is an up-set. But by assumption $h_-^{-1}[(p')^{-1}[x'_+]] \subseteq x_-$, so we should have $b_- \in x_-$, a contradiction. Thus, $(p')^{-1}[x'_+] \cap \downarrow h_-[x_-^c] = \emptyset$, and we can invoke the prime filter theorem for distributive lattices to extend $(p')^{-1}[x'_+]$ to a prime filter x'_- such that $x'_- \cap \downarrow h_-[x_-^c] = \emptyset$. To finish the proof, it is sufficient to observe that $h_-^{-1}[x'_-] \subseteq x_-$. In fact, if $h_-(a_-) \in x'_-$, then $h_-(a_-) \notin \downarrow h_-[x_-^c]$, which entails $a_- \notin x_-^c$ and so $a_- \in x_-$ as required. □

Given two-sorted Priestley spaces $X = \langle X_+, X_-, R_n, R_p\rangle$ and $X' = \langle X'_+, X'_-, R'_n, R'_p\rangle$ and a 2PrSp-morphism $f = \langle f_+, f_-\rangle\colon X \to X'$, the pair $\mathbf{L}(f) = \langle \mathbf{L}(f_+), \mathbf{L}(f_-)\rangle$ of lattice homomorphisms $\mathbf{L}(f_+)\colon \mathbf{L}(X'_+) \to \mathbf{L}(X_+)$, $\mathbf{L}(f_-)\colon \mathbf{L}(X'_-) \to \mathbf{L}(X_-)$ is defined as in Priestley duality: for all clopen up-sets $U'_+ \subseteq X'_+$, $U'_- \subseteq X'_-$,

$$\mathbf{L}(f_+)(U'_+) := f_+^{-1}[U'_+] \qquad \mathbf{L}(f_-)(U'_-) := f_-^{-1}[U'_-].$$

Proposition 4.9 *Let $X = \langle X_+, X_-, R_n, R_p\rangle$ and $X' = \langle X'_+, X'_-, R'_n, R'_p\rangle$ be two-sorted Priestley spaces and $f = \langle f_+, f_-\rangle\colon X \to X'$ a 2PrSp-morphism. Then $\mathbf{L}(f) = \langle \mathbf{L}(f_+), \mathbf{L}(f_-)\rangle$ is a 2Lat-morphism.*

Proof It follows from Priestley duality that $\mathbf{L}(f_+)$ and $\mathbf{L}(f_-)$ are bounded lattice homomorphisms. We will only consider the first additional condition of Definition 4.5 (the other can be shown using a similar reasoning), namely $\mathbf{L}(f_+) \circ \Box_{p'} = \Box_p \circ \mathbf{L}(f_-)$. This means that, for any clopen up-set $U'_- \subseteq X'_-$, we must have $(\mathbf{L}(f_+) \circ \Box_{p'})(U'_-) = (\Box_p \circ \mathbf{L}(f_-))(U'_-)$. Let us compute the left-hand side of the equation, which is

$$\begin{aligned} (\mathbf{L}(f_+) \circ \Box_{p'})(U'_-) &= \mathbf{L}(f_+)(\{x'_+ \in X'_+ : R'_p[x'_+] \subseteq U'_-\}) \\ &= f_+^{-1}[\{x'_+ \in X'_+ : R'_p[x'_+] \subseteq U'_-\}]. \end{aligned}$$

Thus, we have that $x_+ \in (\mathbf{L}(f_+) \circ \Box_{p'})(U'_-)$ if and only if, for all $x'_- \in X'_-$, if $\langle f_+(x_+), x'_-\rangle \in R_{p'}$, then $x'_- \in U'_-$. Likewise, we compute

$$\begin{aligned} (\Box_p \circ \mathbf{L}(f_-))(U'_-) &= \Box_p(f_-^{-1}[U'_-]) \\ &= \{x_+ \in X_+ : R_p[x_+] \subseteq f_-^{-1}[U'_-]\}] \end{aligned}$$

which means that $x_+ \in (\Box_p \circ \mathbf{L}(f_-))(U'_-)$ if and only if, for all $x_- \in X_-$, if $\langle x_+, x_-\rangle \in R_p$, then $f_-(x_-) \in U'_-$. To see that $(\mathbf{L}(f_+) \circ \Box_{p'})(U'_-) \subseteq (\Box_p \circ \mathbf{L}(f_-))(U'_-)$, assume $x_+ \in (\mathbf{L}(f_+) \circ \Box_{p'})(U'_-)$ and $\langle x_+, x_-\rangle \in R_p$. Then, by Definition 4.7.(ii), we have $\langle f_+(x_+), f_-(x_-)\rangle \in R_{p'}$. Thus, by our assumption, we immediately conclude that $f_-(x_-) \in U'_-$ as required. Conversely, to see that $(\Box_p \circ \mathbf{L}(f_-))(U'_-) \subseteq (\mathbf{L}(f_+) \circ \Box_{p'})(U'_-)$, assume $x_+ \in (\Box_p \circ \mathbf{L}(f_-))(U'_-)$ and $\langle f_+(x_+), x'_-\rangle \in R_{p'}$. Then, by Definition 4.7.(iii), there is $x_- \in X_-$ such that $f_-(x_-) \leq'_- x'_-$ and $\langle x_+, x_-\rangle \in R_p$. Then, by our assumption, $f_-(x_-) \in U'_-$ and so, since U'_- is an up-set, we also have $x'_- \in U'_-$, as required. □

Given a two-sorted Priestley space $X = \langle X_+, X_-, R_n, R_p\rangle$, the map $\Phi_X = \langle \Phi_{X_+}, \Phi_{X_-}\rangle$ is defined by

$$\begin{aligned} \Phi_{X_+}(x_+) &:= \{U_+ \in \mathbf{L}(X_+) : x_+ \in U_+\} \\ \Phi_{X_-}(x_-) &:= \{U_- \in \mathbf{L}(X_-) : x_- \in U_-\}. \end{aligned}$$

Proposition 4.10 *For any two-sorted Priestley space $X = \langle X_+, X_-, R_n, R_p\rangle$, the map $\Phi_X\colon X \to X(\mathbf{L}(X))$ is a 2PrSp-isomorphism.*

Proof We know from Priestley duality that Φ_{X_+} and Φ_{X_-} are order homeomorphisms. It remains to check that items (ii) and (iii) of Definition 4.7 are satisfied. Item (ii) follows from Bezhanishvili and Jansana (2011, Proposition 6.7). Let us prove (iii) (a); the proof of (b) is similar. Let then $\langle \Phi_{X_+}(x_+), x'_- \rangle \in X(\Box_{R_p})$ for some $x_+ \in X_+$ and $x'_- \in X(\mathbf{L}(X))$. Since Φ_{X_-} is surjective, we know that $x'_- = \Phi_{X_-}(x_-)$ for some $x_- \in X_-$ (so $\Phi_{X_-}(x_-) \subseteq x'_-$). Then $\langle \Phi_{X_+}(x_+), \Phi_{X_-}(x_-) \rangle \in X(\Box_{R_p})$ which, using again Bezhanishvili and Jansana (2011, Proposition 6.7), entails $\langle x_+, x_- \rangle \in R_p$, as required. □

Joining the previous results, we obtain our first equivalence.

Theorem 4.11 *The categories* 2Lat *and* 2PrSp *are dually equivalent via the functors X and L.*

5 Specialising the duality

Having established a duality between two-sorted lattices of type $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$ and two-sorted Priestley spaces, we can restrict our attention to special subclasses (subcategories) obtained by imposing further structural conditions on $\mathbf{L}_+$, $\mathbf{L}_-$ or on the maps n and p. Focusing on the maps first, we begin by collecting a number of observations that have been established in Bezhanishvili and Jansana (2011).

Proposition 5.1 (Bezhanishvili and Jansana 2011, Corollary 6.12, Theorem 8.9) *Let $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$ be a two-sorted lattice and let*

$$X(\mathbb{L}) = \langle X(\mathbf{L}_+), X(\mathbf{L}_-), X(n), X(p) \rangle$$

be the corresponding two-sorted Priestley space. Then the following hold:

(i) *n preserves joins if and only if $X(n)$ is* functional *(*Bezhanishvili and Jansana 2011*, Definition 6.11), i.e. for all $x_- \in X(\mathbf{L}_-)$ there is $x_+ \in X(\mathbf{L}_+)$ such that $\uparrow x_+ = X(n)[x_-]$.*
(ii) *n is injective if and only $X(n)$ is* onto *(Bezhanishvili and Jansana* 2011*, Definition 8.7), i.e. for all $x_+ \in X(\mathbf{L}_+)$ there is $x_- \in X(\mathbf{L}_-)$ such that $\uparrow x_+ = X(n)[x_-]$.*
(iii) *n is surjective if and only $X(n)$ is* 1-1 *(*Bezhanishvili and Jansana 2011*, Definition 8.7), i.e. for all $x_- \in X(\mathbf{L}_-)$ and for all $U_- \in \mathbf{L}(X(\mathbf{L}_-))$, there is $U_+ \in \mathbf{L}(X(\mathbf{L}_+))$ such that $X(n)[U_-] \subseteq U_+$ and $X(n)[x_-] \nsubseteq U_+$.*

Analogous equivalences hold for p.

In our setting (and this time, differently from that of Bezhanishvili and Jansana 2011), further restrictions can also be imposed on the composition of the maps n and p. To look at this, it will be useful to have the following lemma at hand.

Lemma 5.2 *Let $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$ be a two-sorted lattice and let*

$$X(\mathbb{L}) = \langle X(\mathbf{L}_+), X(\mathbf{L}_-), X(n), X(p) \rangle$$

be the corresponding two-sorted Priestley space. Then for all $x_-, x'_- \in X(\mathbf{L}_-)$, we have $\langle x_-, x'_- \rangle \in X(p) \circ X(n)$ if and only if $(n \circ p)^{-1}[x_-] \subseteq x'_-$, and analogously for composing the two maps the other way round.

Proof Assume $\langle x_-, x'_- \rangle \in X(p) \circ X(n)$. Then there is $x_+ \in X(\mathbf{L}_+)$ with $\langle x_-, x_+ \rangle \in X(n)$ (i.e. $n^{-1}[x_-] \subseteq x_+$) and $\langle x_+, x'_- \rangle \in X(p)$ (i.e. $p^{-1}[x_+] \subseteq x'_-$). If that is the case, then $p^{-1}[n^{-1}[x_-]] = (n \circ p)^{-1}[x_-] \subseteq p^{-1}[x_+] \subseteq x'_-$.

Conversely, assume $(n \circ p)^{-1}[x_-] = p^{-1}[n^{-1}[x_-]] \subseteq x'_-$. Let $(x'_-)^c$ be the complement of x'_- in L_-, which is a (prime) ideal of $\mathbf{L}_-$. Consider the set $p[(x'_-)^c]$, and let us form its down-set $\downarrow p[(x'_-)^c]$ in $\mathbf{L}_+$. Observe that $\downarrow p[(x'_-)^c]$ is an up-directed down-set and therefore an ideal of $\mathbf{L}_+$. Moreover, $n^{-1}[x_-] \cap \downarrow p[(x'_-)^c] = \emptyset$. To see this, suppose there was $a_+ \in n^{-1}[x_-] \cap \downarrow p[(x'_-)^c]$. Then $n(a_+) \in x_-$ and $a_+ \leq_+ p(a_-)$ for some $a_- \notin x'_-$. Then, by monotonicity of n, we would have $n(a_+) \leq_- n(p(a_-))$ and so $n(p(a_-)) \in x_-$ because x_- is an up-set. This means that $a_- \in p^{-1}[n^{-1}[x_-]]$, but $p^{-1}[n^{-1}[x_-]] \subseteq x'_-$ by assumption, so $a_- \in x'_-$: a contradiction. Now, since $n^{-1}[x_-]$ is a lattice filter, we can invoke the prime filter theorem for distributive lattices to extend $n^{-1}[x_-]$ to a prime filter x_+ such that $n^{-1}[x_-] \subseteq x_+$ and $x_+ \cap \downarrow p[(x'_-)^c] = \emptyset$. To prove our claim, it is then sufficient to observe that $p^{-1}[x_+] \subseteq x'_-$. In fact, if there was some $a_- \in p^{-1}[x_+]$ with $a_- \notin x'_-$, then we would have $p(a_-) \in x_+$ and $p(a_-) \in p[(x'_-)^c]$, and so $x_+ \cap \downarrow p[(x'_-)^c] \neq \emptyset$, against what we have shown. Thus, $\langle x_-, x_- \rangle \in (X(p) \circ X(n))$. □

The following proposition studies the behaviour of the composition of n and p in comparison with the identity maps on the two lattices, and we are going to need it to characterise the spaces corresponding to bilattices, semi-De Morgan and quasi-Nelson algebras.

Proposition 5.3 *Let $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$ be a two-sorted lattice and let*

$$X(\mathbb{L}) = \langle X(\mathbf{L}_+), X(\mathbf{L}_-), X(n), X(p) \rangle$$

be the corresponding two-sorted Priestley space. Then the following hold:

(i) *$n \circ p \leq_- Id_{L_-}$ if and only if $\leq_{X(\mathbf{L}_-)} \subseteq (X(p) \circ X(n))$.*

(ii) $p \circ n \leq_+ Id_{L_+}$ *if and only if* $\leq_{X(\mathbf{L}_+)} \subseteq (X(n) \circ X(p))$.
(iii) $Id_{L_-} \leq_- n \circ p$ *if and only if* $(X(p) \circ X(n)) \subseteq \leq_{X_-}$.
(iv) $Id_{L_+} \leq_+ p \circ n$ *if and only if* $(X(n) \circ X(p)) \subseteq \leq_{X_+}$.

Proof Clearly it suffices to show, for example, (i) and (iii).

(i). Assume $n \circ p \leq_- Id_{L_-}$ and observe that, for all $x_-, y_- \in X(\mathbf{L}_-)$ such that $x_- \subseteq y_-$, we have $(n \circ p)^{-1}[x_-] \subseteq y_-$. Indeed, if $a_- \in (n \circ p)^{-1}[x_-]$, then $n(p(a_-)) \in x_-$. Since $n(p(a_-)) \leq_- a_-$ and x_- is an up-set, we have $a_- \in x_- \subseteq y_-$. By Lemma 5.2, this means that $\langle x_-, y_-\rangle \in (X(p) \circ X(n))$ for all $y_- \in X(\mathbf{L}_-)$, as required.

Conversely, assume $\leq_{X(\mathbf{L}_-)} \subseteq (X(p) \circ X(n))$. Let $a_- \in L_-$ and $x_- \in X(\mathbf{L}_-)$ be such that $a_- \notin x_-$. Since $\langle x_-, x_-\rangle \in (X(p) \circ X(n))$, there is $x_+ \in X(\mathbf{L}_+)$ such that $n^{-1}[x_-] \subseteq x_+$ and $p^{-1}[x_+] \subseteq x_-$. From the latter and the assumption that $a \notin x_-$ we have $p(a_-) \notin x_+$, and from $p(a_-) \notin x_+$, we have $n(p(a_-)) \notin x_-$. Thus, contrapositively, $n(p(a_-)) \in x_-$ entails $a \in x_-$ and this holds for any prime filter x_-, which means that $n(p(a_-) \leq_- a_-$.

(iii). Assume $Id_{L_-} \leq_- n \circ p$, and let $x_-, x'_- \in X(\mathbf{L}_-)$ be such that $\langle x_-, x'_-\rangle \in (X(p) \circ X(n))$. This means that there is $x_+ \in X(\mathbf{L}_+)$ with $\langle x_-, x_+\rangle \in X(n)$ and $\langle x_+, x'_-\rangle \in X(p)$. That is, $n^{-1}[x_-] \subseteq x_+$ and $p^{-1}[x_+] \subseteq x'_-$. Let $a_- \in x_-$. By assumption $a_- \leq_- n(p(a_-))$, so $n(p(a_-)) \in x_-$ as well, because x_- is an up-set. This means that $p(a_-) \in n^{-1}[x_-] \subseteq x_+$. Hence, $p(a_-) \in x_+$. Thus, we have $x_- \subseteq p^{-1}[x_+] \subseteq x'_-$ and so $x_- \subseteq x'_-$, as required.

Conversely, suppose $(X(p) \circ X(n)) \subseteq \leq_{X_-}$. Assume, in view of a contradiction, that there exists $a_- \in L_-$ such that $a_- \not\leq_- n(p(a_-))$. Then, by the prime filter theorem, there is a prime filter $x_- \in X(\mathbf{L}_-)$ such that $a_- \in x_-$ and $n(p(a_-)) \notin x_-$. Then $a_- \notin (n \circ p)^{-1}[x_-]$. Since $(n \circ p)^{-1}[x_-]$ is a filter, we can invoke the prime filter theorem again to obtain a prime filter x'_- such that $(n \circ p)^{-1}[x_-] \subseteq x'_-$ and $a_- \notin x'_-$. By Lemma 5.2, $(n \circ p)^{-1}[x_-] \subseteq x'_-$ means that $\langle x_-, x'_-\rangle \in X(p) \circ X(n)$. Hence, applying the assumption that $(X(p) \circ X(n)) \subseteq \leq_{X_-}$, we have $x_- \subseteq x'_-$, which would imply $a_- \in x'_-$: a contradiction. □

Recalling Definition 2.4, one sees that every (tuple corresponding to a) non-involutive bilattice satisfies the left-hand sides of conditions (i) and (ii) of Proposition 5.3; likewise, quasi-Nelson algebras (Definition 2.12) satisfy (i), (iii) and (iv). These observations motivate the following definition.

Definition 5.4 Let $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ be a two-sorted lattice (Definition 4.1). We shall say that:

(i) $\mathbb{L}$ is a *non-involutive bilattice* if (1) $Id_{\mathbf{L}_-} \geq_- n \circ p$ and (2) $Id_{\mathbf{L}_+} \geq_+ p \circ n$.
(ii) $\mathbb{L}$ is a *non-involutive implicative bilattice* if $\mathbb{L}$ is a non-involutive bilattice and $\mathbf{L}_+$ and $\mathbf{L}_-$ and are both Heyting algebras.
(iii) $\mathbb{L}$ is a *semi-De Morgan algebra* if (1) $\mathbf{L}_-$ is a De Morgan algebra, (2) n preserves finite joins, and (3) $Id_{\mathbf{L}_-} = n \circ p$ (no condition is required on $p \circ n$).
(iv) $\langle \mathbb{L}, \nabla\rangle$ is a *quasi-Nelson algebra* if (1) $\mathbf{L}_+$ and $\mathbf{L}_-$ and are both Heyting algebras, (2) n preserves finite joins, (3) ∇ is a lattice filter of $\mathbf{L}_+$ such that $D(\mathbf{L}_+) \subseteq \nabla$, (4) $Id_{\mathbf{L}_-} = n \circ p$, and (5) $Id_{\mathbf{L}_+} \leq_+ p \circ n$.

Note that, in order to simplify our terminology, we have resorted to the slight abuse of language of calling a structure $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$, e.g. a semi-De Morgan algebra, rather than "the tuple corresponding, via the representation theorem, to some semi-De Morgan algebra".

In order to characterise the spaces corresponding to the above-defined algebraic structures, we shall exploit Esakia duality for Heyting algebras and Cornish–Fowler duality for De Morgan algebras (on Esakia duality see, for example, Bezhanishvili and Jansana 2013; see Cornish and Fowler 1977 on the duality for De Morgan algebras). According to these, the category having as objects Heyting algebras (respectively, De Morgan algebras) and as morphisms algebraic homomorphisms is dually equivalent to the category of Esakia spaces (resp., De Morgan spaces), both being special Priestley spaces; the functors establishing these dualities are defined in the same way as in Priestley duality. We recall below the necessary definitions and results.

Definition 5.5 An *Esakia space* is a Priestley space X in which the down-set of each clopen set is clopen (or equivalently, the down-set of each open set is open).

An *Esakia function* between Esakia spaces X and Y is a Priestley function $f\colon X \to Y$ satisfying $f^{-1}[\downarrow \mathcal{O}] = \downarrow f^{-1}[\mathcal{O}]$ for any open set $\mathcal{O} \subseteq Y$ (or equivalently, $\uparrow f(x) \subseteq f[\uparrow x]$ for every $x \in X$).

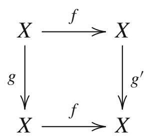
Esakia functions correspond to Heyting algebra homomorphisms. Besides this notion, we shall need a more general one to be able to represent the map p (which need not preserve joins). This is the notion of a *generalised Esakia morphism* introduced in Bezhanishvili and Jansana (2013, Definition 4.2), that we shall more briefly call *Esakia relation*. A relation $R \subseteq X \times Y$ between Esakia spaces X and Y is an *Esakia relation* if R satisfies item (ii) of Definition 4.3, and moreover, for all $x \in X, y \in Y$, if $\langle x, y\rangle \in R$, then there is $z \in \uparrow x$ such that $R[z] = \uparrow y$.

For each Esakia space X, the lattice of clopen up-sets $\mathbf{L}(X)$ is a Heyting algebra in which the implication is given, for all $U, V, \in \mathbf{L}(X)$, by

$$U \to V := X - \downarrow(U - V) = \{x \in X : \uparrow x \cap U \subseteq V\}.$$

Differently from Esakia duality, De Morgan duality is not just a restriction of Priestley duality, because a De Morgan space is a Priestley space enriched with an extra map.

Definition 5.6 A *De Morgan space* is a pair $\langle X, g\rangle$ where X is a Priestley space and $g\colon X \to X$ is an order-reversing homeomorphism such that $g \circ g = Id_X$. A *De Morgan function* between De Morgan spaces $\langle X, g\rangle$ and $\langle X', g'\rangle$ is a Priestley map $f\colon X \to X'$ satisfying $f \circ g = g' \circ f$.



For any De Morgan space $\langle X, g\rangle$, the lattice of clopen up-sets $\mathbf{L}(X)$ is a De Morgan algebra in which the negation is given, for all $U, V, \in \mathbf{L}(X)$, by

$$\neg U := X - g[U],$$

where $g(U) = \{g(x) : x \in U\}$.

Conversely, the Priestley space $X(\mathbf{L})$ of any De Morgan algebra $\mathbf{L}$ can be endowed with a map $g\colon X(\mathbf{L}) \to X(\mathbf{L})$ satisfying the properties in Definition 5.6. This is given, for all $x \in X(\mathbf{L})$, by

$$g(x) := L - \{\neg a : a \in x\}.$$

We are now ready to introduce spaces corresponding to the classes of algebras of Definition 5.4. Given a Priestley space X, we denote by $\max(X)$ the set of points in X that are maximal w.r.t. the Priestley order. (Recall that in a Priestley space every element is below a maximal one.)

Definition 5.7 Let $X = \langle X_+, X_-, R_n, R_p\rangle$ be a two-sorted Priestley space (Definition 4.3). We say that:

(i) X is a *non-involutive bilattice space* if (1) $\leq_{X_+} \subseteq (R_n \circ R_p)$ and (2) $\leq_{X_-} \subseteq (R_p \circ R_n)$.
(ii) X is a *non-involutive implicative bilattice space* if X is a non-involutive bilattice space and X_+ and X_- are both Esakia spaces.
(iii) $\langle X, g\rangle$ is a *semi-De Morgan space* if (1) $\langle X_-, g\rangle$ is a De Morgan space, (2) R_n is functional, and (3) $\leq_{X_-} = (R_p \circ R_n)$.
(iv) $\langle X, \mathcal{C}\rangle$ is a *quasi-Nelson space* if (1) X_+ and X_- are both Esakia spaces, (2) R_n is functional, (3) R_p is an Esakia relation, (4) $\mathcal{C} \subseteq X_+$ is a closed set such that $\mathcal{C} \subseteq \max(X_+)$, (5) $\leq_{X_-} = (R_p \circ R_n)$, and (6) $(R_n \circ R_p) \subseteq \leq_{X_+}$.

A brief comment on the last item of the preceding definition is perhaps in order, and in particular about the closed set $\mathcal{C}$. This is meant to be the spatial counterpart of the filter ∇ appearing in the last item of Definition 5.4, which in turn comes from the representation of quasi-Nelson algebras given in Corollary 2.17. Indeed, the correspondence between lattice filters of a distributive lattice and closed (up-)sets of the corresponding Priestley space is well known, and has been exploited in a similar way in the duality for Nelson algebras introduced in Jansana and Rivieccio (2014). Thus, following Jansana and Rivieccio (2014), given a quasi-Nelson algebra $\langle \mathbb{L}, \nabla\rangle$ with $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$, we let $\mathcal{C}_\nabla := \bigcap\{\Phi_{\mathbf{L}_+}(a) : a \in \nabla\}$.

The next two propositions show that Definition 5.7 is indeed adequate.

Proposition 5.8 *Let $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ be a two-sorted lattice and let*

$$X(\mathbb{L}) = \langle X(\mathbf{L}_+), X(\mathbf{L}_-), X(n), X(p)\rangle$$

be the corresponding two-sorted Priestley space. Then the following hold:

(i) *If $\mathbb{L}$ is a non-involutive bilattice, then $X(\mathbb{L})$ is a non-involutive bilattice space.*
(ii) *If $\mathbb{L}$ is a non-involutive implicative bilattice, then $X(\mathbb{L})$ is a non-involutive implicative bilattice space.*
(iii) *If $\mathbb{L}$ is a semi-De Morgan algebra, then $X(\mathbb{L})$ is a semi-De Morgan space.*
(iv) *If $\langle \mathbb{L}, \nabla\rangle$ is a quasi-Nelson algebra, then $\langle X(\mathbb{L}), \mathcal{C}_\nabla\rangle$ is a quasi-Nelson space.*

Proof (i). Follows from the duality between 2Lat and 2PrSp and from items (i) and (ii) of Proposition 5.3.

(ii). Follows from the preceding item and from Esakia duality.

(iii). Property (1) of Definition 5.7(iii) follows from Cornish–Fowler duality, (2) follows from item (i) of Proposition 5.1, and (3) from items (i) and (iii) of Proposition 5.3.

(iv). Property (1) of Definition 5.7(iv) follows from Esakia duality, (2) from item (i) of Proposition 5.1, (4) follows from Jansana and Rivieccio (2014, Section 3.3), (5) follows from items (i) and (iii) of Proposition 5.3, and (6) from item (iv) of Proposition 5.3. It remains to verify that (3) holds, i.e. that $R_p = X(p) \subseteq X(\mathbf{L}_+) \times X(\mathbf{L}_-)$ is an Esakia relation. So let $\langle x_+, x_-\rangle \in X(\mathbf{L}_+) \times X(\mathbf{L}_-)$ be a pair of prime filters that belongs to $X(p)$, which means $p^{-1}[x_+] \subseteq x_-$. The Esakia condition states that we should find $z_+ \in X(\mathbf{L}_+)$ such that (a) $x_+ \subseteq z_+$, and (b) for all $y_- \in X(\mathbf{L}_-)$, $x_- \subseteq y_-$ iff $p^{-1}[z_+] \subseteq y_-$. Since we can choose $y_- = x_-$ in (b), and because of the prime filter theorem, we are really seeking that $p^{-1}[z_+] = x_-$. We show that $z_+ := n^{-1}[x_-]$ satisfies the two requirements. First note that z_+ is a prime filter of $\mathbf{L}_+$ since n is a lattice homomorphism. Then, because we have $Id_{L_+} \leq_+ p \circ n$ in the twist-product representation of a quasi-Nelson algebra, and since prime filters are upper sets, we know that $(p \circ n)[x_+] \subseteq x_+$, so

$$x_+ \subseteq (p \circ n)^{-1}[(p \circ n)[x_+]] \subseteq (p \circ n)^{-1}[x_+] \\ = n^{-1}[p^{-1}[x_+]] \subseteq n^{-1}[x_-] = z_+$$

which establishes (a). For (b) we exploit $n \circ p = Id_{L_-}$, and indeed, $x_- = (n \circ p)^{-1}[x_-] = p^{-1}[n^{-1}[x_-]] = p^{-1}[z_+]$. □

Proposition 5.9 *Let $X = \langle X_+, X_-, R_n, R_p\rangle$ be a two-sorted Priestley space, and let $\mathbf{L}(X) = \langle \mathbf{L}(X_+), \mathbf{L}(X_-), \Box_{R_n}, \Box_{R_p}\rangle$ be the corresponding two-sorted lattice. Then the following hold:*

(i) *If X is a non-involutive bilattice space, then $\mathbf{L}(X)$ is a non-involutive bilattice.*
(ii) *If X is a non-involutive implicative bilattice space, then $\mathbf{L}(X)$ is a non-involutive implicative bilattice.*
(iii) *If X is a semi-De Morgan space, then $\mathbf{L}(X)$ is a semi-De Morgan algebra.*
(iv) *If $\langle X, \mathcal{C}\rangle$ is a quasi-Nelson space, then $\langle \mathbf{L}(X), \nabla_{\mathcal{C}}\rangle$ is a quasi-Nelson algebra.*

Proof (i). Follows from (the duality between 2Lat and 2PrSp and from) items (i) and (ii) of Proposition 5.3.

(ii). Follows from the preceding item and from Esakia duality.

(iii). Property (1) of Definition 5.4.iii follows from Cornish–Fowler duality, (2) follows from item (i) of Proposition 5.1 and (3) from items (i) and (iii) of Proposition 5.3.

(iv). Property (1) of Definition 5.4.iv follows from Esakia duality, (2) from item (i) of Proposition 5.1, (3) follows from Bezhanishvili and Jansana (2013, Proposition 4.3), (4) follows from Jansana and Rivieccio (2014, Section 3.3), (5) follows from items (i) and (iii) of Proposition 5.3. □

In the next definition, we stipulate how the general definition of morphism for two-sorted lattices specialises (as expected) to the various classes of algebras we are interested in.

Definition 5.10 Let $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ and $\mathbb{L}' = \langle \mathbf{L}'_+, \mathbf{L}'_-, n', p'\rangle$ be two-sorted lattices (Definition 4.1) and $h = \langle h_+, h_-\rangle : \mathbb{L} \to \mathbb{L}'$ a 2Lat-morphism (Definition 4.5). We shall say that:

(i) h is a *non-involutive bilattice morphism* if $\mathbb{L}, \mathbb{L}'$ are non-involutive bilattices.
(ii) h is a *non-involutive implicative bilattice morphism* if $\mathbb{L}, \mathbb{L}'$ are non-involutive implicative bilattices and h_+, h_- preserve the Heyting implication.
(iii) h is a *semi-De Morgan morphism* if $\mathbb{L}, \mathbb{L}'$ are semi-De Morgan algebras and h_- preserves the De Morgan negation.
(iv) h is a *quasi-Nelson morphism* if $\langle \mathbb{L}, \nabla\rangle$, $\langle \mathbb{L}', \nabla'\rangle$ are quasi-Nelson algebras, h_+, h_- preserve the Heyting implication and $h_+[\nabla] \subseteq \nabla'$.

We are now in a position to confirm that the map $\Phi_{\mathbb{L}} = \langle \Phi_{\mathbf{L}_+}, \Phi_{\mathbf{L}_-}\rangle$ is, in each case, an isomorphism in the appropriate category.

Proposition 5.11 *Let $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ be a two-sorted lattice. If $\mathbb{L}$ is a non-involutive bilattice (respectively, a non-involutive implicative bilattice, a semi-De Morgan algebra, or if $\langle \mathbb{L}, \nabla\rangle$ is a quasi-Nelson algebra), then the unit of the adjunction $\Phi_{\mathbb{L}} = \langle \Phi_{\mathbf{L}_+}, \Phi_{\mathbf{L}_-}\rangle$ is an isomorphism of non-involutive bilattices (respectively, of non-involutive implicative bilattices, semi-De Morgan algebras or quasi-Nelson algebras) between $\mathbb{L}$ and $\mathbf{L}(X(\mathbb{L}))$.*

Proof We know from Proposition 4.6 that $\Phi_{\mathbb{L}}$ is a 2Lat-isomorphism. It remains to check that the additional conditions of Definition 5.10 are satisfied in each case. If $\mathbb{L}$ is a non-involutive bilattice, then the result follows from Proposition 5.8.i and Proposition 5.9.i. Similarly, if $\mathbb{L}$ is a non-involutive implicative bilattice, then we apply Proposition 5.8.ii and Proposition 5.9.ii to obtain that $\mathbf{L}(X(\mathbb{L}))$ is a non-involutive implicative bilattice, and we know from Esakia duality that the maps $\Phi_{\mathbf{L}_+}$ and $\Phi_{\mathbf{L}_-}$ preserve the Heyting implication as required. If $\mathbb{L}$ is a semi-De Morgan algebra, then we apply Proposition 5.8.iii and Proposition 5.9.iii to obtain that $\mathbf{L}(X(\mathbb{L}))$ is a semi-De Morgan algebra as well, and we know from Cornish–Fowler duality that $\Phi_{\mathbf{L}_-}$ preserves the De Morgan negation as required. Finally, if $\langle \mathbb{L}, \nabla\rangle$ is a quasi-Nelson algebra, then we use Proposition 5.8.iv and Proposition 5.9.iv to obtain that $\langle \mathbf{L}(X(\mathbb{L})), \nabla_{\mathcal{C}_\nabla}\rangle$ is a quasi-Nelson algebra too; then, we know from Esakia duality that the maps $\Phi_{\mathbf{L}_+}$ and $\Phi_{\mathbf{L}_-}$ preserve the Heyting implication, and we know that $\Phi_{\mathbf{L}_+}[\nabla] = \nabla_{\mathcal{C}_\nabla}$ from Jansana and Rivieccio (2014, Lemma 3.7). □

The next definition fixes the notion of spatial morphism for the various spaces considered, and Proposition 5.13 is the spatial counterpart of Proposition 5.11.

Definition 5.12 Let $X = \langle X_+, X_-, R_n, R_p\rangle$ and $X' = \langle X'_+, X'_-, R'_n, R'_p\rangle$ be two-sorted Priestley spaces and let $f = \langle f_+, f_-\rangle : X \to X'$ be a 2PrSp-morphism (Definition 4.7). We shall say that:

(i) f is a *non-involutive bilattice space function* if X, X' are non-involutive bilattice spaces.
(ii) f is a *non-involutive implicative bilattice space function* if X, X' are non-involutive implicative bilattice spaces and f_+, f_- are Esakia functions.

(iii) f is a *semi-De Morgan space function* if X, X' are semi-De Morgan spaces and f_- is a De Morgan space function.
(iv) f is a *quasi-Nelson space function* if $\langle X, \mathcal{C}\rangle, \langle X', \mathcal{C}'\rangle$ are quasi-Nelson spaces, f_+, f_- are Esakia functions and $f_+[\mathcal{C}] \subseteq \mathcal{C}'$.

Proposition 5.13 *Let $X = \langle X_+, X_-, R_n, R_p\rangle$ be a two-sorted Priestley space. If X is a non-involutive bilattice space (respectively, a non-involutive implicative bilattice space, a semi-De Morgan space, or a quasi-Nelson space), then the unit of the adjunction $\Phi_X = \langle \Phi_{X_+}, \Phi_{X_-}\rangle$ is an isomorphism of non-involutive bilattice spaces (resp., of non-involutive implicative bilattice spaces, semi-De Morgan spaces or quasi-Nelson spaces) between X and $X(\mathbf{L}(X))$.*

Proof We know from Proposition 4.10 that Φ_X is a 2PrSp-isomorphism. It remains to check that the additional conditions of Definition 5.12 are satisfied in each case. If X is a non-involutive bilattice space, then the result follows from Proposition 5.8.i and Proposition 5.9.i. Similarly, if X is a non-involutive implicative bilattice space, then we apply Proposition 5.8.ii and Proposition 5.9.ii to obtain that $X(\mathbf{L}(X))$ is a non-involutive implicative bilattice space, and we know from Esakia duality that Φ_{X_+} and Φ_{X_-} are Esakia isomorphisms as required. If X is a semi-De Morgan space, then we apply Proposition 5.8.iii and Proposition 5.9.iii to obtain that $X(\mathbf{L}(X))$ is a semi-De Morgan space as well, and we know from Cornish–Fowler duality that Φ_{X_-} is a De Morgan space isomorphism, as required. Finally, if $\langle X, \mathcal{C}\rangle$ is a quasi-Nelson space, then we use Proposition 5.8.iv and Proposition 5.9.iv to obtain that $\langle X(\mathbf{L}(X)), \mathcal{C}_{\nabla_\mathcal{C}}\rangle$ is a quasi-Nelson space too; then, we know from Esakia duality that Φ_{X_+} and Φ_{X_-} are Esakia isomorphisms, and we know that $\Phi_{X_+}[\mathcal{C}] = \mathcal{C}_{\nabla_\mathcal{C}}$ from Jansana and Rivieccio (2014, Lemma 3.8). □

The next two propositions show that the functors X and L operate as expected on morphisms.

Proposition 5.14 *Let $\mathbb{L} = \langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ and $\mathbb{L}' = \langle \mathbf{L}'_+, \mathbf{L}'_-, n', p'\rangle$ be two-sorted lattices, let $h = \langle h_+, h_-\rangle \colon \mathbb{L} \to \mathbb{L}'$ be a* 2Lat*-morphism and let $X(h) \colon X(\mathbb{L}') \to X(\mathbb{L})$ be the corresponding* 2PrSp*-morphism. The following hold:*

(i) *if h is a non-involutive bilattice morphism, then $X(h)$ is a non-involutive bilattice space function.*
(ii) *if h is a non-involutive implicative bilattice morphism, then $X(h)$ is a non-involutive implicative bilattice space function.*
(iii) *if h is a semi-De Morgan morphism, then $X(h)$ is a semi-De Morgan space function.*
(iv) *if h is a quasi-Nelson morphism, then $X(h)$ is a quasi-Nelson space function.*

Proof Item (i) follows from Proposition 5.8.i. Item (ii) follows from Proposition 5.8.ii and Esakia duality. Similarly, (iii) follows from Proposition 5.8.iii and Cornish–Fowler duality. Lastly, (iv) follows from Proposition 5.8.iv together with Jansana and Rivieccio (2014, Lemma 3.5). □

Proposition 5.15 *Let $X = \langle X_+, X_-, R_n, R_p\rangle$ and $X' = \langle X'_+, X'_-, R'_n, R'_p\rangle$ be two-sorted Priestley spaces, $f = \langle f_+, f_-\rangle \colon X \to X'$ a* 2PrSp*-morphism and let $\mathbf{L}(f) \colon \mathbf{L}(X') \to \mathbf{L}(X')$ be the corresponding* 2Lat*-morphism. The following hold:*

(i) *if f is a non-involutive bilattice space function, then $\mathbf{L}(f)$ is a non-involutive bilattice morphism.*
(ii) *if f is a non-involutive implicative bilattice space function, then $\mathbf{L}(f)$ is a non-involutive implicative bilattice morphism.*
(iii) *if f is a semi-De Morgan space function, then $\mathbf{L}(f)$ is a semi-De Morgan morphism.*
(iv) *if f is a quasi-Nelson space function, then $\mathbf{L}(f)$ is a quasi-Nelson morphism.*

Proof Item (i) follows immediately from Proposition 5.9.i. Item (ii) follows from Proposition 5.9.ii together with Esakia duality, while (iii) follows from Proposition 5.9.iii and Cornish–Fowler duality. Lastly, (iv) follows from Proposition 5.9.iv and Jansana and Rivieccio (2014, Lemma 3.6). □

Joining the preceding results, we obtain the announced equivalences.

Theorem 5.16 *The following categories are dually equivalent via the functors X and $\mathbf{L}$ introduced in Sect. 3:*

(i) *Non-involutive bilattices and non-involutive bilattice spaces.*
(ii) *Non-involutive implicative bilattices and non-involutive implicative bilattice spaces.*
(iii) *Semi-De Morgan algebras and semi-De Morgan spaces.*
(iv) *Quasi-Nelson algebras and quasi-Nelson spaces.*

6 Future work

As mentioned in the Introduction, a most appealing feature of the approach to duality proposed in the present paper is that it may, in principle, be applied to further classes of algebras which can be presented in a many-sorted fashion. Some obviously interesting cases in this direction are algebras defined over fragments of the algebraic languages considered in the present paper; for instance, the fragment of the quasi-Nelson language containing two negation operations but no implication is considered in Rivieccio (2020a) and Rivieccio et al.

(2020). Indeed, just as the language of quasi-Nelson algebras is a fragment of that of non-involutive bilattices (as it happens, a sufficiently rich one to allow for a two-sorted representation), so the language of De Morgan and Kleene algebras is a fragment of that of Nelson algebras; as mentioned earlier, whether the structure of De Morgan/Kleene algebras is sufficiently rich to admit a twist-algebra presentation and a topological duality is the object of current research of ours (Rivieccio 2020b). As a further example of potential future research direction, we will mention another class of algebras that can indeed be viewed as two-sorted (in more than one way: see, for example, Gehrke and Van Gool 2014), namely general (i.e. not necessarily distributive) lattices.

One final point that needs to be stressed is the connection of our investigations to logic. Although in the present paper we have not dwelled much on this aspect, all the algebras considered in the previous sections are indeed "algebras of logic", and the main interest in their study lies in their being the algebraic counterpart of certain non-classical logics. The many-sorted approach has recently proven to be particularly insightful in the proof-theoretic understanding of a wide array of non-classical logics, including bilattice (Greco et al. 2019) and semi-De Morgan logic (Greco et al. 2017); indeed, in this last case a novel two-sorted algebraic presentation for semi-De Morgan algebras has been introduced precisely with the aim of providing a two-sorted (display) calculus for the corresponding logic. Thus, the research program initiated in the present paper can also be seen as a duality-theoretic counterpart to the wider enterprise (also pursued in Frittella et al. (2016), Frittella et al. (2014), Frittella et al. (2016), Greco and Palmigiano (2017) and Greco and Palmigiano (submitted), to which we refer for further discussion and motivation) of developing a uniform many-sorted framework for non-classical logics and their algebras; in particular, the logics and algebras studied in the papers Greco and Palmigiano (2017) and Greco and Palmigiano (submitted) provide obvious further examples that may be worthwhile investigating from a many-sorted duality perspective.

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Compliance with ethical standards

Conflict of interest Both authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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