A Logical Approach to Stable Domains

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Abstract

Building on earlier work by Guo-Qiang Zhang on disjunctive information systems, and by Thomas Ehrhard, Pasquale Malacaria, and the first author on stable Stone duality, we develop a framework of disjunctive propositional logic in which theories correspond to algebraic L-domains. Disjunctions in the logic can be indexed by arbitrary sets (as in geometric logic) but must be provably disjoint. This raises several technical issues which have to be addressed before clean notions of axiom system and theory can be defined.

We show soundness and completeness of the proof system with respect to distributive disjunctive semilattices, and prove that every such semilattice arises as the Lindenbaum algebra of a disjunctive theory. Via stable Stone duality, we show how to use disjunctive propositional logic for a logical description of algebraic L-domains.
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1 Introduction

This paper takes up a number of research strands that have lain dormant for several years, and while it presents a number of new results it also highlights

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several unresolved issues. The central objects of study are *L-domains*, discovered independently by Th. Coquand [9] and the second author [20]. They occupy a curious position in domain theory; on the one hand, they form one of two maximal cartesian closed categories of algebraic domains and *Scott-continuous* functions [20], on the other hand, they form a large cartesian closed category of *stable* functions [26].[1] To date, no deeper reason is known for this coincidence but it explains and warrants the attention devoted to them in the literature. The specific goal of the present paper is to develop a logical language for describing algebraic L-domains, similar to S. Abramsky's *domain theory in logical form* (or *DTLF* for short), [2], for SFP-domains and Scott-continuous functions.

The possibility of logical descriptions for domains was first proposed and demonstrated by D. Scott in [25], where the logical apparatus is that of *information systems*. Scott showed that domain theory can be based on the notion of a "token of information" together with an entailment relation, thus tying denotational objects very closely to computational concerns. The approach was taken up in a number of publications; the most relevant for us are G.-Q. Zhang's papers on information systems for stable domain theory [28, 29, 30].

In a separate development, Abramsky realised that the language of information systems, while extremely elegant, is too parsimonious to serve as a useful basis for program logics. Indeed, at first approximation, the step from Scott's information systems to Abramsky's DTLF, is to allow information tokens to be combined by propositional connectives. The present paper similarly attempts to enrich Zhang's disjunctive information systems to a *disjunctive propositional logic*. Apart from the technical advantage of having the logical apparatus at one's disposal, one may gain deeper insight into the subject by explicating the connection with lattice theory, Stone duality, and topology.

Abramsky demonstrated the applicability of DTLF to problems in Computer Science in two landmark papers, [1, 4], devoted to concurrent and functional programming, respectively. This paper aims to lay the foundations for similar applications of stable domain theory. For this recall the role stability plays as an approximation to the operational notion of *sequentiality*, [7], in studying computability at higher types, [22], and in the λ-calculus, [5].

[1] For an in-depth discussion of cartesian closure in the stable universe, see [5, Chapter 12].

Obviously, a logic for disjunctive propositions must deviate in some way from classical propositional logic. Here we take our cue from work in category theory, especially from M. Coste's notion of a *lim theory* [10] and P. Johnstone's *disjunctive theories* [16]. In both cases, the set of admissible formulas is restricted by requirements that need to be established in parallel via a proof system. For our purposes we end up with a propositional calculus in which disjunctions can be indexed by sets of arbitrary cardinality but must be shown to be over a "disjoint" set of formulas. Section 2 of our paper is devoted to a careful analysis of the resulting syntactic framework. Three challenges present themselves: firstly, formulas and derivations have to be defined in parallel, through a simultaneous induction; secondly, and because of this, it is not obvious what the correct definition of an axiom system should be; thirdly, by admitting arbitrary infinite disjunctions we are faced with problems of size. Luckily, all three problems can be overcome in what we would deem a satisfying and elegant way.

A useful intermediary step on the way towards a logical description of L-domains is to provide an algebraic semantics for the logic, essentially by factoring valid formulas by interderivability, also known as the Lindenbaum construction. The appropriate lattice-like structures were presented by the first author in [8]; they are called *distributive disjunctive semilattices*. Some care needs to be taken to define the semantics of disjunctive propositions because of the interdependence with derivations, but with the presentation obtained in Section 2 this is not too difficult, and both soundness and completeness can be established following essentially the classical construction.

In Section 3.3 we take a closer look at the category of distributive disjunctive semilattices from a purely algebraic perspective. The supremum operation on these is only defined partially, but the domain of definition is given by equations expressed in the totally defined infimum operation, so we are dealing with an *essentially algebraic theory* in the sense of P. Freyd [12]. The completeness proof of the previous section can now be used to set up an adjunction between certain structured sets and distributive disjunctive semilattices. Somewhat to our surprise, we find that this adjunction is not monadic, though it is known that it can be written as a composition of two monadic adjunctions [19].

In Section 4 we look at the link between disjunctive propositional logic and distributive disjunctive semilattices from the perspective of the latter, and show that every semilattice has a logical presentation. Only axioms of a certain kind are required and we see most clearly the link between Zhang's disjunctive information systems and our logic. It can be argued that the proof of the presentation theorem 4.3 is precisely the price one has to pay for the increased expressivity of the latter over the former. We conclude this section with an application of the presentation theorem by showing that the category of distributive disjunctive semilattices has coequalisers.

In Section 5 we combine the link between logic and semilattices, on the one hand, with a Stone-type duality between disjunctive semilattices and L-domains, established by the first author in [8]. The role of open sets is played by Zhang's "stable neighbourhoods" [31], and we take some care to explore the concept in the realm of general L-domains (rather than dI-domains).

We conclude with a discussion of the problems that need to be overcome if one were to attempt to extend the framework to *continuous* rather than algebraic L-domains.

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2 Disjunctive propositional theories

M. Coste (see Johnstone's paper [16]) introduces the notion of a lim-theory by requiring that its axioms be sequents constructed using the logical operations *true*, $\wedge$ and $\exists$, with the further restriction that existential quantification may be used only when the variable being quantified is provably unique; i.e., $\exists x.\phi(x)$ is a "good" formula only if the sequent $(\phi(x) \wedge \phi(x') \vdash x{=}x')$ is deducible from the axioms.

Johnstone [16] defines a disjunctive theory in a similar manner: he admits all the operations of geometric logic (including infinite disjunctions), subject to the same restriction as before on the use of $\exists$ and the additional requirement that disjunctions must be provably disjoint, i.e., $\bigvee_{i\in I}\phi_i$ is a "good" formula only if $(\phi_i \wedge \phi_j \vdash \mathit{false})$ is provable for each pair of distinct indices (i, j).

The focus of the current paper is the propositional part of Johnstone's disjunctive theory. In other words, we will deal with provably disjoint disjunctions but not the existential.

2.1 Formulas and derivations

Formulas will be built out of atomic propositions using binary conjunctions and arbitrary, but provably disjoint, disjunctions. Because the construction of formulas refers to proofs, we simultaneously define a proof system for establishing disjointness. For this we employ sequents in the style of Gentzen's intuitionistic sequent calculus LJ, [13]. These take the form $\Gamma \vdash \phi$ where Γ is a finite set of formulas and ϕ is a single formula. As usual, the intended meaning is that the conjunction of the propositions in Γ entails ϕ.

Without further assumptions it is not possible to prove the disjointness of any two formulas, unless one of them is equivalent to *false* already. So it is necessary also to allow some disjointness assumptions to be made at the very beginning. Once again, these assumptions have an impact on which formulas can be constructed. This is a rather unusual situation, and we take some care in this section in setting up the formal system and proving its fundamental properties.

Finally, as there is no restriction on the cardinality of the arity of the disjunction operation, we are dealing with a version of *infinitary logic*. As a result, we have to deal with proper *classes* of formulas and derivations, and allow transfinite inductions. Luckily, though, it will turn out that the expressivity of the system is already captured by a *set* of formulas (and derivations).

Definition 2.1 *Let P be a set, the elements of which we call* atomic (disjunctive) propositions. *Likewise, let S_0 be a set of sequents of the form $p_1, \ldots, p_n \vdash F$ where the p_i are atomic propositions, and F is the syntactic constant for "false." We call the elements of S_0* atomic disjointness assumptions, *and the pair (P, S_0) a* disjunctive basis.

The class $\mathcal{L}(P, S_0)$ of disjunctive propositions over P and S_0, *and the class $\mathbf{T}(P, S_0)$ of* valid sequents over P and S_0 *are generated by mutual transfinite induction according to the following rules:*

Disjunctive propositions

$$\text{(At)}\quad \frac{\phi \in P}{\phi \in \mathcal{L}(P, S_0)} \qquad\qquad \text{(Const)}\quad \frac{}{T, F \in \mathcal{L}(P, S_0)}$$

$$\text{(Conj)}\quad \frac{\phi, \psi \in \mathcal{L}(P, S_0)}{\phi \wedge \psi \in \mathcal{L}(P, S_0)}$$

$$\text{(Disj)}\quad \frac{\phi_i \in \mathcal{L}(P, S_0)\ (\text{all } i \in I) \qquad \phi_i, \phi_j \vdash F\ (\text{all } i \neq j \in I)}{\dot{\bigvee_{i \in I}} \phi_i \in \mathcal{L}(P, S_0)}$$

Valid sequents

$$(\mathrm{Ax})\quad \frac{(\Gamma \vdash F) \in S_0}{\Gamma \vdash F} \qquad\qquad (\mathrm{Id})\quad \frac{\phi \in \mathcal{L}(P, S_0)}{\phi \vdash \phi}$$

$$(\mathrm{Lwk})\quad \frac{\Gamma \vdash \psi \qquad \phi \in \mathcal{L}(P, S_0)}{\Gamma, \phi \vdash \psi} \qquad\qquad (\mathrm{Cut})\quad \frac{\Gamma \vdash \phi \qquad \Delta, \phi \vdash \psi}{\Gamma, \Delta \vdash \psi}$$

$$(\mathrm{L}F)\quad \frac{\phi \in \mathcal{L}(P, S_0)}{F \vdash \phi} \qquad\qquad (\mathrm{R}T)\quad \frac{}{\vdash T}$$

$$(\mathrm{L}\wedge)\quad \frac{\Gamma, \phi, \psi \vdash \theta}{\Gamma, \phi \wedge \psi \vdash \theta} \qquad\qquad (\mathrm{R}\wedge)\quad \frac{\Gamma \vdash \phi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \wedge \psi}$$

$$(\mathrm{L}\dot{\vee})\quad \frac{\Gamma, \phi_i \vdash \theta \text{ (all } i \in I) \qquad \phi_i, \phi_j \vdash F \text{ (all } i \neq j \in I)}{\Gamma, \dot{\bigvee_{i \in I}} \phi_i \vdash \theta}$$

$$(\mathrm{R}\dot{\vee})\quad \frac{\Gamma \vdash \phi_{i_0} \text{ (some } i_0 \in I) \qquad \phi_i, \phi_j \vdash F \text{ (all } i \neq j \in I)}{\Gamma \vdash \dot{\bigvee_{i \in I}} \phi_i}$$

Although the inductive definitions produce proper classes of objects, in each formula the nesting of operators is only finite (though may be unbounded); likewise, the length of any path from assumption to conclusion in a derivation is finite (though a derivation may contain paths of arbitrary length). This is because each rule preserves this property.

As in usual elementary proof theory, we can show that the logical rules can be "inverted" (except $\mathrm{R}\dot{\vee}$, because the setting is intuitionistic, with only a single formula allowed on the right).

Proposition 2.2 *(i) $\Gamma, \phi, \psi \vdash \theta$ is derivable, if and only if $\Gamma, \phi \wedge \psi \vdash \theta$ is derivable.*

(ii) $\Gamma \vdash \phi$ and $\Gamma \vdash \psi$ are derivable, if and only if $\Gamma \vdash \phi \wedge \psi$ is derivable.

(iii) Assuming $\phi_i, \phi_j \vdash F$ is derivable for all $i \neq j \in I$, then all $\Gamma, \phi_i \vdash \theta$ are derivable if and only if $\Gamma, \dot{\bigvee}_{i\in I} \phi_i \vdash \theta$ is derivable.

Proof. In each case, the "only if"-part is just an application of the corresponding rule. The "if" part requires use of the cut rule. We only illustrate this for the last statement:

$$\dfrac{\dfrac{\dfrac{}{\phi_{i_0} \vdash \phi_{i_0}}\,\mathrm{Id}}{\phi_{i_0} \vdash \dot{\bigvee}_{i\in I} \phi_i}\,\mathrm{R}\dot{\vee} \qquad \Gamma, \dot{\bigvee}_{i\in I} \phi_i \vdash \theta}{\Gamma, \phi_{i_0} \vdash \theta}\,\mathrm{Cut}$$

∎

In the remainder of this section we will usually treat the side conditions of the rules (L$\dot{\vee}$) and (R$\dot{\vee}$) separately. All our derivations are then entirely standard, except that disjunctions can be indexed by an arbitrary set. We have written the derivations down so that the reader can check that the necessary side conditions have indeed been established.

2.2 Normal forms

The goal of this subsection is to show that every disjunctive formula over a basis (P, S_0) is provably equivalent to a disjunction of conjunctions of atomic formulas. This is in analogy to the theory of frames, see [18, Section II.2.11], or the generation of a topology from a subbasis.

Definition 2.3 *We call disjunctive propositions ϕ and ψ* interderivable, *and write $\phi \dashv\vdash \psi$, if both $\phi \vdash \psi$ and $\psi \vdash \phi$ can be derived.*

We begin with a suitable version of the frame distributivity law.

Proposition 2.4 *Assume $\phi_i, \phi_j \vdash F$ for all $i \neq j \in I$. Then $\phi \wedge (\dot{\bigvee}_{i\in I} \phi_i)$ and $\dot{\bigvee}_{i\in I}(\phi \wedge \phi_i)$ are interderivable.*

Proof. We first show that $\phi \wedge \phi_i, \phi \wedge \phi_j \vdash F$ is derivable whenever $\phi_i, \phi_j \vdash F$ is derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi_i, \phi_j \vdash F}{\phi, \phi_i, \phi_j \vdash F}\,\text{Lwk}}{\phi \wedge \phi_i, \phi_j \vdash F}\,\text{L}\wedge}{\phi \wedge \phi_i, \phi, \phi_j \vdash F}\,\text{Lwk}}{\phi \wedge \phi_i, \phi \wedge \phi_j \vdash F}\,\text{L}\wedge$$

The derivation of $\phi \wedge (\overset{\bullet}{\bigvee}_{i \in I} \phi_i) \vdash \overset{\bullet}{\bigvee}_{i \in I}(\phi \wedge \phi_i)$ is not difficult but we need to be careful with the indices. First note that the following is valid for each $i_0 \in I$:

$$\dfrac{\dfrac{}{\phi \vdash \phi}\,\text{Id} \qquad \dfrac{}{\phi_{i_0} \vdash \phi_{i_0}}\,\text{Id}}{\phi, \phi_{i_0} \vdash \phi \wedge \phi_{i_0}}\,\text{R}\wedge$$

For each i_0 we can therefore apply the rule $\text{R}\overset{\bullet}{\vee}$ and obtain $\phi, \phi_{i_0} \vdash \overset{\bullet}{\bigvee}_{i \in I}(\phi \wedge \phi_i)$. Since the right hand side does not depend on i_0, we can next apply $\text{L}\overset{\bullet}{\vee}$ and get $\phi, \overset{\bullet}{\bigvee}_{i \in I} \phi_i \vdash \overset{\bullet}{\bigvee}_{i \in I}(\phi \wedge \phi_i)$. An application of $\text{L}\wedge$ completes the proof of the first entailment. For the converse we just give the derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \vdash \phi}\,\text{Id} \qquad \dfrac{\dfrac{}{\phi_{i_0} \vdash \phi_{i_0}}\,\text{Id}}{\phi_{i_0} \vdash \overset{\bullet}{\bigvee}_{i \in I} \phi_i}\,\text{R}\overset{\bullet}{\vee}}{\phi, \phi_{i_0} \vdash \phi \wedge \overset{\bullet}{\bigvee}_{i \in I} \phi_i}\,\text{R}\wedge}{\phi \wedge \phi_{i_0} \vdash \phi \wedge (\overset{\bullet}{\bigvee}_{i \in I} \phi_i)}\,\text{L}\wedge}{\overset{\bullet}{\bigvee}_{i \in I} (\phi \wedge \phi_i) \vdash \phi \wedge (\overset{\bullet}{\bigvee}_{i \in I} \phi_i)}\,\text{L}\overset{\bullet}{\vee}$$

∎

We note that despite the interderivability stated in this proposition, distributivity only works in one direction, as we can not infer $\phi_i, \phi_j \vdash F$ from $\phi \wedge \phi_i, \phi \wedge \phi_j \vdash F$.

Next we consider the associativity of disjoint disjunctions. We begin with the disjointness side condition.

Proposition 2.5 *Assume $\phi_i, \phi_{i'} \vdash F$ for all $i \neq i' \in I$, and $\psi_j, \psi_{j'} \vdash F$ for all $j \neq j' \in J$. Then $\dot{\bigvee}_{i\in I} \phi_i, \dot{\bigvee}_{j\in J} \psi_j \vdash F$, if and only if $\phi_i, \psi_j \vdash F$ for all $i \in I$ and $j \in J$.*

Proof. Suppose $\dot{\bigvee}_{i\in I} \phi_i, \dot{\bigvee}_{j\in J} \psi_j \vdash F$. Then for each $i_0 \in I$ and $j_0 \in J$, we have the derivation

$$\dfrac{\dfrac{\dfrac{}{\phi_{i_0} \vdash \phi_{i_0}}\,\mathrm{Id}}{\phi_{i_0} \vdash \dot{\bigvee}_{i\in I} \phi_i}\,\mathrm{R}\dot{\vee} \qquad \dfrac{\dfrac{\dfrac{}{\psi_{j_0} \vdash \psi_{j_0}}\,\mathrm{Id}}{\psi_{j_0} \vdash \dot{\bigvee}_{j\in J} \psi_j}\,\mathrm{R}\dot{\vee} \qquad \dot{\bigvee}_{i\in I} \phi_i, \dot{\bigvee}_{j\in J} \psi_j \vdash F}{\psi_{j_0}, \dot{\bigvee}_{i\in I} \phi_i \vdash F}\,\mathrm{Cut}}{\phi_{i_0}, \psi_{j_0} \vdash F}\,\mathrm{Cut}$$

For the converse, assume $\phi_i, \psi_j \vdash F$ for all $i \in I$ and $j \in J$, $\phi_i, \phi_{i'} \vdash F$ for all $i \neq i' \in I$, and $\psi_j, \psi_{j'} \vdash F$ for all $j \neq j' \in J$. We get

$$\dfrac{\dfrac{\phi_i, \psi_j \vdash F \quad (\text{all } i) \quad (\text{all } j)}{(\dot{\bigvee}_{i\in I} \phi_i), \psi_j \vdash F \quad (\text{all } j)}\,\mathrm{L}\dot{\vee}}{(\dot{\bigvee}_{i\in I} \phi_i), (\dot{\bigvee}_{j\in J} \psi_j) \vdash F}\,\mathrm{L}\dot{\vee}$$

∎

Proposition 2.6 *Let $(I_j)_{j\in J}$ be a partition of the set I, and $(\phi_i)_{i\in I}$ be a disjoint family of propositions. Then $\dot{\bigvee}_{i\in I} \phi_i$ and $\dot{\bigvee}_{j\in J} \dot{\bigvee}_{i\in I_j} \phi_i$ are interderivable.*

Proof. The side conditions having been checked in the previous proposition it suffices to provide the derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi_i \vdash \phi_i \quad (\text{all } i \in I_j) \quad (\text{all } j \in J)}\,\mathrm{Id}}{\phi_i \vdash \dot{\bigvee\limits_{k\in I}} \phi_k \quad (\text{all } i \in I_j) \quad (\text{all } j \in J)}\,\mathrm{R}\dot{\vee}}{\dot{\bigvee\limits_{i\in I_j}} \phi_i \vdash \dot{\bigvee\limits_{k\in I}} \phi_k \quad (\text{all } j \in J)}\,\mathrm{L}\dot{\vee}}{\dot{\bigvee\limits_{j\in J}} \dot{\bigvee\limits_{i\in I_j}} \phi_i \vdash \dot{\bigvee\limits_{k\in I}} \phi_k}\,\mathrm{L}\dot{\vee}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{\phi_k \vdash \phi_k}\,\mathrm{Id}}{\phi_k \vdash \dot{\bigvee\limits_{i\in I(k)}} \phi_i}\,\mathrm{R}\dot{\vee}}{\phi_k \vdash \dot{\bigvee\limits_{j\in J}} \dot{\bigvee\limits_{i\in I_j}} \phi_i}\,\mathrm{R}\dot{\vee}}{\dot{\bigvee\limits_{k\in I}} \phi_k \vdash \dot{\bigvee\limits_{j\in J}} \dot{\bigvee\limits_{i\in I_j}} \phi_i}\,\mathrm{L}\dot{\vee}$$

(In the second derivation we wrote $I(k)$ for the class of the partition to which a given $k \in I$ belongs.) ∎

Proposition 2.7 *(i) Let $\phi = \bigwedge_{i=1}^{n} \phi_i$ be a disjunctive proposition; then all occurrences of subformulas ϕ_i for which $\phi_i \dashv\vdash T$ can be dropped from the conjunction and the resulting formula is interderivable with ϕ. Likewise, any two interderivable subformulas $\phi_i, \phi_{i'}$ can be reduced to one of them.*

(ii) Let $\phi = \dot{\bigvee}_{i\in I} \phi_i$ be a disjunctive proposition; then all occurrences of subformulas ϕ_i for which $\phi_i \dashv\vdash F$ can be dropped from the disjunction and the resulting formula is interderivable with ϕ. Furthermore, any two interderivable subformulas can be dropped entirely.

Proof. Part (i) is standard, as is the first half of (ii). For the second half assume that ϕ_i and $\phi_{i'}$, $i \neq i' \in I$, are interderivable. By definition it must be the case that $\phi_i, \phi_{i'} \vdash F$. Using the cut-rule and interderivability, we get from this $\phi_i \vdash F$ and $\phi_{i'} \vdash F$, which implies that both ϕ_i and $\phi_{i'}$ are interderivable with the constant F.

Since the index set in a disjunction can be an arbitrary set, it is also noteworthy that all occurrences of subformulas which are interderivable with F can be dropped *in one step*. To this end set $I_0 := \{i \in I \mid \phi_i \dashv\vdash F\}$ and

$I_1 := I \setminus I_0$. We have the derivations

$$\forall i \in I_1 : \quad \dfrac{\phi_i \vdash \phi_i}{\phi_i \vdash \dot{\bigvee}_{i \in I_1} \phi_i}\,\mathrm{R}\dot{\vee} \qquad\qquad \forall i \in I_0 : \quad \dfrac{\phi_i \vdash F \quad F \vdash \dot{\bigvee}_{i \in I_1} \phi_i}{\phi_i \vdash \dot{\bigvee}_{i \in I_1} \phi_i}\,\mathrm{Cut}$$

and one application of $\mathrm{L}\dot{\vee}$ shows that $\dot{\bigvee}_{i \in I} \phi_i \vdash \dot{\bigvee}_{i \in I_1} \phi_i$. The other direction is trivial. ∎

Theorem 2.8 *Let (P, S_0) be a disjunctive basis. Every disjunctive proposition over P and S_0 is interderivable with a formula of the form $\dot{\bigvee}_{i \in I} \bigwedge_{j \in M_i} p_j$, where each M_i is finite and all p_j are elements of P. Furthermore, the formula can be chosen in such a way that the sets $F_i := \{p_j \mid j \in M_i\}$ are all different from each other.*

Proof. This is shown by induction on the derivation of the given formula ϕ; if ϕ is equal to F then choose $I = \emptyset$, if it is equal to T, then choose $I = \{*\}$, $M_* = \emptyset$; if $\phi = p \in P$, then set $I = \{*\} = M_*$, $p_* = p$.

For conjunction assume $\phi = \phi_1 \wedge \phi_2 \in \mathcal{L}(P, S_0)$ and by induction hypothesis $\phi_1 \dashv\vdash \dot{\bigvee}_{i \in I} \bigwedge_{j \in M_i} q_j$ and $\phi_2 \dashv\vdash \dot{\bigvee}_{i' \in I'} \bigwedge_{j \in M_{i'}} q_j$. For ϕ consider $\dot{\bigvee}_{i \in I, i' \in I'} \bigwedge_{j \in M_i \cup M_{i'}} q_j$ which is certainly well-formed and interderivable with the given formula, but may not be quite what we want as some $F_{i_1, i'_1} = \{q_j \mid j \in M_i \cup M_{i'}\}$ may be equal to another F_{i_2, i'_2} without $(i_1, i'_1) = (i_2, i'_2)$. However, by the preceding proposition, such instances can all be dropped from the disjunction without affecting its logical strength.

For disjunction assume ϕ is of the form $\dot{\bigvee}_{k \in K} \phi_k$ and we have already established propositions of the desired form interderivable with each ϕ_k, say $\phi_k \dashv\vdash \dot{\bigvee}_{i \in I_k} \bigwedge_{j \in M_i} q_j$. We assume that ϕ is well-formed, which yields $\dot{\bigvee}_{i \in I_k} \bigwedge_{j \in M_i} q_j, \dot{\bigvee}_{i \in I_{k'}} \bigwedge_{j \in M_i} q_j \vdash F$ for all $k \neq k' \in K$, and hence $\bigwedge_{j \in M_i} q_j, \bigwedge_{j \in M_i} q_j \vdash F$ for all $i \in I_k$, $i' \in I_{k'}$, $k \neq k' \in K$, by Proposition 2.2-(iii). This means that we can form the disjunctive proposition $\dot{\bigvee}_{i \in I'} \bigwedge_{j \in M_i} q_j$

where $I' = \dot{\bigcup}_{k \in K} I_k$. As in the previous case, we may need to apply Proposition 2.7 to remove repeated conjunctions. ∎

We call formulas of the form $\dot{\bigvee}_{i \in I} \bigwedge_{j \in M_i} p_j$ with $\{p_j \mid j \in M_i\} \neq \{p_j \mid j \in M_{i'}\}$ for $i \neq i' \in I$, *flat disjunctive propositions*, or *disjunctive normal forms*. Likewise, a sequent will be called flat if all formulas occurring in it are flat.

Corollary 2.9 *For sets P and S_0 there exists a set $\ell(P, S_0)$ of disjunctive propositions over P such that every element of the class $\mathcal{L}(P, S_0)$ is interderivable with an element of $\ell(P, S_0)$.*

Proof. The disjunctive normal forms of the theorem above can be put into 1-1 correspondence with a subset of the powerset of the finite powerset of P. ∎

We note that the flat disjunctive proposition defined in the proof of Theorem 2.8 is not necessarily the only such formula that is interderivable with a given ϕ. Thus the development above does not amount to true "normal forms", but what we have is certainly sufficient for the purposes of this paper.

2.3 Axiom sets and disjunctive theories

The normal form theorem allows us to answer a question that may have occurred to the reader in Definition 2.1 already, namely, whether it is possible and meaningful to postulate more general disjointness assumptions than those allowed as members of S_0 when constructing formulas. As it turns out, this would not add anything in terms of expressiveness; a disjointness sequent $\phi, \psi \vdash F$ can be replaced with one in which ϕ and ψ are flat. The outer disjunctions can then be stripped off (by Proposition 2.2-(iii)) and we obtain a set of disjointness conditions between conjunctions of atomic formulas. The conjunctions, in turn, can be replaced by commas as shown in Proposition 2.2-(i). We end up with a set of atomic disjointness conditions. This means that all meaningful sets of disjunctive propositions are already covered by Definition 2.1.

On the other hand, within a given set (or class) of disjunctive propositions we can ask whether there are additional sequents that can be assumed as axioms. This is indeed the case, but the reasoning of the previous paragraph still applies, and so we only need to consider axioms of the form $p_1, \ldots, p_n \vdash \dot{\bigvee}_{i \in I} \bigwedge_{j \in M_i} q_j$. Of course, with each such axiom we must require that the disjunction on the right is well-formed. Let us make this precise:

Definition 2.10 *A* disjunctive axiom system *over a set of atomic propositions P is a set S of disjunctive sequents of the form*

$$p_1, \ldots, p_n \vdash \dot{\bigvee_{i \in I}} \bigwedge_{j \in M_i} q_j$$

where all p_k and q_j are elements of P. Furthermore, with each sequent of this form, the sequents

$$q_{j_1}, \ldots, q_{j_m}, q_{j'_1}, \ldots, q_{j'_{m'}} \vdash F$$

for each $i \neq i' \in I$ and $M_i = \{j_1, \ldots, j_m\}$, $M_{i'} = \{j'_1, \ldots, j'_{m'}\}$ must also belong to S. We call the subset of axioms where the right-hand side is F the set of disjointness assumptions.

Definition 2.11 *For S a disjunctive axiom system over atomic propositions P we denote with $\mathcal{L}(P, S)$ ($\ell(P, S)$) the set of (flat) disjunctive propositions, and with $\mathbf{T}(P, S)$ the set of sequents that can be derived with the simultaneous rules of Definition 2.1. We call $\mathbf{T}(P, S)$ the* disjunctive propositional theory *generated by S, and the elements of $\mathbf{T}(P, S)$ the* valid sequents *of the theory.*

To improve readability we will often leave the set P of atomic propositions implicit and only write $\mathbf{T}(S)$ or even $\mathbf{T}$.

This definition requires us to adjust rule (Ax) of 2.1 to

$$(\mathrm{Ax}') \quad \frac{(\Gamma \vdash \phi) \in S}{\Gamma \vdash \phi}$$

There is also a slight subtlety with the requirement that a disjunctive axiom system contain all disjointness assumptions that are needed to build the formulas that appear in an axiom. Obviously, it ensures that all formulas that are mentioned somewhere in a derived sequent are in fact members of $\mathcal{L}(P,S)$, but on the other hand, the disjointness assumptions of a disjunctive axiom system on their own are not necessarily enough to generate all of $\mathcal{L}(P,S)$ or even $\ell(P,S)$: consider the simple example $S = \{(p \vdash q), (p' \vdash q'), (q, q' \vdash F)\}$ in which not only $q \overset{\bullet}{\vee} q'$ but also $p \overset{\bullet}{\vee} p'$ is generated as a legal disjunctive proposition. In other words, additional disjointness assumptions for atomic propositions may be derivable from the given axioms in S.

Finally, this is a good moment to explicate the link between our logic and G.-Q. Zhang's *disjunctive information systems*. Looking again at the shape of sequents in a disjunctive axiom system, one may notice that if one allows additional atomic propositions to be created, then even simpler axioms will suffice. To this end one introduces a fresh atomic proposition r_i for every subexpression $\bigwedge_{j \in M_i} q_j$ together with the axioms $r_i \vdash q_j$ for all $j \in M_i$, and $q_1, \ldots, q_n \vdash r_i$ (where $\{q_1, \ldots, q_n\} = \{q_j \mid j \in M_i\}$). Furthermore, one could allow *disjoint* sequences of formulas on the right and render the axioms in the form

$$p_1, \ldots, p_n \vdash r_1, \ldots, r_i, \ldots$$

which avoids all connectives. Together with those derivation rules that do not introduce or eliminate a connective, one obtains in this way exactly a *disjunctive information system* in the sense of [30].

3 Algebraic semantics

3.1 Disjunctive semilattices

In order to give a representation of L-domains in the style of frames, and a Stone-type duality for the category of L-domains and stable functions, Chen [8] introduced the notion of D-semilattice. We briefly recall the relevant definitions.

Definition 3.1 *Let $(L; 0, 1, \sqcap)$ be a meet-semilattice with least element 0 and greatest element 1.*

- *For $x, y \in L$ we say that x and y are* disjoint *if $x \sqcap y = 0$.*
- *A subset B of L is disjoint if each pair of distinct elements x and y in B are disjoint.*

- *The semilattice is called* disjunctive *(for short, a* D-semilattice*) if every disjoint subset has a supremum. Joins of disjoint subsets B are denoted by $\dot{\bigsqcup} B$.*

Finite D-semilattices are lattices but in the infinite case the difference becomes apparent, see Figure 1.

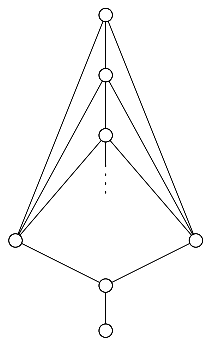

Figure 1: A D-semilattice which is not a lattice.

Definition 3.2 *Let L, N be* D*-semilattices. A map $f\colon L \to N$ is called a D-semilattice homomorphism if it preserves finite meets and disjoint suprema. In particular, it preserves least and largest element.*

We will write **DSL** *for the category of D-semilattices and D-semilattice homomorphisms.*

A D-semilattice L is called distributive *(or a* dD-semilattice*) if*

$$a \sqcap (\dot{\bigsqcup} B) = \dot{\bigsqcup_{b \in B}} a \sqcap b$$

is true for each element $a \in L$ and disjoint subset B of L.

The full sub-category of dD-semilattices in **DSL** *will be denoted by* **dDSL**.

3.2 Structures, soundness, and completeness

dD-semilattices are the appropriate structures for interpreting disjunctive propositional logic.

Definition 3.3 *Let L be a dD-semilattice. A* structure *M for a disjunctive basis (P, S_0) in L is a function $M\colon P \to L$ such that $\prod_{i=1}^{n} M(p_i) = 0$ for all $(p_1, \ldots, p_n \vdash F) \in S_0$.*

Given a structure M one defines a semantics $[\![\cdot]\!]_M$ for disjunctive propositions in $\mathcal{L}(P, S_0)$ by transfinite induction in the obvious way:

1. $[\![p]\!]_M := M(p)$;
2. $[\![T]\!]_M := 1$, $[\![F]\!]_M := 0$;
3. $[\![\phi \wedge \psi]\!]_M := [\![\phi]\!]_M \sqcap [\![\psi]\!]_M$;
4. $[\![\dot{\bigvee}_{i \in I} \phi_i]\!]_M := \dot{\bigsqcup}_{i \in I} [\![\phi_i]\!]_M$.

For the last clause to make sense we must prove that the supremum is over a disjoint subset. This can be done straightforwardly by transfinite induction over the rules with which we derive valid sequents starting from a set of atomic disjointness assumptions. One shows that that for all valid sequents $\phi_1, \ldots, \phi_n \vdash \psi$ it holds that $\bigsqcap_{i=1}^{n} [\![\phi_i]\!]_M \sqsubseteq [\![\psi]\!]_M$, so in particular, if $\phi_1, \ldots, \phi_n \vdash F$, then $\bigsqcap_{i=1}^{n} [\![\phi_i]\!]_M = 0$.

In general, whenever $\bigsqcap_{i=1}^{n} [\![\phi_i]\!]_M \sqsubseteq [\![\psi]\!]_M$ holds for a structure M in a dD-semilattice L, we say that M *satisfies* the sequent $\phi_1, \ldots, \phi_n \vdash \psi$.

If S is a disjunctive axiom system over P, then we can consider the set $S_0 \subseteq S$ of disjointness assumptions and thus establish whether a given map $M \colon P \to L$ is a structure for (P, S_0). If so, then the semantics $[\![\cdot]\!]_M$ will assign a meaning at least to all formulas appearing in the sequents of S. This allows us to check whether M satisfies the sequents in S, in which case we call M a *model* of S. As a model, M will also satisfy all derived sequents, i.e., all of $\mathbf{T}(P, S)$, and therefore the semantic function can be extended to all of $\mathcal{L}(P, S)$ (which, as we saw at the end of Section 2.3, can be bigger than $\mathcal{L}(P, S_0)$). Suppressing these subtleties, we can summarise:

Theorem 3.4 (Soundness) *If M is a model of a disjunctive axiom system S in a dD-semilattice L, then M satisfies all valid sequents of the disjunctive propositional theory $\mathbf{T}(S)$ generated by S.*

Let's now turn to completeness: suppose P is a set of atomic propositions and S a disjunctive axiom system according to Definition 2.10, with $\mathbf{T} := \mathbf{T}(P, S)$ the disjunctive theory generated. We would like to follow the usual procedure and show that $\mathcal{L}(P, S)$, quotiented by interderivability, is a dD-semilattice that satisfies exactly those sequents that are derivable from the axiom system, but we must be conscious of the problem of size. One shows easily that for every formula ϕ, the equivalence class

$$[\phi]_{\mathbf{T}} := \{\psi \in \mathcal{L}(P,S) \mid \phi \dashv\vdash \psi\}$$

is a proper class. Luckily, it is still the case that there are only set-many such equivalence classes. We see this by considering

$$[\phi]'_{\mathbf{T}} := [\phi]_{\mathbf{T}} \cap \ell(P,S)$$

which is always small as it is a subset of $\ell(P,S)$. Furthermore, in Theorem 2.8 we showed that $[\phi]'_{\mathbf{T}}$ is always non-empty, and from this we infer

$$\phi \dashv\vdash \psi \quad \text{if and only if} \quad [\phi]'_{\mathbf{T}} = [\psi]'_{\mathbf{T}} \; .$$

In other words, the (small) equivalence classes on $\ell(P,S)$ are fully representative of the (big) equivalence classes on $\mathcal{L}(P,S)$. Consequently we set

$$A(\mathbf{T}) := \{[\phi]'_{\mathbf{T}} \mid \phi \in \mathcal{L}(P,S)\} \; .$$

(Note that every element of $A(\mathbf{T})$ still has a *class* of different names.) We can now define the dD-semilattice operations on $A(\mathbf{T})$ in an entirely straightforward fashion:

$$\begin{array}{rcl} 0 & := & [F]'_{\mathbf{T}} \\ 1 & := & [T]'_{\mathbf{T}} \\ [\phi]'_{\mathbf{T}} \sqcap [\psi]'_{\mathbf{T}} & := & [\phi \wedge \psi]'_{\mathbf{T}} \\ \dot{\bigsqcup}_{i\in I}[\phi_i]'_{\mathbf{T}} & := & [\dot{\bigvee}_{i\in I}\phi_i]'_{\mathbf{T}} \end{array}$$

We should briefly reassure ourselves that the disjoint disjunction in the last clause can be formed for all disjoint subsets of $A(\mathbf{T})$; indeed:

$$\begin{array}{ll} & [\phi_i]'_{\mathbf{T}} \sqcap [\phi_{i'}]'_{\mathbf{T}} = 0 \\ \text{iff} & [\phi_i \wedge \phi_{i'}]'_{\mathbf{T}} = [F]'_{\mathbf{T}} \\ \text{iff} & \phi_i \wedge \phi_{i'} \vdash F \end{array}$$

Proposition 3.5 *$A(\mathbf{T})$ is a dD-semilattice. Furthermore, the associated order $\sqsubseteq$ satisfies $[\phi]'_{\mathbf{T}} \sqsubseteq [\psi]'_{\mathbf{T}}$ if and only if $\phi \vdash \psi$.*

Proof. The only interesting bit of the first statement, suprema for disjoint subsets, we showed already. To support the second statement, we give the following sequence of transformations:

$$
\begin{aligned}
& [\phi]'_{\mathbf{T}} \sqsubseteq [\psi]'_{\mathbf{T}} \\
\text{iff}\quad & [\phi]'_{\mathbf{T}} = [\phi]'_{\mathbf{T}} \sqcap [\psi]'_{\mathbf{T}} \\
\text{iff}\quad & [\phi]'_{\mathbf{T}} = [\phi \wedge \psi]'_{\mathbf{T}} \\
\text{iff}\quad & \phi \dashv\vdash \phi \wedge \psi \\
\text{iff}\quad & \phi \vdash \psi
\end{aligned}
$$

∎

Theorem 3.6 (Completeness) *The valid sequents in a disjunctive propositional theory $\mathbf{T}(P, S)$ are precisely those that are satisfied in every model of the disjunctive axiom system S.*

Proof. We define a structure $M \colon P \to A(\mathbf{T})$ by $M(p) := [p]'_{\mathbf{T}}$. This satisfies the axioms by construction, and therefore gives rise to a denotational function $[\![\cdot]\!]_M \colon \mathcal{L}(P, S) \to A(\mathbf{T})$. Using the explicit description of operations on $A(\mathbf{T})$ above, it is immediate that $[\![\phi]\!]_M = [\phi]'_{\mathbf{T}}$ holds for all formulas, not just the atomic ones. Now:

$$
\begin{aligned}
& \Gamma \vdash \phi \text{ is a valid sequent in } \mathbf{T} \\
\text{iff}\quad & \textstyle\bigwedge \Gamma \vdash \phi \text{ is a valid sequent in } \mathbf{T} \text{ by Proposition 2.2-(i)} \\
\text{iff}\quad & [\textstyle\bigwedge \Gamma]'_{\mathbf{T}} \sqsubseteq [\phi]'_{\mathbf{T}} \text{ as seen above} \\
\text{iff}\quad & [\![\textstyle\bigwedge_{\gamma \in \Gamma} \gamma]\!]_M \sqsubseteq [\![\phi]\!]_M \text{ as } [\![\cdot]\!]_M = [\cdot]'_{\mathbf{T}} \\
\text{iff}\quad & \textstyle\bigsqcap_{\gamma \in \Gamma} [\![\gamma]\!]_M \sqsubseteq [\![\phi]\!]_M \text{ by the definition of } [\![\cdot]\!]_M.
\end{aligned}
$$

From this we see that the valid sequents in $\mathbf{T}(P, S)$ are precisely those that are satisfied by M in $A(\mathbf{T})$. ∎

3.3 Categories of algebras

The disjunctive bases of Definition 2.1 can easily be turned into a category.

Definition 3.7 *Let* (P, S_0) *and* (P', S_0') *be disjunctive bases. A function* $f\colon P \to P'$ *is said to be* disjointness preserving *if for each sequent* $p_1, \ldots, p_n \vdash F$ *in* S_0*, the sequent* $f(p_1), \ldots, f(p_n) \vdash F$ *belongs to* S_0'*.*

We denote the category of disjunctive bases and disjointness preserving maps with $\mathbf{DB}_0$*.*

There is an obvious forgetful functor U from $\mathbf{dDSL}$ to $\mathbf{DB}_0$, which assigns to a dD-semilattice L the pair $(L, S_0(L))$, where $S_0(L)$ consists of all sequents $x_1, \ldots, x_n \vdash F$ for which $\prod_{i=1}^{n} x_i = 0$ holds in L. It assigns to a homomorphism $f\colon L \to L'$ the function f itself. What we have called a "structure" in Section 3.2, Definition 3.3, can now be rendered more conspicuously as a disjointness preserving map $M\colon (P, S_0) \to U(L)$.

For a functor in the opposite direction one can employ the construction of Section 3.2, that is, assign to a disjunctive basis (P, S_0) the dD-semilattice $A(\mathbf{T}_0)$ where $\mathbf{T}_0$ is the disjunctive propositional theory generated by S_0. Its action on morphisms derives from the following:

Proposition 3.8 *The assignment* $\eta_P\colon p \mapsto [p]'_{\mathbf{T}_0}$ *is a universal arrow from* (P, S_0) *to* U*.*

Proof. Let $f\colon (P, S_0) \to (L, S_0(L))$ be a disjointness preserving map. We need to show that f can be lifted to a $\mathbf{dDSL}$ homomorphism $\bar{f}$ from $A(\mathbf{T}_0)$ to L, such that $U(\bar{f}) \circ \eta_P = f$. As a diagram:

$$
\begin{tikzcd}
\mathbf{DB}_0 & (P, S_0) \arrow[r, "\eta_P"] \arrow[dr, "f"'] & (A(\mathbf{T}_0), S_0(A(\mathbf{T}_0))) \arrow[d, dotted, "U(\bar{f})"] & A(\mathbf{T}_0) \arrow[d, dotted, "\bar{f}"] & \mathbf{dDSL} \\
& & (L, S_0(L)) & L &
\end{tikzcd}
$$

We extend f to all disjunctive propositions by transfinite induction

$$f(\bigwedge_{i=1}^{n} \phi_i) := \bigsqcap_{i=1}^{n} f(\phi_i) \qquad\qquad f(\dot{\bigvee}_{i\in I} p_i) := \dot{\bigsqcup}_{i\in I} f(p_i)$$

(The disjoint supremum exists because f is assumed to preserve disjointness.) The extension translates interderivability to equality because of soundness (Theorem 3.4) and so the definition

$$\bar{f}([\phi]'_{\mathbf{T}_0}) := f(\phi)$$

is well-defined.

Since the elements of $A(\mathbf{T}_0)$ are generated by the atomic propositions in P, there is no other choice for a homomorphic lifting of f. ∎

We now invoke general category theory (e.g., [23, Theorem IV-2(ii)]) and obtain:

Theorem 3.9 *The forgetful functor $U\colon \mathbf{dDSL} \to \mathbf{DB}_0$ has a left adjoint F. It assigns to a disjunctive basis (P, S_0) the dD-semilattice $A(\mathbf{T}_0)$, with $\mathbf{T}_0 = \mathbf{T}(P, S_0)$, and to a disjointness preserving map $f\colon (P, S_0) \to (P', S'_0)$ the homomorphism $\overline{\eta_{P'} \circ f}$.*

An obvious question at this stage is to ask whether the forgetful functor is monadic. This property can be interpreted in more than one way, but for the situation at hand we prefer to paraphrase it as saying that **dDSL** is a "category of algebras" over $\mathbf{DB}_0$. Unfortunately, though, this is not the case; U appears to "forget" too much of the disjointness information that describes the domain of definition of disjoint suprema.

Example 3.10 *Consider the four element set $P = \{0, a, b, 1\}$ with the single disjointness assumption $0 \vdash F$. The effect of $U \circ F$ on this basis is the set $P' = \{[F] = [0], [a], [b], [1], [a \wedge b], [a \wedge 1], [b \wedge 1], [a \wedge b \wedge 1], [T]\}$ with disjointness assumption $[0] \vdash F$. Note that P' does not have any nontrivial terms involving $\dot{\vee}$.*

Now consider the Eilenberg-Moore algebra $\alpha\colon (P', S'_0) \to (P, S_0)$ *given by the following assignments:*

$$\begin{array}{rcl} [0],[a\wedge b],[a\wedge b\wedge 1] & \mapsto & 0 \\ [a],[a\wedge 1] & \mapsto & a \\ [b],[b\wedge 1] & \mapsto & b \\ [T],[1] & \mapsto & 1 \end{array}$$

It is isomorphic to the four-element lattice $0 \le a, b \le 1$ *but the existence of the supremum of* a *and* b *is coincidental and not given explicitly, since the term* $a \overset{\bullet}{\vee} b$ *does not belong to* $\mathcal{L}(P, S_0)$*. Indeed, it would be easy to construct an Eilenberg-Moore algebra that is isomorphic to the ordered set shown in Figure 1 with the least element removed, and thus provide an example that can not be rescued by extending the domain of definition of* $\overset{\bullet}{\vee}$ *to those subsets that happen to have a supremum in the quotient.*

We leave it as an open problem whether the situation can be improved by changing the concept of dD-semilattice (for example, by specifying the domain of definition of disjoint supremum explicitly), or by adjusting the definition of $\mathbf{DB}_0$.

On the other hand, the concept of dD-semilattice is clearly *essentially algebraic*, as the domain of definition of $\overset{\bullet}{\sqcup}$ is specified by an equation involving only $\sqcap$ and 0. In discussing the same issue in the case of preframes, Johnstone and Vickers, [19], point out that essentially algebraic theories can always be factored as a *tower of monadic adjunctions*. In our case, the tower has two stories: the adjunction between **Set** and meet semilattices, and the adjunction between the latter and **dDSL**.[2] The problem that arises with the Eilenberg-Moore algebra in our counterexample above can not arise in the adjunction between semilattices and **dDSL** because structure maps are now semilattice isomorphisms between X and the image of X in $U \circ F(X)$ under η_X.

This is a good moment to point out another open question that we have to leave unanswered in this report. It concerns the construction of free dD-semilattices itself, which we have carried out via term algebras. One may

[2] It is not quite so automatic as we pretend, as the arity of the disjoint supremum operation is unbounded, but this is only a minor irritation, also addressed in [19].

wonder whether it is possible to do this in a similar vein to Johnstone's use of "coverings" for the construction of free frames. One of the attractions would be that it follows the two stages of our tower of monadic adjunctions between **Set** and **dDSL**; another that it addresses in a direct way the problem of size caused by the unbounded arity of $\dot{\bigvee}$.

4 Presentations

A *presentation* is a description of a mathematical structure via *generators* and *relations*. For example, one can specify the group $\mathbb{Z}_3$ of residue classes of integers modulo 3 by a one-element set $\{a\}$ of generators and the one-element set $\{a \cdot a \cdot a = e\}$ of relations. The principle works equally well for general algebraic systems (i.e., sets with a system of operations of fixed arity), see [24] for example. The theory of frames is not subsumed by this, but there, too, "presentations always present," see [18, 27]. Here we study this question for dD-semilattices.

4.1 Categorical considerations

In case one has a monadic adjunction between **Set** and a category **Alg** of "algebras," one can argue that a presentation over a set X of generators amounts to a parallel pair of morphisms $Uf, Ug\colon UFUFX \to UFX$, where Uf is the multiplication of the associated monad $T = UF$ and g is the transpose of a function $g'\colon UFX \to UFX$ which "picks out representatives" among each equivalence class. The parallel pair becomes *contractible* (cf. [23, Exercise 2, Section VI-6]) because of the map η_{UFX} which goes in the opposite direction, i.e., from UFX to $UFUFX$. Beck's Theorem states that the monadicity of the adjunction implies that U creates a coequaliser for f and g. This coequaliser, then, is the algebra presented by the parallel pair.

The analysis above applies to varieties of algebraic systems, because they are monadic over **Set**. Given a presentation, the map g' can be defined using the Axiom of Choice.

We stated earlier that there is a tower of two monadic adjunctions linking **Set** and **dDSL**, with the category **SL** of meet semilattices acting as the intermediate category. Now, it is easy to see that a contractible pair in **SL** will give rise to a coequaliser in **dDSL** (by transporting it first down to **Set**, and then lifting it to **SL** and then **dDSL**), but we have not found a convincing argument why a contractible pair of meet semilattice homomorphisms should be a useful notion of presentation for dD-semilattices. In particular, it is not clear to us that a presentation via a disjunctive axiom system (Definition 2.10) can always be translated into such a contractible pair. We have to leave this issue as (yet another) open question.

A more mundane categorical treatment of presentations for algebraic systems can be given by using the fact that varieties have all coequalisers:

Diagram 4.1 *Consider the following diagram:*

$$
\begin{tikzcd}
\mathbf{Set} & & R \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & FR \arrow[d, shift right, "\bar{\pi}_1"'] \arrow[d, shift left, "\bar{\pi}_2"] & \mathbf{Alg} \\
& & UFX \arrow[dl, "U(e)"'] \arrow[dd, "U(\bar{f})"] & FX \arrow[dr, "e"] \arrow[dd, "\bar{f}"'] & \\
X \arrow[urr, "\eta"] \arrow[r, "\eta'"] \arrow[drr, "f"'] & UA \arrow[dr, dotted, "U(m)"] & & & A \arrow[dl, dotted, "m"] \\
& & UB & B &
\end{tikzcd}
$$

We read this as follows: R is a set of pairs of terms, that is, a subset of $(UFX)^2$, and π_1, π_2 are the projections restricted to this subset. We assume that $e\colon FX \to A$ is the coequaliser of the transposed maps $\bar{\pi}_1, \bar{\pi}_2$. The image under the forgetful functor U gives us the map $U(e)\colon UFX \to UA$, and we get $\eta'\colon X \to UA$ by composition with η.

We claim that η' is a universal arrow. So suppose that we are given a map f from X to the carrier set of an algebra B such that the transpose $\bar{f}$ coequalises $\bar{\pi}_1$ and $\bar{\pi}_2$. The rules for adjunctions tell us that then $U(\bar{f})$ coequalises π_1, π_2, so B can indeed be said to satisfy the given relations. Because e is assumed to be the coequaliser, we have a (unique) mediating homomorphism m in **Alg**, *and its image under U makes the two resulting triangles on the left commute.*

From a practical point of view, however, it appears that showing the existence of coequalisers in **Alg** is no simpler than showing that presentations present, and indeed, we do not know of a straightforward argument for the existence of coequalisers of dD-semilattices. The considerations above illustrate that coequalisers and presentations are closely related, and indeed, we will show below first that there is a meaningful notion of presentation for dD-semilattices and then prove as a corollary that **dDSL** has coequalisers.

4.2 Presenting dD-semilattices

At this stage, it will come as no surprise to the reader that a *presentation of a dD-semilattice* consists of a set P of generators and a disjunctive axiom system S over P (cf. Definition 2.10). Also, in Section 3.2 we have already shown considerable detail of the construction of the dD-semilattice $A(\mathbf{T})$ from such a presentation. From an algebraic point of view, $A(\mathbf{T})$ is a "universal solution" to the given presentation. This is made precise in the same way as we did for disjunctive bases in Section 3.3. We set up a category **DB** of presentations by defining morphisms from (P, S) to (P', S') as maps $h\colon P \to P'$ for which

$$(p_1, \ldots, p_n \vdash \dot{\bigvee_{i\in I}} \bigwedge_{j\in M_i} q_j) \in S$$

implies

$$(h(p_1), \ldots, h(p_n) \vdash \dot{\bigvee_{i\in I}} \bigwedge_{j\in M_i} h(q_j)) \in \mathbf{T}(P', S') \; .$$

For a forgetful functor U from **dDSL** to **DB** we assign to a dD-semilattice L the set of generators $|L| := \{\lceil x\rceil \mid x \in L\}$, and as axioms the set $S(L)$ which consists of the sequents[3]

$$\lceil x_1\rceil, \ldots, \lceil x_n\rceil \vdash \dot{\bigvee_{i\in I}} \lceil y_i\rceil$$

for which

$$x_1 \sqcap \ldots \sqcap x_n \sqsubseteq \dot{\bigsqcup_{i\in I}} y_i$$

[3]Note that the right hand side of these sequents does not involve finite infima of generators, as allowed in the general form of axioms in Definition 2.10. This is the result of there being sufficiently many generators to fix the meet semilattice structure by those sequents where the right hand side is a single generator. Cf. the discussion just before Section 3.

(In keeping with our convention throughout this paper, this specialises to $\lceil x_1 \rceil, \ldots, \lceil x_n \rceil \vdash F$ iff $x_1 \sqcap \ldots \sqcap x_n = 0$ when $I = \emptyset$.) The following is then shown in exactly the same way as Proposition 3.8:

Proposition 4.2 *The assignment $\eta_P: p \mapsto [p]'_{\mathbf{T}}$ is a universal arrow from (P, S) to U.*

As before, this proposition provides us with a left adjoint F to U. In the language of Section 3.2, we can express this adjunction as saying that for every dD-semilattice L there is a natural isomorphism between *models* of $\mathbf{T}$ in L, and *D-semilattice homomorphisms* from $A(\mathbf{T})$ to L.

The presentation theorem can now be expressed as follows:

Theorem 4.3 *The composition $F \circ U$ is naturally isomorphic to the identity functor on* **dDSL**.

Proof. We abbreviate $\mathbf{T}(|L|, S(L))$ by $\mathbf{T}$ throughout this proof. The plan is to show that the components of the counit $\epsilon: F \circ U \stackrel{\bullet}{\to} Id$ have inverses. To make this concrete, the result of applying ϵ_L to an element $[\phi]'_{\mathbf{T}}$ of $A(\mathbf{T})$ is $[\![\phi]\!]_E$, where $E: |L| \to L$ "strips off the quotes". For an inverse to ϵ_L consider $s_L: L \to A(\mathbf{T})$, given by $s_L(x) := [\lceil x \rceil]'_{\mathbf{T}}$. It will follow that this is a homomorphism of D-semilattices if we can show that it is an inverse to ϵ_L on the underlying sets, because the theory of dD-semilattices is essentially algebraic.[4]

With these definitions we compute

$$\epsilon_L \circ s_L(x) = \epsilon_L([\lceil x \rceil]'_{\mathbf{T}}) = [\![\lceil x \rceil]\!]_E = E(\lceil x \rceil) = x$$

and

$$s_L \circ \epsilon_L([\phi]'_{\mathbf{T}}) = s_L([\![\phi]\!]_E) = [\lceil [\![\phi]\!]_E \rceil]'_{\mathbf{T}}$$

and we are left with the task of showing $\lceil [\![\phi]\!]_E \rceil \dashv\vdash \phi$ in the theory $\mathbf{T}$. This we have to do by induction over the structure of ϕ.

- $\phi = F$: We have $F \vdash \lceil [\![F]\!]_E \rceil = \lceil 0 \rceil$ by Rule (LF), and $(\lceil 0 \rceil \vdash F) \in S(L)$ because $0 \sqsubseteq \dot{\bigsqcup} \emptyset = 0$.
- $\phi = T$: We have $\lceil [\![T]\!]_E \rceil = \lceil 1 \rceil \vdash T$ by (RT), and $(\vdash \lceil 1 \rceil) \in S(L)$ because $\bigsqcap \emptyset = 1 \sqsubseteq 1$, and so $(T \vdash \lceil 1 \rceil) \in \mathbf{T}$ by (Lwk).

[4]In other words, for general partial algebras this need not be true.

- $\phi = \lceil x \rceil$: We have $[\![\lceil x \rceil]\!]_E = E(\lceil x \rceil) = x$ and $\lceil x \rceil \dashv\vdash \lceil x \rceil$ by (Id).

- $\phi = \phi_1 \wedge \phi_2$: We can assume $\lceil [\![\phi_i]\!]_E \rceil \dashv\vdash \phi_i$, $i = 1, 2$, by induction hypothesis, and from this we get $\lceil [\![\phi_1]\!]_E \rceil \wedge \lceil [\![\phi_2]\!]_E \rceil \dashv\vdash \phi_1 \wedge \phi_2$ by an application of (L$\wedge$) and (R$\wedge$). It remains to show that $\lceil [\![\phi_1]\!]_E \rceil \wedge \lceil [\![\phi_2]\!]_E \rceil \dashv\vdash \lceil [\![\phi_1 \wedge \phi_2]\!]_E \rceil$. Well, since $[\![\phi_1 \wedge \phi_2]\!]_E = [\![\phi_1]\!]_E \sqcap [\![\phi_2]\!]_E \sqsubseteq [\![\phi_i]\!]_E$, $i = 1, 2$, we have that both $\lceil [\![\phi_1 \wedge \phi_2]\!]_E \rceil \vdash \lceil [\![\phi_1]\!]_E \rceil$ and $\lceil [\![\phi_1 \wedge \phi_2]\!]_E \rceil \vdash \lceil [\![\phi_2]\!]_E \rceil$ belong to $S(L)$; hence $\lceil [\![\phi_1 \wedge \phi_2]\!]_E \rceil \vdash \lceil [\![\phi_1]\!]_E \rceil \wedge \lceil [\![\phi_2]\!]_E \rceil$ belongs to $\mathbf{T}$ by (R$\wedge$). On the other hand, $\lceil [\![\phi_1]\!]_E \rceil \wedge \lceil [\![\phi_2]\!]_E \rceil \vdash \lceil [\![\phi_1 \wedge \phi_2]\!]_E \rceil$ belongs to $S(L)$ by definition.

- $\phi = \dot{\bigvee}_{i \in I} \phi_i$: We assume $\lceil [\![\phi_i]\!]_E \rceil \dashv\vdash \phi_i$, $i \in I$, by induction hypothesis, and we also have $\phi_i, \phi_{i'} \vdash F$ for all $i \neq i' \in I$ because ϕ is well-formed. Hence we also have $\lceil [\![\phi_i]\!]_E \rceil, \lceil [\![\phi_{i'}]\!]_E \rceil \vdash F$ and, using the rules (R$\dot{\vee}$) and (L$\dot{\vee}$), we get $\dot{\bigvee}_{i \in I} \lceil [\![\phi_i]\!]_E \rceil \dashv\vdash \dot{\bigvee}_{i \in I} \phi_i$. It remains to show that $\lceil [\![\dot{\bigvee}_{i \in I} \phi_i]\!]_E \rceil \dashv\vdash \dot{\bigvee}_{i \in I} \lceil [\![\phi_i]\!]_E \rceil$. Now, since $[\![\phi_i]\!]_E \sqsubseteq \dot{\bigsqcup}_{i \in I} [\![\phi_i]\!]_E = [\![\dot{\bigvee}_{i \in I} \phi_i]\!]_E$, $i \in I$, the sequents $\lceil [\![\phi_i]\!]_E \rceil \vdash \lceil [\![\dot{\bigvee}_{i \in I} \phi_i]\!]_E \rceil$, $i \in I$, belong to $S(L)$ by definition, and hence Rule (L$\dot{\vee}$) implies that $\dot{\bigvee}_{i \in I} \lceil [\![\phi_i]\!]_E \rceil \vdash \lceil [\![\dot{\bigvee}_{i \in I} \phi_i]\!]_E \rceil$ belongs to $\mathbf{T}$. On the other hand, $\lceil [\![\dot{\bigvee}_{i \in I} \phi_i]\!]_E \rceil \vdash \dot{\bigvee}_{i \in I} \lceil [\![\phi_i]\!]_E \rceil$ belongs to $S(L)$ already.

∎

Of course, there is always more than one presentation for a given dD-semilattice, so we can not expect the composition $U \circ F$ to be equivalent to Id as well.

Corollary 4.4 *The category* **dDSL** *of dD-semilattices has coequalisers.*

Proof. Given homomorphisms $f, g\colon L \to M$ we add to $S(M)$ the sequents $\lceil f(x) \rceil \vdash \lceil g(x) \rceil$ and $\lceil g(x) \rceil \vdash \lceil f(x) \rceil$ for all $x \in L$. In this way we obtain a disjunctive axiom system $S(M)^+$, and together with the generators $|M|$ we thus have a presentation of a dD-semilattice C. There is an obvious embedding of $(|M|, S(M))$ into $(|M|, S(M)^+)$, and its image under F is a homomorphism m from M to C. Given the concrete descriptions we computed for these constructions above, it is easy to see that m is indeed the coequaliser of f and g. ∎

We note that the proof of this statement suggests that there is also a universal "*in*equaliser" for a parallel pair f, g, which is obtained by adding only the sequents $\lceil f(x) \rceil \vdash \lceil g(x) \rceil$ to the theory of M.

5 Domains as theories

The goal of this section is to show how our framework for disjunctive propositional logic can be used to give a logical description of L-domains, analogous to Abramsky's celebrated *domain theory in logical form* for SFP-domains, [2]. A main ingredient for this application is the first author's *stable Stone duality* for L-domains, [8], which establishes a dual equivalence between certain distributive D-semilattices and algebraic L-domains. We begin by reviewing and generalising the main ingredients of this work.

5.1 L-domains and stable open sets

Our domain-theoretic terminology and notation follows [3], specifically, a *dcpo* is a poset in which every directed subset has a supremum. *Scott-continuous* maps between dcpos preserve these suprema. A subset of a dcpo is called *Scott-closed* if it is a lower set and closed under the formation of directed suprema. They are the closed sets of the *Scott topology*.

A dcpo D is called an *L-domain* if for every $x \in D$, $\downarrow x$ is a complete lattice.[5] An alternative definition can be given via *consistent subsets*, which are non-empty subsets that are bounded above.

Proposition 5.1 *A dcpo D is an L-domain if and only if every consistent subset has an infimum.*

[5]This terminology is somewhat in conflict with the convention in both [3] and [14] where the word "domain" is reserved for *continuous* dcpos, but "L-domain" is too deeply ingrained to make a change at this stage.

Proof. If D is an L-domain and A bounded by x then consider the infimum y of A in the complete lattice $\downarrow x$. Any lower bound of A also belongs to $\downarrow x$ because $A \neq \emptyset$, and hence must be below y. Conversely, all non-empty subsets of $\downarrow x$ are consistent and have a (global) infimum by assumption. Relative to $\downarrow x$ the (local) infimum of the empty set is x. This shows that $\downarrow x$ is a complete lattice. ∎

A dcpo D is called an *algebraic domain* if every element of D is the directed supremum of the compact elements below it. Algebraic L-domains were discovered independently by Coquand, [9], and the second author, [20]. In the latter work it is shown that they form one of two maximal cartesian subcategories of the category of pointed algebraic domains with Scott-continuous maps. See [3, Section 4] or [5, Section 5] for a discussion of this result. Our interest in the present paper, however, is the combination of L-domains with *stable* functions. This notion originated in the work of G. Berry on models of sequential programming languages [6]; for an up-to-date and comprehensive presentation of categories of stable functions see [5, Chapter 12].

Definition 5.2 *Let D, E be dcpos. A Scott-continuous function $f\colon D \to E$ is called* stable *if for all $x \in D$ and $y \leq f(x)$ there exists a least element $x' \leq x$ with $y \leq f(x')$.*

Proposition 5.3 *Let D, E be L-domains and f a Scott-continuous function from D to E. The following are equivalent:*

(i) f is stable;

(ii) f preserves infima of consistent sets.

Proof. (i) $\Longrightarrow$ (ii): If $A \leq x$ is consistent then so is $\{f(a) \mid a \in A\}$ because Scott-continuous maps are order-preserving. Set $y := \bigwedge_{a\in A} f(a)$ and let $x' \leq x$ be minimal with $y \leq f(x')$. Since all $a \in A$ are mapped above y, x' is a lower bound of A, from which we see that $f(\bigwedge A) \geq y = \bigwedge_{a\in A} f(a)$. The other inequality is automatic for order-preserving functions.

(ii) $\Longrightarrow$ (i): Assume $y \leq f(x)$. The set $A := \{x' \mid x' \leq x \text{ and } y \leq f(x')\}$ is consistent. We get $f(\bigwedge A) = \bigwedge_{a\in A} f(a) \geq y$, so $\bigwedge A$ is the desired minimal element below x. ∎

It is well known that Scott continuity of functions is a topological notion; a similar statement holds in the present context. Call a subset of an L-domain *stable*, if it is Scott-open and closed under infima of consistent subsets.

Proposition 5.4 *For a Scott-open subset O of an L-domain the following are equivalent:*

(i) O is stable;

(ii) Every element x of O is above a unique minimal compact element $\mu(x, O)$ of O;

(iii) O can be written as a disjoint union $\dot{\bigcup}_{i \in I} \uparrow \alpha_i$ with all α_i compact.

Proof. (i) $\Longrightarrow$ (ii): Let x be an element of the stable open set O. The set $\downarrow x \cap O$ is consistent and $\mu(x, O) := \bigwedge(\downarrow x \cap O)$ belongs to O. We must show that it is a compact element, so let $(x_i)_{i \in I}$ be a directed family with $\mu(x, O) \leq \bigvee^{\uparrow}_{i \in I} x_i$. As the supremum is in the open set O, for some i_0 we must have $x_{i_0} \in O$ already. Now consider the element $\mu(\bigvee^{\uparrow}_{i \in I} x_i, O)$; it must be below $\mu(x, O)$ but because of minimality this can only mean that the two are the same. It follows that x_{i_0} is above $\mu(x, O)$.

(ii) $\Longrightarrow$ (i): Let $A \leq x$ be a consistent subset of the Scott-open set O. The minimal element $\mu(x, O)$ below x is a lower bound for A in O.

The equivalence of (ii) with (iii) is immediate. ∎

With this characterisation it is clear that the points of an algebraic L-domain are separated by stable open sets, whereas for a general L-domain this need not be the case:

Example 5.5 *The L-domain D in Figure 2 has only one compact element, $\perp$, and consequently only two stable open sets, $\emptyset$ and D.*

Proposition 5.6 *Let D, E be L-domains and f a function from D to E. If f is stable, then for every stable open set O of E, $f^{-1}(O)$ is a stable open set of D. If E is algebraic then the converse is true, too.*

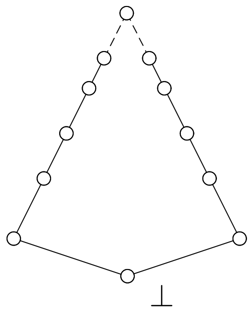

Figure 2: A non-algebraic L-domain.

Proof. For the first statement we know already from general domain theory that $f^{-1}(O)$ is Scott-open as stability subsumes Scott continuity. If $A \leq x$ is a consistent subset of $f^{-1}(O)$ then $f(\bigwedge A) = \bigwedge_{a \in A} f(a)$ and this point belongs to O as well. If follows that $\bigwedge A \in f^{-1}(O)$.

The second statement is shown in three nearly identical stages. We only give the first one which establishes that f is monotone. So let $x \leq y$ in D and α a compact element of E below $f(x)$. The open set $\uparrow\alpha$ is stable, so $f^{-1}(\uparrow\alpha)$ is stable, too, and contains x and hence y. Consequently, $\alpha \leq f(y)$. By forming the directed supremum of all compact elements below $f(x)$ we get $f(x) \leq f(y)$. ∎

5.2 Stable Stone duality

The appropriate structure for the stable Stone dual of an L-domain is suggested by the following observation:

Proposition 5.7 *The stable open sets of an L-domain D form a distributive disjunctive semilattice $\mathsf{sn}(D)$ when ordered by inclusion. Finite meets are given by intersection and disjoint suprema by disjoint union.*

Proof. The intersection of finitely many Scott-open sets is again Scott-open; stability is also preserved because it is given by a closure property. Next let $(O_i)_{i \in I}$ be a collection of pairwise disjoint stable open sets. A consistent subset $A \leq x$ of the union $O := \dot{\bigcup}_{i \in I} O_i$ must belong entirely to one component O_i because open sets are upper. It follows that $\bigwedge A$ is also contained in $O_i \subseteq O$.

Distributivity is inherited from the powerset. ∎

For a functional view of Stone duality we observe that stable open sets on an L-domain D are in one-to-one correspondence to stable functions from D to $\mathbb{S}$, the two-element L-domain $\bot < *$. This is an immediate consequence of Proposition 5.6 as $\{*\}$ is the only non-trivial stable open set of $\mathbb{S}$. We note that the order between stable functions into $\mathbb{S}$ which corresponds to inclusion between stable open sets is the pointwise one, not the "stable order" that is usually considered in studies of stability.

As usual, the functional view allows us to give a short definition of the contravariant functor sn from the category **LDom** of L-domains and stable functions to the category **DSL** of disjunctive semilattices and D-semilattice homomorphisms. The action on a stable map $f\colon D \to D'$ is given by $\chi \mapsto \chi \circ f$ for $\chi\colon D' \to \mathbb{S}$. Alternatively, we can write $\mathsf{sn}(f)(O) := f^{-1}(O)$ and it is this form from which one sees most easily that $\mathsf{sn}(f)$ is a D-semilattice homomorphism.

In order to recover the points from a D-semilattice we make the following definition:

Definition 5.8 *Let L be a disjunctive semilattice. A* disjunctive completely prime filter *F of L is a subset that is closed under finite meets and is inaccessible by disjoint suprema. For brevity we often use* (abstract) point *instead of disjunctive completely prime filter. The set of all abstract points is denoted by $\mathsf{pt}(L)$. We view it as an ordered set where the order relation is given by inclusion between the filters.*

We note that the empty set is deemed to be disjoint in our framework, so a disjunctive completely prime filter can not contain the least element 0 of the D-semilattice. Conversely, the greatest element 1 is always a member.

Proposition 5.9 *Disjunctive completely prime filters on a D-semilattice are in one-to-one correspondence to D-semilattice homomorphisms from L to 2, the two-element D-semilattice $0 < 1$.*

Classically, there is a third representation of abstract points, namely, by meet-prime elements of the lattice, cf. [3, Section 7.1.3]. This is not available in the disjunctive setting because the set $L \setminus F$ need not be disjoint, and hence may not have a supremum. On the other hand, the following proposition is stronger than what is available in standard Stone duality.

Proposition 5.10 *For any D-semilattice L the ordered set $(\mathsf{pt}(L), \subseteq)$ is an L-domain.*

Proof. Obviously, and analogous to the classical situation, $\mathsf{pt}(L)$ is a dcpo. For the L-domain condition we employ the characterisation given in Proposition 5.1. So let $(F_i)_{i\in I}$ be a non-empty collection of disjunctive completely prime filters contained in another such filter F. The intersection $G := \bigcap_{i\in I} F_i$ is clearly the greatest lower bound for the F_i provided we can show that it satisfies the conditions for an abstract point. Certainly, G is a filtered upper set, so let A be a disjoint set of elements with $\dot{\bigsqcup} A \in G$. For every $i \in I$ there is then at least one element $a_i \in A \cap F_i$ but in actual fact, there is precisely one element a of A that is contained in *all* the F_i. Indeed, any a_i belongs to the enclosing filter F, and if $a_i \neq a_j$ then $a_i \sqcap a_j = 0 \in F$, too, contradicting primality. ∎

In order to obtain a functor from **DSL** to **LDom** (the category of L-domains and stable functions) we define $\mathsf{pt}(f)\colon \mathsf{pt}(L') \to \mathsf{pt}(L)$ as $\mathsf{pt}(F)(\chi) := \chi \circ f$ for any D-semilattice homomorphism $\chi\colon L' \to 2$. It is the equivalent definition $\mathsf{pt}(f)(F) := f^{-1}(F)$ in terms of disjunctive prime filters, though, which makes it apparent that $\mathsf{pt}(f)$ is a stable function.

To summarise:

Theorem 5.11 *The assignments sn and pt form a dual adjunction between the categories* **LDom** *and* **DSL**.

It is somewhat unorthodox to view the set of abstract points as an ordered set rather than as a topological space. We chose this approach because stable open sets do not necessarily form a topology. Still, it is worthwhile to explore the behaviour of the usual definition of the spectrum, which employs the following sets for arbitrary elements x of the D-semilattice L:

$$O_x := \{F \in \mathsf{pt}(L) \mid x \in F\}$$

Proposition 5.12 *Every O_x is a stable open set of $\mathsf{pt}(L)$. Furthermore, for every $F \in \mathsf{pt}(L)$ and every stable open set O containing F, there exists $x \in L$ such that $\mu(F, O_x) = \mu(F, O)$.*

Proof. The first statement is established with two chains of equivalences:

$$\bigvee_{i\in I}^{\uparrow} F_i = \bigcup_{i\in I} F_i \in O_x \iff x \in \bigcup_{i\in I} F_i \iff \exists i_0 \in I.\, x \in F_{i_0} \iff \exists i_0 \in I.\, F_{i_0} \in O_x$$

and for $(F_i)_{i\in I} \subseteq F$ a non-empty consistent set of abstract points

$$\forall i \in I.\, F_i \in O_x \iff \forall i \in I.\, x \in F_i \iff x \in \bigcap_{i\in I} F_i = \bigwedge_{i\in I} F_i \iff \bigwedge_{i\in I} F_i \in O_x$$

For the second, assume F belongs to the stable open set O. By Proposition 5.4, $\mu(F,O)$ is a compact element G of $\mathsf{pt}(L)$. Consider the stable open sets O_x, $x \in G$; they form a downward directed family and consequently, the collection $F_x := \mu(F, O_x)$ is upward directed in $\mathsf{pt}(L)$. Note that $F_x \subseteq G$ as $G \in O_x$, and also that $x \in F_x$. Therefore the union of the F_x is a filter that equals G. By compactness, for some $x \in G$, $\mu(F, O_x) = F_x = G = \mu(F, O)$. ∎

In general, it is not the case that a disjunctive semilattice has names for *all* the stable open sets of $\mathsf{pt}(L)$, though:

Example 5.13 *Let L be the frame of open sets of C, the Cantor set. We show that every disjunctive completely prime filter F is already completely prime in the usual sense. To this end let $O := \bigcup_{i\in I} O_i \in F$. O is the disjoint union of clopen sets, so some clopen U belongs to F already. Because closed implies compact, U is covered by finitely many O_i already. Now, each O_i is the disjoint union of clopen sets and therefore U is covered by finitely many of these. Although the covering collection $V_0, \ldots, V_n$ may not be disjoint, they can easily be replaced by a disjoint collection by setting $V_0' := V_0$, $V_k' := V_k \setminus (V_0' \cup \ldots \cup V_{k-1}')$. One of the V_k' must belong to F and hence the same is true about the corresponding O_i from the original collection.*

Thus the abstract points of L are exactly the elements of Cantor space C. The order between these is trivial and therefore every *subset of C is a stable open set.*

As a consequence of the adjunction between **LDom** and **DSL**, for every L-domain D there is a stable function into the second dual, given concretely by

$$\eta(x) := \{O \in \mathsf{sn}(D) \mid x \in O\}$$

In general, it need not be injective as Example 5.5 illustrates (where the second dual is the one-point L-domain). On the other hand, and in contrast to the classical situation, [17, 15], surjectivity always holds.

Theorem 5.14 *For any L-domain D the canonical embedding $\eta\colon D \to \mathsf{pt}(\mathsf{sn}(D))$ is surjective. If D is algebraic then it is an order isomorphism.*

Proof. Characterisation 5.4(iii) tells us that every stable open set of D is of the form $\dot{\bigcup}_{i\in I} \uparrow\alpha_i$ where the α_i's are compact elements. Consequently, O belongs to a disjunctive completely prime filter F if and only if $\uparrow\alpha$ belongs to F for some compact $\alpha \in O$. Furthermore, the sets of this shape form a downward directed collection in F, which means that the corresponding α's are directed in D. Let x be their supremum, and it is immediate that $F = \{O \in \mathsf{sn}(D) \mid x \in O\} = \eta(x)$.

Injectivity in the algebraic case is trivial as there the stable open sets $\uparrow\alpha$, α compact in D, separate the points. ∎

It remains to characterise the stable Stone duals of algebraic L-domains.

Definition 5.15 *An element c of a D-semilattice L is called* disjunctively completely coprime *(or simply* coprime*) if $c \sqsubseteq \dot{\bigsqcup} A$ always implies $c \sqsubseteq a$ for some element a of the disjoint subset A of L. We denote with $\mathsf{cop}(L)$ the set of these elements.*

We call L coprime generated *if every element x is the disjoint supremum of coprimes; we call it* stable *if in addition the top element 1 is coprime.*

Proposition 5.16 *Let L be a coprime generated D-semilattice.*

(i) L is distributive (in the sense of Definition 3.2).

(ii) For each element x of L there is a unique set A of coprimes such that $x = \dot{\bigsqcup} A$.

The proof of this is entirely straightforward and should not distract us from stating the main result of this section.

Theorem 5.17 *The dual adjunction between* **LDom** *and* **DSL** *restricts to a dual equivalence between* *(i) algebraic L-domains and coprime generated D-semilattices;*

(ii) algebraic L-domains with least element and stable D-semilattices.

Proof. For every compact element α of an L-domain D the stable open set $\uparrow\alpha$ is disjunctively completely coprime in $\mathsf{sn}(D)$, so it is clear that the stable Stone dual of an algebraic L-domain has enough coprimes. For the converse we observe that $\uparrow c$ is an abstract point whenever c is disjunctively completely coprime in a D-semilattice L. Furthermore, it is compact in the L-domain $\mathsf{pt}(L)$. If L is coprime generated, then for every disjunctive completely prime filter F, the set of coprimes $F_c := \mathsf{cop}(L) \cap F$ is downward directed with $F = \uparrow F_c$. In other words, $F = \bigvee^{\uparrow}_{c \in F_c} \uparrow c$ holds in the L-domain $\mathsf{pt}(L)$, and this is sufficient for establishing algebraicity. ∎

5.3 Logical description of L-domains

Our general presentation theorem 4.3 for distributive D-semilattices allows us immediately to define a disjunctive propositional logic that characterises a given algebraic L-domain. However, the stable Stone duals of algebraic L-domains are coprime generated and this suggests that a more compact representation should be possible. So let us reconsider the construction of Section 4.2 for the coprime-generated situation. There, the forgetful functor $U\colon \mathbf{dDSL} \to \mathbf{DB}$ created an atomic proposition for every element of the lattice; now we will try to make do with coprime elements alone. We set $|L|_c := \{\lceil x \rceil \mid x \in \mathsf{cop}(L)\}$ and let the axioms be

$$\lceil x_1 \rceil, \ldots, \lceil x_n \rceil \vdash \dot{\bigvee_{i \in I}} \lceil y_i \rceil \quad \text{iff} \quad x_1 \sqcap \ldots \sqcap x_n \sqsubseteq \dot{\bigsqcup_{i \in I}} y_i$$

as before but restricted to coprime elements. Denote the set of these with $S_c(L)$ and the derived theory $\mathbf{T}(|L|_c, S_c(L))$ with $\mathbf{T}_c(L)$.

Our goal is to show that from $\mathbf{T}_c(L)$ the semilattice can be reconstructed by computing the canonical model $A(\mathbf{T}_c(L))$ (see Section 3.2). For this we adjust the proof of Theorem 4.3. The inverse $s_L\colon L \to A(\mathbf{T}_c)$ to ϵ_L is now given by $x \mapsto [\dot{\bigvee}_{i \in I} \lceil x_i \rceil]'_{\mathbf{T}}$ where $\{x_i \mid i \in I\}$ is the unique set of coprime elements such that $x = \dot{\bigsqcup}_{i \in I} x_i$. The rest of the argument is changed as follows:

$$\epsilon_L \circ s_L(x) = \epsilon_L([\dot{\bigvee_{i\in I}} \lceil x_i \rceil]'_{\mathbf{T}_c}) = [\![\dot{\bigvee_{i\in I}} \lceil x_i \rceil]\!]_E = \dot{\bigsqcup_{i\in I}} [\![\lceil x_i \rceil]\!]_E = \dot{\bigsqcup_{i\in I}} x_i = x$$

Composition the other way round yields:

$$s_L \circ \epsilon_L([\phi]'_{\mathbf{T}_c}) = s_L([\![\phi]\!]_E) = [\dot{\bigvee_{i\in I}} \lceil x_i \rceil]'_{\mathbf{T}_c} \quad \text{where} \quad [\![\phi]\!]_E = \dot{\bigsqcup_{i\in I}} x_i$$

Our revised task is to show that $\dot{\bigvee}_{i\in I} \lceil x_i \rceil \dashv\vdash \phi$ in the theory $\mathbf{T}_c(L)$. As before, we do so by induction over the structure of ϕ.

- $\phi = F$: We have $[\![F]\!]_E = 0 = \dot{\bigsqcup} \emptyset$ and therefore $s_L([\![F]\!]_E) = \dot{\bigvee}_{i\in\emptyset} \lceil x_i \rceil = F$ and the statement becomes trivial.

- $\phi = T$: We have $[\![T]\!]_E = 1$ and assume $a = \dot{\bigsqcup}_{i\in I} x_i$. By definition, $\vdash \dot{\bigvee}_{i\in I} \lceil x_i \rceil$ belongs to the axioms from which $T \vdash \dot{\bigvee}_{i\in I} \lceil x_i \rceil$ follows by rule Lwk. The reverse implication follows in the same way from RT.

- $\phi = \lceil x \rceil$: Now applied only to elements of $\mathsf{cop}(L)$, the argument remains the same.

- $\phi = \phi_1 \wedge \phi_2$: By induction hypothesis, we can assume $[\![\phi_1]\!]_E = \dot{\bigsqcup}_{i\in I} x_i$, $[\![\phi_2]\!]_E = \dot{\bigsqcup}_{j\in J} y_j$, and $\dot{\bigvee}_{i\in I} \lceil x_i \rceil \dashv\vdash \phi_1$, $\dot{\bigvee}_{j\in J} \lceil y_j \rceil \dashv\vdash \phi_2$. Now, $[\![\phi]\!]_E = [\![\phi_1 \wedge \phi_2]\!]_E = [\![\phi_1]\!]_E \sqcap [\![\phi_2]\!]_E = (\dot{\bigsqcup}_{i\in I} x_i) \sqcap (\dot{\bigsqcup}_{j\in J} y_j) = \dot{\bigsqcup}_{i\in I, j\in J} (x_i \sqcap y_j)$ and each element $x_i \sqcap y_j$ can be written in a unique way as $\dot{\bigsqcup}_{k\in K_{ij}} z_k$ with all z_k coprime. This last equality is coded in the axioms because only coprime elements are mentioned: $\lceil x_i \rceil, \lceil y_j \rceil \vdash \dot{\bigvee}_{k\in K_{ij}} \lceil z_k \rceil$, $\lceil z_k \rceil \vdash \lceil x_i \rceil$, and $\lceil z_k \rceil \vdash \lceil y_j \rceil$. Using the rules of the proof system we can derive

$$\begin{array}{rcl} \phi & = & \phi_1 \wedge \phi_2 \\ \text{(induction hypothesis)} & \dashv\vdash & (\dot{\bigvee}_{i\in I}\lceil x_i\rceil) \wedge (\dot{\bigvee}_{j\in J}\lceil y_j\rceil) \\ \text{(Proposition 2.4)} & \dashv\vdash & \dot{\bigvee}_{i\in I, j\in J}(\lceil x_i\rceil \wedge \lceil y_j\rceil) \\ \text{(axioms in } S_c(L)) & \dashv\vdash & \dot{\bigvee}_{i\in I, j\in J}\dot{\bigvee}_{k\in K_{ij}}\lceil z_k\rceil \\ \text{(Propositions 2.5 and 2.6)} & \dashv\vdash & \dot{\bigvee}_{k\in\bigcup_{i\in I,j\in J}K_{ij}}\lceil z_k\rceil \end{array}$$

as required.

- $\phi = \dot{\bigvee}_{k\in K}\phi_k$: We assume $[\![\phi_k]\!]_E = \dot{\bigsqcup}_{i\in I_k} x_i^k$ and $\phi_k \dashv\vdash \dot{\bigvee}_{i\in I_k}\lceil x_i^k\rceil$ for all $k \in K$ by induction hypothesis. Now we note that for $k \neq k'$ we necessarily have $x_i^k \sqcap x_j^{k'} = 0$ because the well-formedness of ϕ requires that $\phi_k, \phi_{k'} \vdash F$ holds, and soundness translates this to $[\![\phi_k]\!]_E \sqcap [\![\phi_{k'}]\!]_E = 0$. For the theory $\mathbf{T}_c(L)$ this means that $\lceil x_i^k\rceil, \lceil x_j^{k'}\rceil \vdash F$ is an axiom. Together this says that we can apply propositions 2.5 and 2.6 about the associativity of disjoint disjunctions and without further ado we obtain

$$\phi = \dot{\bigvee_{k\in K}}\phi_k \dashv\vdash \dot{\bigvee_{k\in K}}\dot{\bigvee_{i\in I_k}}\lceil x_i^k\rceil \dashv\vdash \dot{\bigvee_{k\in K, i\in I_k}}\lceil x_i^k\rceil$$

∎

Thus we have shown:

Theorem 5.18 *For L a coprime generated D-semilattice, there is a D-semilattice isomorphism between L and $A(\mathbf{T}_c(L))$.*

If the semilattice is given as the set of stable open sets of an algebraic L-domain D, as in Theorem 5.17, then the disjunctive basis for $\mathbf{T}_c(L)$ can be derived directly from the domain: The generators $|L|_c$ are in one to one correspondence with the compact elements of D and the axioms are defined as

$$\lceil\alpha_1\rceil, \ldots, \lceil\alpha_n\rceil \vdash \dot{\bigvee_{i\in I}}\lceil\beta_i\rceil \quad \text{iff} \quad \uparrow\alpha_1 \cap \ldots \cap \uparrow\alpha_n \subseteq \dot{\bigcup_{i\in I}}\uparrow\beta_i$$

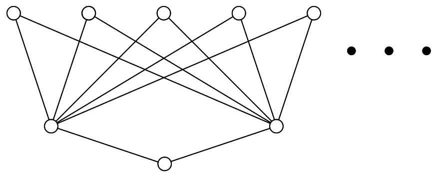

Figure 3: An L-domain that is not SFP.

In this formulation it becomes clear that it is not possible to restrict disjoint disjunctions to finite index sets (as is the case in Abramsky's logic for SFP-domains), as there are examples of L-domains where a finite set of compact elements has infinitely many minimal upper bounds. We depict the simplest example in Figure 3.

We conclude by assembling the various equivalences together for a logical representation theorem.

Theorem 5.19 *Let D be an algebraic L-domain. Then D is isomorphic to the set of models of the disjunctive propositional theory $\mathbf{T}_c(\mathsf{sn}(D))$ in the two-element D-semilattice* 2.

Proof. Remember that a "model" is just a mapping $M\colon |\mathsf{sn}(D)|_c \to 2$ of the atomic propositions into the target D-semilattice, which validates all axioms in $S_c(\mathsf{sn}(D))$. Soundness (Theorem 3.4) says that models are in one-to-one correspondence to models of the whole of the generated theory, $\mathbf{T}_c(\mathsf{sn}(D))$. Because of the adjunction between disjunctive bases and dD-semilattices, Proposition 4.2, such assignments are in one-to-one correspondence to D-semilattice homomorphisms from $A(\mathbf{T}_c(\mathsf{sn}(D)))$ to 2. We have just shown above that the construction $A \circ \mathbf{T}_c$ returns an isomorphic copy for every coprime generated D-semilattice, so we are down to D-semilattice homomorphisms from $\mathsf{sn}(D)$ to 2. In stable Stone duality, then, such maps define disjunctive completely prime filters — the abstract points of $\mathsf{sn}(D)$, and so we arrive at $\mathsf{pt}(\mathsf{sn}(D))$, which, according to Theorem 5.17 is isomorphic to the algebraic L-domain D that we started with. ∎

5.4 Towards continuity

From a semantics point of view, the construction of our logic is guided by the collection of stable open sets of algebraic L-domains. An alternative approach, presented by Th. Ehrhard and P. Malacaria in [11], focuses instead on what we would like to call *cm-open sets*; these are those Scott-open sets which are closed under infima of *finite* consistent subsets. A characterisation analogous to Proposition 5.4 is possible: cm-open sets are exactly the disjoint unions of Scott-open filters. In the following we discuss the relationship between the two frameworks, which — following [5] — we label as "stable" and "conditionally multiplicative (cm)", respectively.

Ehrhard and Malacaria identify the Stone duals of L-domains in the cm sense as "S-structures," which in addition to 0,1, $\sqcap$, and $\dot{\bigsqcup}$ also admit the formation of directed joins. In fact, they show that $\dot{\bigsqcup}$ and $\bigsqcup^{\uparrow}$ can be subsumed under one operation, the supremum of "disjoint directed (dd) subsets" where the latter is defined as those non-empty subsets which contain upper bounds for all pairs x, y for which $x \sqcap y \neq 0$. With a suitable notion of prime filter on S-structures they establish a Stone-type representation theorem.

With regards to morphisms, cm-open sets are naturally associated with conditionally multiplicative functions, i.e. those Scott-continuous maps that preserve meets of *finite* consistent subsets. On continuous L-domains cm functions are characterised by preserving cm-open sets, analogous to Proposition 5.6. The representation theorem can thus be extended to a duality between cm maps and S-structure homomorphisms.

We make the following observations. On dI-domains[6] there is no difference between stable and cm-open sets. Consequently, a Scott-continuous function between dI-domains is stable if and only if it is conditionally multiplicative, and therefore the two approaches coincide.

For the larger category of algebraic L-domains the two approaches are different (in the sense that cm maps and stable maps are not the same) but the Stone representation theorems are still closely related. Semantically, we know that every cm-open set is the directed join of stable open sets. The corresponding Stone duals are coprime generated S-structures and coprime generated D-semilattices, respectively, and one would expect there to be a left adjoint to the forgetful functor from the former to the latter. This is indeed possible: For a give coprime generated D-semilattice L one considers the collection of *dd-ideals* (which are lower dd-sets) in the poset $\mathsf{cop}(L)$. Adapting the proof of [11, Proposition 1] we get:

[6]Distributive algebraic Scott-domains for which the subposet of compact elements satisfies the descending chain condition.

Proposition 5.20 *For L a coprime generated D-semilattice, the set of dd-ideals of $\mathsf{cop}(L)$ together with the empty set form a coprime generated S-structure.*

Proof. The smallest element is obviously given by the empty set and the largest by $\mathsf{cop}(L)$ itself.

Meet is given by intersection, and we must show that this is disjoint directed. So assume we are given dd-ideals A, B and $a, b \in A \cap B$ with $a \sqcap b \neq 0$. There are upper bounds $c \in A$ and $c' \in B$ and the infimum $c \sqcap c'$ is an upper bound in L. By coprime generation we have $c \sqcap c' = \dot{\bigsqcup}_{i \in I} c_i$ with all c_i coprime. Since a and b are coprime themselves, they are covered by some c_i, $c_{i'}$, respectively, but because the collection $\{c_i \mid i \in I\}$ is disjoint, we must in fact have $c_i = c_{i'}$ and this is the desired upper bound in $A \cap B$.

Next consider a dd-set $(A_i)_{i \in I}$ of dd-ideals. We claim that the union is again disjoint directed. For this let $a \in A_i$, $b \in A_j$ be such that $a \sqcap b \neq 0$ in L. By prime generation there are then coprime elements below that meet and we find that $A_i \cap A_j \neq \emptyset$. The collection itself being disjoint directed we conclude that there is an upper bound A_k of which both a and b are a member; so they have an upper bound there.

The coprime dd-ideals are those of the form $\downarrow a$ for $a \in \mathsf{cop}(L)$. ∎

For a universal arrow we assign to an element $x \in L$ the dd-ideal generated by the set A of coprime elements for which $x = \dot{\bigsqcup} A$. Proposition 5.16(ii) assures us that this is well defined. A D-semilattice map f from L to an S-structure M is then extended to dd-ideals in the obvious way:

$$\bar{f}(A) := \bigsqcup_{a \in A} f(a)$$

The supremum is over a dd-set because f preserves 0 and meets, and so $f(a) \sqcap f(a') \neq 0$ implies $a \sqcap a' \neq 0$. The extension preserves 0 and 1 because because f preserves them. For the preservation of meets we calculate $\bar{f}(A \cap B) = \bigsqcup\{f(c) \mid c \in A \cap B\} = \bigsqcup\{f(c) \mid c \in \mathsf{cop}(L) \text{ and } \exists a \in A, b \in B.\ c \leq a \sqcap b\} = \bigsqcup\{f(a) \sqcap f(b) \mid a \in A, b \in B\} = \bigsqcup_{a \in A} f(a) \sqcap \bigsqcup_{b \in B} f(b)$. The preservation of suprema of dd-subsets, finally, is shown using the general associativity of supremum.

If we extend the setting even further to include all *continuous* L-domains, then the notion of stable open set becomes too sparse to be of any use. Indeed, the unit interval $([0,1], \leq)$, for example, has only the two trivial stable open sets $\emptyset$ and $[0,1]$. However, there are still plenty of cm-open sets to validate Ehrhard's and Malacaria's representation theorem. For a logical description of continuous L-domains, though, one would have to develop a syntax for S-structures. This would have to capture the definition of a disjoint directed set, which would necessitate keeping track of when two formulas do *not* entail *false*.

Alternatively, we would like to ask whether the disjunctive propositional logic of the present paper can be extended in such a way that all continuous L-domains are covered. The hope that such a programme could be successful is founded on [21]; there a finitary propositional logic is given that captures all (coherent) continuous domains. The necessary adjustment to Abramsky's *domain theory in logical form* was to drop the identity axiom $\phi \vdash \phi$.

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