On the Duality of Compact vs. Open

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Abstract

It is a pleasant fact that Stone-duality may be described very smoothly when restricted to the category of compact spectral spaces: The Stone-duals of these spaces, arithmetic algebraic lattices, may be replaced by their sublattices of compact elements thus discarding infinitary operations.

We present a similar approach to describe the Stone-duals of coherent spaces, thus dropping the requirement of having a base of compact-opens (or, alternatively, replacing algebraicity of the lattices by continuity). The construction via *strong proximity lattices* is resembling the classical case, just replacing the order by an *order of approximation.*

Our development enlightens the fact that "open" and "compact" are dual concepts which merely happen to coincide in the classical case.

Introduction

Pointless topology relies on the duality between the category of sober spaces with continuous functions and the category of spatial frames with frame-homomorphisms. The origin of this subject, however, dates back to the classical Representation Theorem by Marshall Stone [Sto36], which establishes a duality between compact zero-dimensional Hausdorff spaces and Boolean algebras. So the original theory deals just with *finitary* operations on the algebraic side, whereas one has to pay a price for the broader generality of frames: *infinitary* operations are involved.

The key property of Stone spaces which makes it possible to restrict to finitary operations is the fact that their lattices of open sets are *algebraic.* More

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precisely, they are exactly the lattices arising as ideal completions of Boolean algebras. Having realized this, one can immediately generalize Stone's Theorem to distributive lattices, yielding spectral spaces as their duals (see e.g. [Joh82, II-3]). But this is the end of the road: Ideal completions of posets are always algebraic domains, and hence the spaces which are described in this fashion will always have a base of compact-open sets.

The way out of this cul-de-sac is via the theory of *R-structures* or *abstract bases* as introduced in [Smy77] and developed in [AJ94]. These sets are 'ordered' by an *order of approximation* which need not be reflexive. Ideal completion of these structures lead to continuous domains, and, in our setting, to continuous lattices.

Note the shift of emphasis as we pass to the continuous setting: While Stone was concerned with a representation of Boolean algebras in topological spaces, we are more interested in the spaces and seek to *describe* them via a finitary algebraic structure. It is our goal, then, to make this description as faithful as possible.

Similar attempts already exist in the literature, we mention [GK81] and [Smy92] where one can find pointers to yet earlier work. Our approach differs from theirs in that we require stronger axioms; so strong in fact, that the lattice of open sets no longer qualifies. It is a particularly striking result of our work that there is nevertheless a representation theorem for all coherent spaces. The representation takes both open and compact subsets into account. We feel that this sheds fresh light on the foundations of pointless topology.

Quite a number of equivalences between certain topological spaces and certain complete lattices have been established and it is therefore no surprise that there is no commonly accepted naming convention for the concepts involved. In what follows, we will essentially use the terminology of [AJ94]. It is also our main reference for background information.

1 Coherent spaces

We begin with the following corollary to the Hofmann-Mislove Theorem (see [HM81, KP94]):

Theorem 1 *Let X be a sober space.*

1. *If K is compact and if $(O_i)_{i\in I}$ is a directed family of open sets such that $K \subseteq \bigcup_{i\in I} O_i$ then K is contained in some O_i already.*
2. *If O is open and if $(K_i)_{i\in I}$ is a filtered family of compact saturated sets such that $\bigcap_{i\in I} K_i \subseteq O$ then some K_i is contained in O already.*

Except for the necessity to restrict to *saturated* compact subsets, this theorem highlights a fundamental duality between open and compact sets. (Recall that a subset of a topological space is called *saturated,* if it coincides with the intersection of all its neighborhoods.) A more familiar formulation would employ arbitrary families rather than directed ones. But this does not work as the (even finite) intersection of compact saturated sets is not necessarily compact again. If we require this property then we have indeed a complete lattice of compact saturated sets and the following formulation of Theorem 1 is true:

Theorem 2 *Let X be a sober space for which finite intersections of compact saturated subsets are compact.*

1. *Every open cover of a compact set contains a finite subcover.*
2. *Whenever the intersection of compact saturated sets is contained in an open set then the same is true for an intersection of finitely many of them.*

More connections between opens and compacts appear if we also require local compactness. We arrive at what we choose to call *coherent spaces*, our principal objects of interest.

Definition 3 *A topological space X is called* coherent *if it is sober, locally compact and if intersections of finite families of compact saturated sets are compact.*

We denote the set of open sets of X by $\mathcal{T}_X$ and order it by inclusion. The set of compact saturated sets is denoted by $\mathcal{K}_X$ and is ordered by reversed inclusion.

Theorem 4 *Let X be a coherent space.*

1. *$\mathcal{T}_X$ and $\mathcal{K}_X$ are arithmetic lattices.*
2. *In $\mathcal{T}_X$ we have $O \ll O'$ if and only if there is $K \in \mathcal{K}_X$ with $O \subseteq K \subseteq O'$.*
3. *In $\mathcal{K}_X$ we have $K \ll K'$ if and only if there is $O \in \mathcal{T}_X$ with $K' \subseteq O \subseteq K$.*

The terminology *arithmetic lattice* is adopted from [AJ94] (it was used in [GHK$^+$80] for the algebraic version). It comprises completeness, distributivity, continuity, and the following multiplicativity property:

$$x \ll y, z \Longrightarrow x \ll y \wedge z \ .$$

We will also assume $1 \ll 1$ for convenience. We cite the following converse to Theorem 4 from [GHK$^+$80]:

Theorem 5 *Arithmetic lattices are spatial and their spaces of points are coherent.*

Our theory of *strong proximity lattices* will connect up with coherent spaces via arithmetic lattices. In particular, we will not need to use the Axiom of Choice for the correspondence. It is present, of course, through the duality of coherent spaces and arithmetic lattices.

The basic topological concepts on the two sides of Stone duality take the following forms:

(spatial) frame	(sober) space
completely prime filter	element
Scott-open filter	compact saturated set
element	open set

We may point out that on the side of frames the duality between opens and compacts is rather opaque. Proximity lattices, as we will see below, behave far better in this respect.

2 Proximity lattices

Definition 6 *A* proximity lattice *is given by a distributive bounded lattice* $(B; \vee, \wedge, 0, 1)$ *together with a binary relation* $\prec$ *on* B *satisfying* $\prec^2 = \prec$. *We call* $(B; \vee, \wedge, 0, 1)$ *the* algebraic structure *of the proximity lattice and* $(B; \prec)$ *the* approximation structure. *The two structures are connected through the following two axioms:*

($\vee$-$\prec$) $\forall a \in B\ \forall M \subseteq_{\mathsf{fin}} B.\ M \prec a \iff \bigvee M \prec a$;

($\prec$-$\wedge$) $\forall a \in B\ \forall M \subseteq_{\mathsf{fin}} B.\ a \prec M \iff a \prec \bigwedge M$;

Here we write $a \prec M$ *for* $\forall m \in M.\ a \prec m$ *and* $M \prec a$ *for* $\forall m \in M.\ m \prec a$. *If* $x \prec y$, *we also say that* x approximates y.

There are various sets of axioms in the literature featuring under the name of proximity lattices. The particular choice we made is very close to the ones found in [GK81, Smy86, Smy92]. Unlike these previous accounts, we do not require the order of approximation $\prec$ to be contained in the order $\leq$ derived from the lattice structure. In fact, we rather think of $\prec$ *replacing* $\leq$. We do not employ the lattice order in our arguments. This is reflected by the fact that the notations $\uparrow$ and $\downarrow$ refer to $\prec$ rather than to $\leq$: For subsets A of B, we define $\uparrow A = \{x \in X \mid \exists a \in A.\ a \prec x\}$ and $\downarrow A = \{x \in X \mid \exists a \in A.\ x \prec a\}$. Moreover, $\uparrow x$ and $\downarrow x$ stand for $\uparrow\{x\}$ and $\downarrow\{x\}$, respectively.

Note that our definition is self-dual, i.e. $B^{op} = (B; \wedge, \vee, 1, 0; \succ)$ is a proximity lattice if B is. Let us start with stating some simple consequences of the axioms enlightening the interplay of algebraic and approximation structure.

Lemma 7 *If* $(B; \vee, \wedge, 0, 1; \prec)$ *is a proximity lattice and* $a, b, a', b' \in B$, *then:*

1. $0 \prec a \prec 1$,

2. $a \prec b \Longrightarrow a \prec b \vee b'$,

3. $a \prec b \Longrightarrow a \wedge a' \prec b$,

4. $a \prec a'$ & $b \prec b' \Longrightarrow a \vee b \prec a' \vee b'$,

5. $a \prec a'$ & $b \prec b' \Longrightarrow a \wedge b \prec a' \wedge b'$.

Proof. Put $M = \emptyset$ in ($\vee$-$\prec$) and ($\prec$-$\wedge$) to get (1). The second assertion holds by ($\prec$-$\wedge$) and the fact that $b = b \wedge (b \vee b')$. If we assume $a \prec a'$ and $b \prec b'$, then (2) gives us $\{a, b\} \prec a' \vee b'$. Now ($\vee$-$\prec$) yields (4). Finally, (3) and (5) follow by symmetry. ■

3 Ideals and open sets

Definition 8 *Suppose $(B; \vee, \wedge, 0, 1; \prec)$ is a proximity lattice. We define the set of all* ideals *on B by*

$$\mathsf{Idl}(B) = \{I \subseteq B \mid I = {\downarrow} I,\ M \subseteq_{\mathsf{fin}} I \Longrightarrow \bigvee M \in I\}\ .$$

Observe that the condition $I = {\downarrow} I$ implies that for each $x \in I$ there is $y \in I$ with $x \prec y$. Since $\prec$ is not necessarily reflexive this is a non-trivial condition. Some authors emphasize this by using the term 'round ideal'.

Lemma 9 *Let B be a proximity lattice and $I \in \mathsf{Idl}(B)$. Then for all $a, b \in B$:*

1. $a \vee b \in I \iff (a \in I$ & $b \in I)$,

2. $a \in I \Longrightarrow a \wedge b \in I$.

Proof. To prove (1), assume $a \vee b \in I$. Then there is $x \in I$ with $a \vee b \prec x$ since $I \subseteq {\downarrow} I$. Now ($\vee$-$\prec$) implies $a \prec x$ and $b \prec x$. Thus $\{a, b\} \subseteq I$ since $x \in I$ and ${\downarrow} I \subseteq I$. The reverse implication holds by definition. By $a = a \vee (a \wedge b)$, (2) is a trivial consequence of (1). ■

Let us now have a closer look at the set of ideals.

Lemma 10 *Suppose $(B; \vee, \wedge, 0, 1; \prec)$ is a proximity lattice. Then ${\downarrow} x$ is an ideal for each $x \in B$. Moreover $I = \bigcup_{x \in I} {\downarrow} x$ for every $I \in \mathsf{Idl}(B)$, and this union is directed.*

Proof. The first claim is immediate from the axioms. Clearly, ${\downarrow} x \subseteq {\downarrow} I \subseteq I$ for each $x \in I$, hence $\bigcup_{x \in I} {\downarrow} x \subseteq I$. In order to see that this union is directed suppose $x, y \in I$. Then $x \vee y \in I$. Since $I \subseteq {\downarrow} I$, this implies the existence of $z \in I$ with $x \vee y \prec z$, hence $x \prec z$ and $y \prec z$. Therefore we have ${\downarrow} x \cup {\downarrow} y \subseteq {\downarrow} z$, i.e. directedness. Finally, choosing $x = y$ in this proof gives for each $x \in I$ a $z \in I$ with $x \in {\downarrow} z$, thus $I \subseteq \bigcup_{x \in I} {\downarrow} x$. ■

Theorem 11 *Suppose that* $(B; \vee, \wedge, 0, 1; \prec)$ *is a proximity lattice. Then* $(\mathsf{Idl}(B), \subseteq)$ *is an arithmetic lattice. Finite infima are intersections, general infima are given by*

$$\bigwedge_{j \in J} I_j = \downarrow \bigcap_{j \in J} I_j \ .$$

Directed suprema are unions, general suprema are calculated by

$$\bigvee_{j \in J} I_j = \downarrow \{ \bigvee M \mid M \subseteq_{\mathsf{fin}} \bigcup_{j \in J} I_j \} \ .$$

The order of approximation is given by $I \ll J \iff \exists x \in B.\, I \subseteq \downarrow x \subseteq J$.

Proof. By Lemma 7(4), $\downarrow C$ is closed under suprema if $C \subseteq B$ is. Hence $\downarrow \bigcap_{j \in J} I_j$ is an ideal, clearly being the greatest lower bound of $\{I_j \mid j \in J\}$. It is a standard observation that the directed union of a family of ideals is an ideal, hence directed suprema are indeed just unions. To see that finite infima are intersections, it suffices to prove $A \cap A' = \downarrow (A \cap A')$. One always has $\downarrow (A \cap A') \subseteq \downarrow A \cap \downarrow A' = A \cap A'$. For the other inclusion observe that if $x \in A \cap A'$ then there exist $a \in A, a' \in A'$ with $x \prec a, a'$, so $x \prec a \wedge a' \in A \cap A'$.

By Lemma 10, we have $I = \bigvee^{\uparrow} \{ \downarrow x \mid x \in I\}$. This implies that $I \ll J$ iff there is some $x \in B$ with $I \subseteq \downarrow x \subseteq J$ and moreover that the lattice is continuous. Multiplicativity follows from the characterization of the approximation order and Axiom ($\prec$-$\wedge$).

Next we verify the correctness of the second formula. To this end let $(I_j)_{j \in J}$ be a collection of ideals and denote $\downarrow \{ \bigvee M \mid M \subseteq_{\mathsf{fin}} \bigcup_{j \in J} I_j \}$ by A. We show that this defines an ideal. It is clear that $\downarrow A = A$ because $\prec^2 = \prec$. For the closure under suprema let $N \subseteq_{\mathsf{fin}} A$. By definition, each $n \in N$ approximates the supremum of some $M_n \subseteq_{\mathsf{fin}} \bigcup_{j \in J} I_j$. By Lemma 7(4), $\bigvee N \prec \bigvee \bigcup_{n \in N} M_n$ and so $\bigvee N \in A$.

Each I_j is contained in $\bigcup_{j \in J} I_j$ and as $\downarrow$ is monotone we get $I_j = \downarrow I_j \subseteq \downarrow \bigcup_{j \in J} I_j$. The last set is contained in A because the definition includes suprema of singleton sets. So A is an upper bound for the I_j.

If, on the other hand, A' is an ideal which contains all I_j then it must also contain suprema of finite subsets of $\bigcup_{j \in J} I_j$. Hence it will contain A.

Finally distributivity. We have to show $A \wedge (C \vee C') \subseteq (A \wedge C) \vee (A \wedge C')$. So assume that $x \in B$ belongs to $A \wedge (C \vee C') = A \cap (C \vee C')$. Then $x \in A$ and $x \prec \bigvee M$ for some $M \subseteq_{\mathsf{fin}} C \cup C'$. Because $A \subseteq \downarrow A$ there exists $a \in A$ with $x \prec a$. Now we can use ($\prec$-$\wedge$) and we get $x \prec a \wedge \bigvee M = \bigvee_{m \in M} (a \wedge m)$. All elements $a \wedge m$ belong to either $A \cap C = A \wedge C$ or $A \cap C' = A \wedge C'$. Hence x is in $(A \wedge C) \vee (A \wedge C')$. ∎

From Theorem 5 it follows that proximity lattices are indeed a finitary description of coherent spaces:

Corollary 12 *Suppose that* $(B; \vee, \wedge, 0, 1; \prec)$ *is a proximity lattice. Then* $\mathsf{pt}(\mathsf{Idl}(B))$ *is a coherent space and* $\mathsf{Idl}(B)$ *is isomorphic to its lattice of open subsets.*

4 Filters and compact saturated subsets

Filters in a proximity lattice are defined dually to ideals. They are denoted by $\mathsf{filt}(B)$. Because our definition of proximity lattice is self-dual the results of the preceding section hold also for filters and the filter completion.

We want to show that filters correspond to compact saturated subsets of the coherent space described. In preparation for this we look at a more general correspondence.

Definition 13 *Let $(B; \vee, \wedge, 0, 1; \prec)$ be a proximity lattice. We call a subset U of B* upper *if $U = \uparrow U$ holds. The collection of all upper sets is denoted by $\mathsf{upper}(B)$.*

Lemma 14 *Let $(B; \vee, \wedge, 0, 1; \prec)$ be a proximity lattice. Then $\sigma_{\mathsf{Idl}(B)}$ (the Scott-topology on $\mathsf{Idl}(B)$) is isomorphic to the set of upper sets of B. The isomorphisms are*

$$\phi\colon \sigma_{\mathsf{Idl}(B)} \to \mathsf{upper}(B), \quad \mathcal{U} \mapsto \{x \in B \mid \downarrow x \in \mathcal{U}\}$$

and

$$\psi\colon \mathsf{upper}(B) \to \sigma_{\mathsf{Idl}(B)}, \quad U \mapsto \{A \in \mathsf{Idl}(B) \mid A \cap U \neq \emptyset\} .$$

Proof. We first show that ϕ is well-defined. Clearly, $\phi(\mathcal{U})$ is upwards closed with respect to $\prec$. It equals $\uparrow\phi(\mathcal{U})$ because $\downarrow x \in \mathcal{U}$ and $\downarrow x = \bigvee^{\uparrow}_{y \prec x} \downarrow y$ imply $\downarrow y \in \mathcal{U}$ for some $y \prec x$ as $\mathcal{U}$ is Scott-open.

Next, let us check the well-definedness of ψ. Again, it is clear that $\psi(U)$ is an upwards closed subset of $\mathsf{Idl}(B)$. We show that $\psi(U)$ is Scott-open. Assume $\bigvee^{\uparrow}_{j \in J} A_j = \bigcup_{j \in J} A_j \in \psi(U)$. This means $\bigcup_{j \in J} A_j \cap U \neq \emptyset$ and so there is some A_j for which the intersection with U is non-empty. This ideal then belongs to $\psi(U)$.

It is clear that both ϕ and ψ are monotone.

The two functions compose to identities. We first check this for $\psi \circ \phi$:

$$\begin{aligned}
\psi \circ \phi(\mathcal{U}) &= \{A \mid A \cap \phi(\mathcal{U}) \neq \emptyset\} \\
&= \{A \mid A \cap \{x \mid \downarrow x \in \mathcal{U}\} \neq \emptyset\} \\
&= \{A \mid \exists x \in A. \downarrow x \in \mathcal{U}\}
\end{aligned}$$

The last set is equal to $\mathcal{U}$. One inclusion is true because $\mathcal{U}$ is upwards closed, the other because $A = \bigvee^{\uparrow}_{x \in A} \downarrow x$ and $\mathcal{U}$ is Scott-open.

The calculation for $\phi \circ \psi$ reads

$$\begin{aligned}
\phi \circ \psi(U) &= \{x \mid \downarrow x \in \psi(U)\} \\
&= \{x \mid \downarrow x \in \{A \mid A \cap U \neq \emptyset\}\} \\
&= \{x \mid \downarrow x \cap U \neq \emptyset\}
\end{aligned}$$

The last set is equal to U. For one inclusion use that $\uparrow U \subseteq U$, for the other that $U \subseteq \uparrow U$. This completes the proof. ■

Let us now turn to compact saturated sets. By the Hofmann-Mislove Theorem (cf. [AJ94], Theorem 7.2.9, and also [KP94, HM81]) we know that $\mathcal{K}_X$ is isomorphic to the set of Scott-open filters on the lattice of opens. We relate these to filters on proximity lattices as follows:

Lemma 15 *Let* $(B; \vee, \wedge, 0, 1; \prec)$ *be a proximity lattice. The isomorphisms in Lemma 14 cut down to isomorphisms between the set of Scott-open filters on* $\mathsf{Idl}(B)$ *and* $\mathsf{filt}(B)$.

Proof. All we need to show is the well-definedness of ϕ and ψ. First of all, both functions preserve the property that the set they are applied to is non-empty, simply because they are isomorphisms and map the empty set onto the empty set. In the case of ϕ we use the fact that $\downarrow\!(x \wedge y) = \downarrow\! x \cap \downarrow\! y$ by ($\prec$-$\wedge$). Hence $x \wedge y$ belongs to $\phi(\mathcal{F})$ if x and y do. Finally, let us show that $\psi(F)$ is a filter. Assume $A, A' \in \psi(F)$. This is by definition equivalent to $A \cap F \neq \emptyset$ and $A' \cap F \neq \emptyset$. Let a and a' be elements from the intersections. By Lemma 9(2), it follows that $a \wedge a'$ belongs to $(A \cap A') \cap F$. ■

Theorem 16 *If* $(B; \vee, \wedge, 0, 1; \prec)$ *is a proximity lattice then* $(\mathcal{K}_{\mathsf{pt}(\mathsf{Idl}(B))}, \supseteq) \cong (\mathsf{filt}(B), \subseteq)$.

5 Strong proximity lattices

The results achieved so far are quite pleasing. The duality between compact and open sets is apparent. There are two issues, however, in which we fall short of a good finitary description of coherent spaces. The first is that we haven't yet said how to recover the points themselves from a proximity lattice. Indeed, a rather complicated definition is needed as can be seen from [Smy92]. The second shortcoming is more subtle. We ask what the tokens of a proximity lattice stand for. One would think that an element $a \in B$ represents the open set $\downarrow\! a$ and, dually, also the compact saturated set $\uparrow\! a$. This, however, conflicts with the algebraic structure of the proximity lattice, unless we add further axioms.

Proposition 17 *Suppose* $(B; \vee, \wedge, 0, 1; \prec)$ *is a proximity lattice. Then the map* $\downarrow : B \to \mathsf{Idl}(B)$ *preserves finite infima and* 0. *It preserves binary suprema (and hence is a lattice homomorphism) iff* B *satisfies*

($\prec$-$\vee$) $\forall a, x, y \in B.\ a \prec x \vee y \Longrightarrow \exists x', y' \in B.\ x' \prec x,\ y' \prec y \ \&\ a \prec x' \vee y'$.

Proof. Preservation of 0 and 1 is immediate. Moreover $\downarrow\!(a \wedge b) = \downarrow\! a \cap \downarrow\! b$ by ($\prec$-$\wedge$). By Theorem 11, the latter equals $\downarrow\! a \wedge \downarrow\! b$.

By the formula for the supremum in Theorem 11, we have $x \in \downarrow\! a \vee \downarrow\! b$ iff there are $a' \prec a$, $b' \prec b$ with $a \prec a' \vee b'$. Requiring $\downarrow$ to preserve $\vee$ means requiring that this holds iff $x \in \downarrow\!(a \vee b)$, i.e. iff $x \prec a \vee b$. This condition is exactly ($\prec$-$\vee$). ■

It appears that we should add ($\prec$-$\vee$) to the list of our axioms. But an analogous result holds for filters and the map $\uparrow\colon B \to \mathsf{filt}(B)$. So we should also include

($\wedge$-$\prec$) $\forall a, x, y \in B.\ x \wedge y \prec a \Longrightarrow \exists x', y' \in B.\ x \prec x',\ y \prec y'\ \&\ x' \wedge y' \prec a.$

We arrive at our main definition:

Definition 18 *A* strong proximity lattice *is a proximity lattice additionally satisfying* ($\prec$-$\vee$) *and its dual* ($\wedge$-$\prec$)*.*

From Proposition 17 we see that the tokens of a strong proximity lattice are very concrete in the sense that each of them stands for a particular open set and also for a particular compact saturated set. The algebraic operations in the lattice translate into actual union and intersection under this reading. Furthermore, the order of approximation corresponds to the way-below relation on the respective lattices. As a by-product, we can now also recover the points of the coherent space in a simple fashion:

Definition 19 *Suppose* $(B; \vee, \wedge, 0, 1; \prec)$ *is a strong proximity lattice. Let the* spectrum *of B comprise all* prime filters *of B:*

$$\mathsf{spec}(B) = \{F \in \mathsf{filt}(B) \mid (M \subseteq_{\mathsf{fin}} B\ \&\ \bigvee M \in F) \Longrightarrow M \cap F \neq \emptyset\}.$$

For $x \in B$ define the basic open set

$$\mathcal{O}_x = \{F \in \mathsf{spec}(B) \mid x \in F\}.$$

Finally, let $\mathcal{T}_B$ denote the topology on $\mathsf{spec}(B)$ *generated by the sets $\mathcal{O}_x$, $x \in B$. We refer to it as the* canonical topology*.*

Lemma 20 *Suppose* $(B; \vee, \wedge, 0, 1; \prec)$ *is a strong proximity lattice and $a, b \in B$. Then $\mathcal{O}_a \cap \mathcal{O}_b = \mathcal{O}_{a\wedge b}$ and $\mathcal{O}_a \cup \mathcal{O}_b = \mathcal{O}_{a\vee b}$. Hence the $\mathcal{O}_a$ form indeed a base for the canonical topology on* $\mathsf{spec}(B)$ *(rather than merely a subbase).*

Proof. The first assertion holds by the dual of Lemma 9(1), the second by primeness of elements of $\mathsf{spec}(B)$. ■

Theorem 21 *Let* $(B; \vee, \wedge, 0, 1; \prec)$ *be a strong proximity lattice. The isomorphism in Lemma 14 cuts down to a homeomorphism between* $\mathsf{pt}(\mathsf{Idl}(B))$ *and* $\mathsf{spec}(B)$*.*

Proof. For bijectivity it remains to show that ϕ and ψ preserve finitary primeness of filters. To this end let $M \subseteq_{\mathsf{fin}} B$ and $\bigvee M \in \phi(\mathcal{F})$. We use the fact that $\downarrow$ preserves finite suprema and get $\bigvee_{m\in M} {\downarrow} m = {\downarrow}\bigvee M \in \mathcal{F}$. Since $\mathcal{F}$ is prime, some ${\downarrow} m_0$ belongs to it and its generator m_0 belongs to $\phi(\mathcal{F})$.

Assuming that F is a prime filter in B, we show that $\psi(F)$ is completely prime. Assume $\bigvee_{j\in J} A_j \in \psi(F)$. This means $\bigvee_{j\in J} A_j \cap F \neq \emptyset$ and so there is $M \subseteq_{\mathsf{fin}} \bigcup_{j\in J} A_j$ and $a \in F$ with $a \prec \bigvee M$. Since F is upwards closed it also contains $\bigvee M$ and since it is prime it contains some $m_0 \in M$. This m_0 came from some A_{j_0} which therefore belongs to $\psi(F)$.

Recall that the topology on $\mathsf{pt}(\mathsf{Idl}(B))$ is given by the collection of all $\mathcal{O}_A = \{\mathcal{F} \in \mathsf{pt}(\mathsf{Idl}(B)) \mid A \in \mathcal{F}\}$, $A \in \mathsf{Idl}(B)$. We check that it translates into the canonical topology on $\mathsf{spec}(B)$ under ψ^{-1}:

$$\begin{aligned}
\psi^{-1}(\mathcal{O}_A) &= \psi^{-1}(\{\mathcal{F} \in \mathsf{pt}(\mathsf{Idl}(B)) \mid A \in \mathcal{F}\}) \\
&= \{F \in \mathsf{spec}(B) \mid A \in \psi(F)\} \\
&= \{F \in \mathsf{spec}(B) \mid A \cap F \neq \emptyset\} \\
&= \bigcup_{a\in A} \{F \in \mathsf{spec}(B) \mid a \in F\} \;=\; \bigcup_{a\in A} \mathcal{O}_a \,.
\end{aligned}$$

The translation for ϕ^{-1} reads:

$$\begin{aligned}
\phi^{-1}(\mathcal{O}_a) &= \phi^{-1}(\{F \in \mathsf{spec}(B) \mid a \in F\}) \\
&= \{\mathcal{F} \in \mathsf{pt}(\mathsf{Idl}(B)) \mid {\downarrow} a \in \mathcal{F}\} \;=\; \mathcal{O}_{\downarrow a} \,.
\end{aligned}$$ ■

We summarize the situation in the following table which should be contrasted to the one at the end of Section 1.

strong proximity lattice	coherent space
prime filter	element
ideal	open set
filter	compact saturated set

6 The representation theorem

So far, we did only one half of the work due: all strong proximity lattices are descriptions of certain coherent spaces. What is missing is a construction which assigns to an arbitrary coherent space X a strong proximity lattice B with $\mathsf{spec}(B) \cong X$. In the classical case of spectral spaces one takes the lattice of all compact-open subsets of X. There is no base of compact-opens in our situation, so this does not work. The naive approach of taking for B (a base of) the topology $\mathcal{T}_X$ of X together with the lattice operations on $\mathcal{T}_X$ and $\ll$ as order of approximation (i.e. $O \prec O'$ iff there is some $K \in \mathcal{K}_X$ with $O \subseteq K \subseteq O'$) does not work either; Axiom ($\wedge$-$\prec$) is violated.

Example 22 *The unit interval with the standard Hausdorff topology is a coherent space. One has* $[0,\frac{1}{2}) \cap (\frac{1}{2},1] = \emptyset \ll \emptyset$*, but for no open sets* O*,* O' *with* $[0,\frac{1}{2}) \ll O$ *and* $(\frac{1}{2},1] \ll O'$*, the intersection* $O \cap O'$ *is empty.*

This failure is not surprising; the classical construction relies on *both* properties of compact-open sets, openness and compactness. If we want to capture both, we have to consider pairs (O, K) consisting of an open set O and a compact set K. What ought to be the order of approximation, replacing subset inclusion in the classical case? Now $O \subseteq O'$ for compact-opens means in fact that O' is a compact neighborhood of O. These thoughts lead to the idea that one should define (O, K) to approximate (O', K') if $K \subseteq O'$. These definitions do indeed work. We define

- $B := \{(O, K) \in \mathcal{T}_X \times \mathcal{K}_X \mid O \subseteq K\}$
- $(O, K) \vee (O', K') := (O \cup O', K \cup K')$
- $(O, K) \wedge (O', K') := (O \cap O', K \cap K')$
- $0 := (\emptyset, \emptyset)$; $1 := (X, X)$
- $(O, K) \prec (O', K') :\iff K \subseteq O'$

Theorem 23 *If* X *is a coherent space, then the above defined structure is a strong proximity lattice with* $X \cong \mathsf{spec}(B)$*.*

Proof. It is clear that $(B; \vee, \wedge, 0, 1)$ is a distributive lattice and that $\prec \circ \prec \subseteq \prec$. On the other hand, local compactness of X implies that whenever $K \subseteq O$ for compact K and open O, there are $O' \in \mathcal{T}_X$ and $K' \in \mathcal{K}_X$ with $K \subseteq O' \subseteq K' \subseteq O$, hence $\prec$ is interpolating.

Now for the axioms: ($\vee$-$\prec$) and ($\prec$-$\wedge$) hold trivially since all operations and relations involved are set-theoretic.

For ($\prec$-$\vee$) assume that $K \subseteq O_1 \cup O_2$. Each $x \in K$ belongs to either O_1 or O_2. In each case we have O_x and K_x with $x \in O_x \subseteq K_x \subseteq O_{1/2}$ by local compactness. Since K is compact, it is covered by finitely many O_x which may be grouped as belonging to O_1 or O_2. This gives the interpolating neighborhoods.

For ($\wedge$-$\prec$) assume $K_1 \cap K_2 \subseteq O$ for an open set O and compact saturated sets K_1, K_2. Since $\mathcal{K}_X$ is arithmetic, we have $K_i = \bigcap\{K'_i \mid K'_i \ll K_i\}$, $i = 1, 2$, and hence $K_1 \cap K_2 = \bigcap\{K'_1 \cap K'_2 \mid K'_1 \ll K_1, K'_2 \ll K_2\}$. This is a filtered intersection of compact saturated sets and Theorem 1 tells us that some $K'_1 \cap K'_2$ is contained in O already.

Finally, we have to verify $X \cong \mathsf{spec}(B)$. As both spaces are sober, it suffices to prove their topologies isomorphic. To this end, define $\Psi \colon (\mathcal{T}_X, \subseteq) \to \mathsf{Idl}(B, \subseteq)$ and $\Phi \colon \mathsf{Idl}(B, \subseteq) \to (\mathcal{T}_X, \subseteq)$ by

$$\Psi(O) = {\downarrow}(O, X) \qquad \text{and} \qquad \Phi(I) = \bigcup\{O \mid \exists K \in \mathcal{K}_X.\ (O, K) \in I\}.$$

It is clear that these maps are well-defined and monotone. It remains to verify them being inverses: $\Phi(\Psi(O)) = \bigcup\{O' \mid \exists K \in \mathcal{K}_X.\ O' \subseteq K \subseteq O\} = O$ holds by local compactness, hence $\Phi \circ \Psi = \mathsf{id}_{\mathcal{T}_X}$. Moreover, we have

$$\begin{aligned}(O,K) \in \Psi(\Phi(I)) &\iff (O,K) \prec (\Phi(I), X)\\ &\iff K \subseteq \bigcup\{O' \in \mathcal{T}_X \mid \exists K' \in \mathcal{K}_X.\ (O',K') \in I\}\\ &\iff K \subseteq O' \text{ for some } O' \text{ with } (O',K') \in I\\ &\iff (O,K) \in I,\end{aligned}$$

where the last equivalence holds since $I = {\downarrow}I$ and the second last by compactness of K and I's being an ideal. Thus $\Psi \circ \Phi = \mathsf{id}_{\mathsf{Idl}(B)}$. ■

Theorem 24 *The spectra of strong proximity lattices are precisely the coherent spaces.*

In case the order of approximation on the strong proximity lattice is reflexive, we find ourselves in the world of algebraic lattices and totally disconnected spaces. The Stone duality of this situation has been described in [Joh82, II-3]. Note the strong resemblance between our constructions and the classical situation. Indeed, if we take the lattice-order as order of approximation, then all axioms are satisfied automatically and the situation reduces to the classical one.

7 Morphisms

We have already emphasized that it is not our purpose to represent proximity lattices in the category of topological spaces but rather the other way round, we developed strong proximity lattices to get a finitary and faithful representation of coherent spaces. It is therefore our task to describe arbitrary continuous functions between coherent spaces. It is well-known that this can not be done with functions between proximity lattices but that one has to resort to certain relations.

Definition 25 *Let* $(A; \vee, \wedge, 0, 1; \prec_A)$ *and* $(B; \vee, \wedge, 0, 1; \prec_B)$ *be strong proximity lattices. A binary relation* $G \subseteq A \times B$ *is called* approximable *if the following conditions are satisfied:*

(G-$\prec$) $G \circ \prec_B = G$;

($\prec$-G) $\prec_A \circ G = G$;

($\vee$-G) $\forall M \subseteq_{\mathsf{fin}} A\ \forall b \in B.\ M\ G\ b \iff \bigvee M\ G\ b$;

(G-$\wedge$) $\forall a \in A\ \forall M \subseteq_{\mathsf{fin}} B.\ a\ G\ M \iff a\ G \bigwedge M$;

(G-$\vee$) $\forall a \in A\, \forall M \subseteq_{\mathsf{fin}} B.\, a\, G \bigvee M \Longrightarrow \exists N \subseteq_{\mathsf{fin}} A.\, a \prec_A \bigvee N \;\&\; \forall n \in N$
$\exists m \in M.\, n\, G\, m.$

We write $G\colon A \to B$ for an approximable relation from A to B. Composition of approximable relations is via the relational product $\circ$.

Note the strong resemblance between these axioms and the definition of strong proximity lattices themselves. (We could have also included the empty supremum and the empty infimum in the axioms ($\prec$-$\vee$) and ($\wedge$-$\prec$), respectively: These cases trivially hold.) Indeed, the order of approximation on a strong proximity lattice gives rise to the identity approximable relation. We also observe that the definition does not include an analogue of ($\wedge$-$\prec$), the dual of (G-$\vee$). This is due to the fact that the concept of a continuous function is inherently non-symmetric.

Theorem 26 *The category of strong proximity lattices and approximable relations is equivalent to the category of coherent spaces and continuous functions.*

Proof. One first has to check that approximable relations indeed give rise to a category. As we have said already, the orders of approximation themselves play the role of identity morphisms. The laws for a category are then straightforward to verify.

We show the one-to-one correspondence between approximable relations and continuous functions by once again making use of the already established duality with the category of arithmetic lattices and frame homomorphisms.

If $G \subseteq A \times B$ is an approximable relation then we let

$$h_G\colon \mathsf{Idl}(B) \to \mathsf{Idl}(A), \quad h_G(I) := \{a \in A \mid \exists b \in I.\, a\, G\, b\}$$

be the corresponding frame homomorphism. This result of applying h_G gives an ideal because of ($\prec$-G) and ($\vee$-G). It is clear that h_G is monotone. Simple calculations show that it preserves finite infima and arbitrary suprema.

For the converse, assume that $h\colon \mathsf{Idl}(B) \to \mathsf{Idl}(A)$ is a frame homomorphism. We define a relation $G_h \subseteq A \times B$ as follows:

$$a\, G_h\, b :\Longleftrightarrow\; a \in h(\downarrow\!b)\ .$$

Again, it is straightforward that this relation satisfies the axioms for an approximable relation.

The two translations are inverses of each other:

$$\begin{array}{rcl} h_{G_h}(I) & = & \{a \in A \mid \exists b \in I.\, a\, G_h\, b\} \\ & = & \{a \in A \mid \exists b \in I.\, a \in h(\downarrow\!b)\} \\ & = & \bigcup_{b \in I} h(\downarrow\!b) \quad = \quad \bigvee_{b \in I} h(\downarrow\!b) \\ & = & h(\bigvee_{b \in I} \downarrow\!b) \quad = \quad h(I) \end{array}$$

and

$$\begin{aligned} a\,G_{h_G}\,b &\iff a \in h_G(\mathord{\downarrow} b) = \{x \in A \mid \exists b' \prec_B b.\; x\,G\,b'\} \\ &\iff \exists b' \prec_B b.\; a\,G\,b' \\ &\iff a\,G\,b \end{aligned}$$

where the last equivalence holds because of (G-$\prec$). ■

8 The cocompact topology

Coherent spaces have been studied in many different guises and there are at least three different names for them in the literature. The first appearance is in [Nac65] where they are called *compact ordered spaces*. Connections with Stone duality are contained in [GHK$^+$80], Section V.5 and VII.3. Finally, they have been characterized as *supersober spaces* in [GHK$^+$80], Section VII.1 and [Law88]. In more recent work, coherent spaces are investigated as the adequate substitute for compact Hausdorff spaces in non-symmetric topology [Law91, Kop94]. From all these sources, we collect the following facts:

Facts. If $(X, \mathcal{T})$ is a coherent space, then $\mathcal{T}_c$, the collection of all complements of compact saturated sets forms the *cocompact topology* on X. The *patch topology* $\mathcal{T}_p = \mathcal{T} \vee \mathcal{T}_c$ is a compact Hausdorff topology on X such that $(X, \mathcal{T}_p, \leq_{\mathcal{T}})$ is a compact ordered space. Starting with a compact ordered space $(X, \mathcal{T}, \leq)$, the open upper sets form a topology $\mathcal{T}_\#$ such that $(X, \mathcal{T}_\#)$ is coherent. The cocompact topology for $\mathcal{T}_\#$ consist of all $\mathcal{T}$-open lower subsets and $(\mathcal{T}_\#)_p$ coincides with $\mathcal{T}$.■

In this section, we want to make the relationship between a strong proximity lattice B, its spectrum $\mathsf{spec}(B)$, its dual B^{op}, and the cocompact topology on $\mathsf{spec}(B)$ explicit. We know from Theorem 16 that for any proximity lattice B, the set $\mathcal{K}_{\mathsf{pt}(\mathsf{Idl}(B))}$ is isomorphic to $\mathsf{filt}(B)$. So the cocompact topology on $\mathsf{pt}(\mathsf{Idl}(B))$ equals $\mathsf{filt}(B)$ which in turn is the topology on $\mathsf{pt}(\mathsf{Idl}(B^{op}))$. For strong proximity lattices, this isomorphism is manifested on the point-level, too. There it turns out to be essentially complementation of prime filters/ideals. But let us first turn our attention to the compact saturated subsets.

Theorem 27 *Let* $(B; \vee, \wedge, 0, 1; \prec)$ *be a strong proximity lattice. Then the map* $\mathsf{comp}\colon (\mathsf{filt}(B), \subseteq) \to (\mathcal{K}_{\mathsf{spec}(B)}, \supseteq)$ *with*

$$\mathsf{comp}(K) = \{F \in \mathsf{spec}(B) \mid K \subseteq F\}$$

is an isomorphism. Its inverse is given by $\mathfrak{K} \mapsto \bigcap \mathfrak{K} : \mathcal{K}_{\mathsf{spec}(B)} \to \mathsf{filt}(B)$.

Proof. The isomorphism of the Hofmann-Mislove-Theorem maps a compact saturated set $\mathfrak{K}$ to its filter of open neighborhoods; if the elements of $\mathfrak{K}$ are given as completely prime filters, then this is their intersection. Thus we get for $\mathfrak{K} \in \mathcal{K}_{\mathsf{spec}(B)}$:

$$\begin{array}{rcll} \mathfrak{K} & \longmapsto & \{\psi(F) \mid F \in \mathfrak{K}\} & \text{via Theorem 21} \\ & \longmapsto & \bigcap\{\psi(F) \mid F \in \mathfrak{K}\} & \text{via the HM-Theorem} \\ & \longmapsto & \phi(\bigcap\{\psi(F) \mid F \in \mathfrak{K}\}) & \text{via Lemma 15} \\ & = & \bigcap\{\phi(\psi(F)) \mid F \in \mathfrak{K}\} & \text{since } \phi \text{ preserves } \bigcap \\ & = & \bigcap \mathfrak{K} & \text{since } \phi = \psi^{-1}. \end{array}$$

Going the other way, the Hofmann-Mislove-Isomorphism maps a Scott-open filter G of opens to its intersection which consists of all points with neighborhood filter containing G. Therefore, we calculate for $K \in \mathsf{filt}(B)$:

$$\begin{array}{rcll} K & \longmapsto & \psi(K) & \text{via Lemma 15} \\ & \longmapsto & \{\mathcal{F} \in \mathsf{pt}(\mathsf{Idl}(B)) \mid \psi(K) \subseteq \mathcal{F}\} & \text{via the HM-Theorem} \\ & = & \{\mathcal{F} \in \mathsf{pt}(\mathsf{Idl}(B)) \mid K \subseteq \phi(\mathcal{F})\} & \text{since } \phi = \psi^{-1} \\ & \longmapsto & \{\phi(\mathcal{F}) \mid \mathcal{F} \in \mathsf{pt}(\mathsf{Idl}(B));\, K \subseteq \phi(\mathcal{F})\} & \text{via Theorem 21.} \end{array}$$

As $\phi\colon \mathsf{pt}(\mathsf{Idl}(B)) \to \mathsf{spec}(B)$ is in particular surjective, the latter set equals $\mathsf{comp}(K)$. ■

Theorem 28 *Let $(B; \vee, \wedge, 0, 1; \prec)$ be a strong proximity lattice. Then the spaces $(\mathsf{spec}(B), (\mathcal{T}_{\mathsf{spec}(B)})_c)$ and $(\mathsf{spec}(B^{op}), \mathcal{T}_{\mathsf{spec}(B^{op})})$ are homeomorphic via*

$$F \mapsto \downarrow(B \setminus F).$$

Moreover, the frame isomorphism between $\mathsf{filt}(B) \cong (\mathcal{T}_{\mathsf{spec}(B)})_c$ and $\mathsf{Idl}(B^{op}) \cong \mathcal{T}_{\mathsf{spec}(B^{op})}$ arising from this homeomorphism is the identity.

Proof. We first have to check that $I := \downarrow(B \setminus F)$ is a prime ideal for $F \in \mathsf{spec}(B)$. Clearly $I = \downarrow I$. Moreover, $B \setminus F$ is closed under suprema since F is prime. Hence I is indeed an ideal by Lemma 7(4). To see primeness, we have to employ ($\wedge$-$\prec$): Suppose $a \wedge b \in I$. Then there is $x \in B \setminus F$ with $a \wedge b \prec x$. Axiom ($\wedge$-$\prec$) gives us $a', b' \in B$ with $a \prec a'$, $b \prec b'$ and $a' \wedge b' \prec x$. The last relation implies $a' \wedge b' \notin F$ since otherwise we had $x \in F$. Hence one of a', b' is not an element of F because this is a filter. Thus $a \in I$ or $b \in I$.

By symmetry, $\uparrow(B \setminus I)$ is a prime filter if I is a prime ideal. For bijectivity, it remains to check $\uparrow(B \setminus \downarrow(B \setminus F)) = F$ which is routine.

By Theorem 21, every open set on $\mathsf{spec}(B^{op})$ is of the form

$$\mathcal{O}_K^{op} = \{I \in \mathsf{spec}(B^{op}) \mid I \cap K \neq \emptyset\}$$

for some $K \in \mathsf{Idl}(B^{op}) = \mathsf{filt}(B)$. We calculate for $F \in \mathsf{spec}(B)$ and $K \in \mathsf{filt}(B)$:

$$\begin{array}{rcll} \downarrow(B \setminus F) \in \mathcal{O}_K^{op} & \iff & \downarrow(B \setminus F) \cap K \neq \emptyset & \\ & \iff & (B \setminus F) \cap K \neq \emptyset & \text{since } K = \uparrow K \\ & \iff & K \not\subseteq F & \\ & \iff & F \in \mathsf{spec}(B) \setminus \mathsf{comp}(K) & \end{array}$$

So the map is indeed a homeomorphism and gives rise to the identity as the corresponding frame-isomorphism. ■

Hence the table of Section 5 should be augmented by a third column:

strong proximity lattice	spectrum	dual spectrum
prime filter or prime ideal	element	element
ideal	open set	compact saturated set
filter	compact saturated set	open set

Again, restricting to the reflexive case leads to familiar territory. *Priestley duality* combines the correspondence between spectral spaces and distributive lattices with that between spectral spaces and their patches, totally order-disconnected compact ordered spaces. An account of this theory may be found in Chapter 10 of [DP90].

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