A bitopological point-free approach to compactifications

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Abstract

We study structures called *d-frames* which were developed by the last two authors for a bitopological treatment of Stone duality. These structures consist of a pair of frames thought of as the opens of two topologies, together with two relations which serve as abstractions of disjointness and covering of the space. With these relations, the topological separation axioms regularity and normality have natural analogues in d-frames. We develop a bitopological point-free notion of complete regularity and characterise all compactifications of completely regular d-frames. Given that normality of topological spaces does not behave well with respect to products and subspaces, probably the most surprising result is this: The category of d-frames has a normal coreflection, and the Stone-Čech compactification factors through it. Moreover, any compactification can be obtained by first producing a regular normal d-frame and then applying the Stone-Čech compactification to it. Our bitopological compactification subsumes all classical compactifications of frames as well as Smyth's stable compactification.

1. Introduction

The real line, by the general theory of topological compactification, has many compactifications, ranging from the one-point compactification to the Stone-Čech compactification. In the case of the reals, however, the two-point compactification, i.e. the extended reals, is very natural and arguably the most used in applications. Certainly the one-point compactification, topologically a circle, is an interesting outcome, and has been generalised to arbitrary locally compact spaces by Fell [4]. Although the two-point compactification is available as one of the possible compactifications of $\mathbb{R}$, it is not in any obvious way canonical. The problem seems to be that the natural order on the reals is not accounted for in topological compactification. If one makes the order the primitive notion and puts the topology of upper or lower semicontinuity on the reals, then the space is not sober any more. For the topology of, say, lower semicontinuity, it appears as if the reals have a point at infinity. The topology and the order of the reals seem to be at odds. Bitopology provides a useful way to remedy the problems described above. Indeed, the join of the upper and lower topologies on the reals is the Euclidean topology, and in some sense together the two topologies can make the point at infinity disappear.

The present paper brings together ideas from four different fields of mathematics. We try to highlight some conceptual similarities and exploit these in our treatment of compactification.

The first is domain theory, which arose from the need of mathematical models of computation. The objects of study are partially ordered sets (posets) where the order, called the *information order*, is derived from the distinction between termination and non-termination of programs. The information order is commonly written as $\sqsubseteq$. Computer programs must preserve this order, so the functions modelling programs are monotone with respect to the information order. It is desirable to model programming features such as recursion and fixed points. This leads to the requirement that the poset models have least upper bounds for all ascending chains. Just as sequences in topology were generalised to nets, it became apparent that requiring the existence of suprema for all directed subsets is a reasonable axiom. Scott, Hofmann and Stralka were among the first who realised the usefulness of an order relation coarser than the information order. One says that "a approximates b" if every computation that produces b as a limit must have produced a at some finite stage. In the poset model, this is captured by a relation $\ll$ called the *way-below* relation. It is contained in the information order but may fail to be reflexive. Concretely, define $a \ll b$ if for any subset D which is directed with respect to $\sqsubseteq$, the supremum (join) $\bigsqcup D$ being above b implies that $a \sqsubseteq d$ for some $d \in D$. The way-below relation is an instance of an *auxiliary relation* which in [6] is defined as a relation $\prec$ on a poset satisfying

(i) $\prec$ is contained in the poset order $\sqsubseteq$.
(ii) $x' \sqsubseteq x \prec y \sqsubseteq y'$ implies $x' \prec y'$.

Such an auxiliary relation is called *approximating* if every x is the supremum of all the y with $y \prec x$. If the poset carries some finitary algebraic structure then one is typically interested in those auxiliary relations which are compatible with the algebraic operations. On a bounded distributive lattice one thus defines a *quasi-proximity* to be an auxiliary relation which satisfies

(iii) $x \prec z$ and $y \prec z$ implies $x \sqcup y \prec z$. The least element 0 satisfies $0 \prec x$ for any x.
(iv) $x \prec y$ and $x \prec z$ implies $x \prec y \sqcap z$. The greatest element 1 satisfies $x \prec 1$ for any x.
(v) The relation $\prec$ is *interpolative*, meaning $x \prec z$ implies that there exists some y with $x \prec y \prec z$.

A *domain* is a poset in which all of directed subsets have a join and every element is the directed join of the elements way-below it. Such posets carry a natural topology called the *Scott topology* whose members are subsets U with the property that b is an element of U if and only if there exists an approximant $a \ll b$ which is in U already.

The second field we rely on in this work is *Stone duality* which provides the link between classical point-set topology and point-free topology. The latter became known as *locale theory*. Stone-type duality today is the name of a certain type of contravariant duality between categories. A Stone duality between two categories features a *dualising object* which carries the structure of both categories. In case of topology and locale theory the dualising object is the Sierpinski space $2 = \{0, 1\}$. It is an ordered set where $0 \sqsubseteq 1$ and also carries a topology where $\{1\}$ is the only non-trivial open set. One establishes a functor from one category to the other by endowing the set of morphisms into the dualising object with the structure of the other category. For instance, the continuous maps $X \to 2$ for any topological space can be ordered point-wise using the order structure on the Sierpinski space. Under this order, the set of continuous maps from X to 2 is order-isomorphic to the lattice of open sets $\mathcal{O}X$. Lattices which have the algebraic structure of open set lattices are called *frames*. More precisely, frames are posets possessing finite meets and arbitrary joins, and the finite meets distribute over arbitrary joins. Any continuous map of spaces $f : X \to Y$ gives rise to a frame homomorphism $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$ between the open set lattices, meaning it preserves the finite meets and arbitrary joins. Notice that the direction is reversed. Conversely, given a frame A, one turns the set of frame homomorphisms $A \to 2$ into a topological space called the *spectrum* of A by declaring the sets $\{h : A \to 2 \,|\, h(a) = 1\}$ as open where a ranges over the elements of A.

The original Stone duality is a representation theorem for Boolean algebras published by M.H. Stone in 1936 and 1937. The homomorphisms $L \to 2$ from a Boolean algebra into the two-element chain, endowed with a basis as above, yield a topological space where clopen sets form a basis of the topology. The topology of the spectrum of L can be constructed, without referring to the points, as the *ideal completion* of L. An ideal I of a poset is a subset which is downward closed and directed with respect to $\sqsubseteq$. The set of all ideals of L, ordered by inclusion, is called the ideal completion and provides a link back to domain theory.

Indeed, domains and their auxiliary relation $\ll$ have a curious property: Call an ideal I of a domain *round* with respect to $\ll$ if $x \in I$ implies that there exists a $y \in I$ with $x \ll y$. The only ideals of a domain which are round with respect to $\ll$ are of the form $\{y \,|\, y \ll x\}$ for some element x of the domain. Moreover, there is an order-isomorphism between the domain itself and the set of round ideals. The order-isomorphism still holds if one forms the round ideal completion of a basis of the domain with $\ll$ restricted to it. Smyth used this fact in the concept of *R-structures* to present domains, which is called *abstract bases* in [1].

Vickers [13] disposed of the information order altogether and thus defined *information systems*, which is the third source of ideas our work is based on. An information system is a set X together with a binary relation $\prec$ which is transitive and interpolative. Every information system gives rise to a domain, by forming the set of ideals with respect to the relation $\prec$. Concretely, define the *round ideal completion* $\mathrm{Idl}^{\prec} X$ to be the collection of all subsets I of X which satisfy $x \in I \Leftrightarrow \exists y \in I.\, x \prec y$ and whenever a finite set M is contained in I then there is some $x \in I$ with $m \prec x$ for all $m \in M$. Conversely, every domain X together with its way-below relation $\ll$ is an information system.

Round ideal completions, finally, lead to the fourth concept employed in the present paper. In [5] Freudenthal constructed a compactification from a binary relation on the lattice of opens, where the binary relation satisfies axioms (i)–(iv). Later it was shown that the compactifications of a completely regular space are in bijective correspondence to certain quasi-proximities on the powerset of the space. Within the theory of Hausdorff compactifications, these relations are called *proximities* and the property *approximating* explained above corresponds to an axiom called *admissible* here. If $e : X \hookrightarrow Y$ is a dense embedding of a completely regular space X into a compact Hausdorff space Y, then the open set lattice $\mathcal{O}Y$ is a domain where $U' \ll U$ if there exists a compact set K with $U' \subseteq K \subseteq U$. One defines the proximity on the powerset of X as $A \prec B$ if there exist opens $U' \ll U$ with $A \subseteq e^{-1}(U')$ and $e^{-1}(U) \subseteq B$. The open set lattice $\mathcal{O}Y$ is now order-isomorphic to the set of round ideals of the powerset of X with respect to the proximity relation $\prec$ we just defined. Notice that nowhere in the construction of the relation $\prec$ above the points of X or Y are mentioned. Those auxiliary relations on the powerset which give rise to Hausdorff compactifications satisfy the axioms (i)–(v) above and in addition

(vi) Every open set V of X is the union of points x with $\{x\} \prec V$.

(vii) If $A \prec B$ then the closure of A is contained in B.

(viii) If $A \prec B$ then $X \setminus B \prec X \setminus A$.

Meanwhile, with the emergence of point-free topology, Banaschewski [3] proved the corresponding result for frames: The point-free compactifications of a frame A are in bijective correspondence to what he calls *strong inclusions* on A. Of course one has to modify the axioms (vi)-(viii) accordingly. For frames, the admissibility axiom (vi) is replaced by the relation $\prec$ being approximating in the sense of domain theory. The set-theoretic complement in axiom (viii) is replaced by the *pseudocomplement* of the frame. The pseudocomplement is given as $\neg a = \bigsqcup\{b \in A \mid a \sqcap b = 0\}$. If $U \in \mathcal{O}X$ is an open of some space X, then the pseudocomplement of U in the frame $\mathcal{O}X$ is the complement of the closure of U. Hence the additional axioms for a strong inclusion on a frame read

(vi) Every a is the join of all the elements b with $b \prec a$.

(vii) If $a \prec b$ then $\neg a \sqcup b = 1$.

(viii) If $a \prec b$ then $\neg b \prec \neg a$.

We will subsequently use the term *proximity* for a strong inclusion on a frame. Quasi-proximities satisfying only (i)-(vi) also give rise to compactifications of a space X, but the compact spaces so obtained are no longer Hausdorff. They are known today as *stably compact spaces* and, closing the circle, play an important role as models of computation. Stably compact spaces can be characterised as those sober topological spaces where the way-below relation on the open set lattice satisfies (i)-(vi). A stably compact space X can be turned into to a bitopological space in a natural way as follows. A subset of a topological space is called *saturated* if it is an intersection of open sets. The complements of compact saturated subsets of X yield another topology on the set X known as the *de-Groot dual* or *cocompact topology*. The common refinement of a stably compact topology and its de-Groot dual is a compact Hausdorff topology on X. In fact Kopperman [9] provides a theory of bitopological compactifications of this kind.

Summing up, all four topics introduced above feature a binary relation $\prec$ of some sort providing a notion of approximation. In all cases, *round ideal completion* with respect to the relation $\prec$ is a useful construction. The main conceptual contribution of our work is breaking the relation $\prec$ down into a composition of two relations between two different sets. We use topological separation axioms to motivate this step. At the same time, the two relations between two sets are the ingredients of a bitopological version of Stone duality. Thus we can work with domain theoretic tools but translate our results to topology whenever desired.

1.1. Contributions

The category we employ has a number of features which one does not find in topology or locale theory. For instance, although there is a dual adjunction to bitopological spaces, there are even finite structures which do not correspond to any bitopological space. In locale theory this can only happen for infinite objects. More prominently, our category admits a normal coreflection; a feature which to our knowledge is absent from both the category of spaces and locales. This normalisation is easy to express and serves as one stage of our compactification construction. Another neat feature of our theory is that there is a bitopological representation of the real numbers which has a unique compactification, the space of extended reals $[-\infty, +\infty]$.

1.2. Organisation of the paper

In the first section we introduce the structures *d-lattices* and *d-frames* and prove some basic properties. The second section contains the coreflection which we use to obtain point-free compactifications of d-frames. In the third section we develop appropriate notions of complete regularity and proximity of d-frames and characterise all compactifications by their associated proximities. We conclude the paper by linking the category of d-frames with the category of frames via an adjoint pair of functors. The adjunction enables us to exhibit Banaschewski's compactification of frames as a special case of our compactification.

2. d-Lattices, d-Frames, regularity and normality

Throughout this paper, the order on lattices will be denoted by the symbol $\sqsubseteq$ and joins by $\sqcup$. In order to avoid confusion, joins and meets on sets of ideals or filters will have symbols $\vee$ and $\wedge$. The order-dual of a lattice L is denoted by L^∂. The relational composition is denoted by ; and we write composition from left to right. For any order symbol, e.g. $\lhd$ we have arrow symbols with the same tip, e.g. $\Downarrow$ which symbolises taking the lower set with respect to that order relation.

Some separation axioms for topological spaces can be formulated without mentioning points, using opens only. For other separation axioms this requires some machinery of locale theory, for example the T_1 and T_2 axioms. But for T_3 and T_4 a point-free formulation is straightforward: A topological space X is T_3 if and only if every open U is the union of opens U' with the property that there exists an open V such that $U' \cap V = \emptyset$ and $U \cup V = X$. In this situation one says the U' are *well inside* U and writes $U' \lhd U$. The open V serves as a witness for the fact that the relation $U' \lhd U$ holds. A space X is T_4 if and only if whenever U and V are opens with $U \cup V = X$ then there exist opens U' and V' such that $U \cup V' = X$, $V' \cap U' = \emptyset$ and $U' \cup V = X$. Notice that only binary intersections to $\emptyset$ and binary unions to X are used.

We formalise this situation as follows. Instead of a lattice of opens we consider an arbitrary bounded distributive lattice $(L_-, \sqsubseteq_-)$ with typical elements u' and u. The witnesses v and v' are not elements of L_- but of another bounded distributive lattice $(L_+, \sqsubseteq_+)$. We formalise disjointness by a relation $\mathsf{con} \subseteq L_+ \times L_-$ called *consistency*, and likewise formalise covering of the space by another relation $\mathsf{tot} \subseteq L_- \times L_+$ called *totality*.

Definition 1. A *d-lattice* is a structure $L_- \underset{\mathsf{con}}{\overset{\mathsf{tot}}{\rightleftarrows}} L_+$ where L_- and L_+ are bounded distributive lattices and tot and con are relations satisfying the axioms of Figure 1. Morphisms between such structures are pairs of homomorphisms (h_-, h_+) between bounded distributive lattices preserving the relations, meaning $u \,\mathsf{tot}\, v$ implies $h_-(u) \,\mathsf{tot}\, h_+(v)$ and $v \,\mathsf{con}\, u$ implies $h_+(v) \,\mathsf{con}\, h_-(u)$. The category of d-lattices and d-lattice morphisms is denoted by dLat.

As promised in the introduction we use the relations con and tot on a d-lattice to build auxiliary relations. Indeed, the composition $\lhd_+ = \mathsf{con};\mathsf{tot}$ is an auxiliary relation on L_+ because it is contained in the lattice order $\sqsubseteq_+$ by axiom (con-tot) and satisfies $\sqsubseteq_+; \lhd_+; \sqsubseteq_+ = \lhd_+$ because of (con-$\downarrow$) and (tot-$\uparrow$). Moreover, the down-set operation $\Downarrow$ derived from it has ideals as values by (con-$\vee$). Likewise, the up-set operation $\Uparrow$ derived from $\lhd_+$ has filters as values because of (tot-$\wedge$). In other words, the relation $\lhd_+$ is a bounded sub-lattice of $L_+ \times L_+$. Clearly the same is true for the relation $\lhd_- = \mathsf{con}^{-1};\mathsf{tot}^{-1}$. The relations $\lhd_+$ and $\lhd_-$ are called the *well-inside* relations on L_+ and L_-, respectively.

There is a contravariant involution on the category of d-lattices, which extends the order-dual operation of lattices. Observe that the axioms in Figure 1 are self-dual in the following sense. Swapping con and tot, L_+ and L_-, and reversing the lattice order on both sides yields the same set of axioms. This motivates the following definition and result:

(con-$\downarrow$)	con is a lower set in $L_- \times L_+$,
(con-$\vee$)	$v \mathrel{\mathsf{con}} u$ and $v' \mathrel{\mathsf{con}} u'$ implies $v \sqcup v' \mathrel{\mathsf{con}} u \sqcap u'$, $0 \mathrel{\mathsf{con}} 1$
(con-$\wedge$)	$v \mathrel{\mathsf{con}} u$ and $v' \mathrel{\mathsf{con}} u'$ implies $v \sqcap v' \mathrel{\mathsf{con}} u \sqcup u'$, $1 \mathrel{\mathsf{con}} 0$
(i.e. con is a bounded sub-lattice of $L_+ \times L_-^\partial$)	
(tot-$\uparrow$)	tot is an upper set in $L_- \times L_+$,
(tot-$\wedge$)	$u \mathrel{\mathsf{tot}} v$ and $u' \mathrel{\mathsf{tot}} v'$ implies $u \sqcup u' \mathrel{\mathsf{tot}} v \sqcap v'$, $0 \mathrel{\mathsf{tot}} 1$
(tot-$\vee$)	$u \mathrel{\mathsf{tot}} v$ and $u' \mathrel{\mathsf{tot}} v'$ implies $u \sqcap u' \mathrel{\mathsf{tot}} v \sqcup v'$, $1 \mathrel{\mathsf{tot}} 0$
(i.e. tot is a bounded sub-lattice of $L_-^\partial \times L_+$)	
(con-tot)	$\mathsf{con}; \mathsf{tot}$ is contained in the lattice order on L_+, $\mathsf{con}^{-1}; \mathsf{tot}^{-1}$ is contained in the lattice order on L_-.

Figure 1: Axioms for a d-lattice

Definition 2. If $\mathcal{L}$ denotes the d-lattice $L_- \underset{\mathsf{con}}{\overset{\mathsf{tot}}{\rightleftarrows}} L_+$ then $\mathcal{L}^\partial$ denotes the structure $L_+^\partial \underset{\mathsf{tot}}{\overset{\mathsf{con}}{\rightleftarrows}} L_-^\partial$ which we call the *order-dual* of $\mathcal{L}$.

Lemma 1. 1. *The order-dual of a d-lattice is a d-lattice. On the component lattices of the order-dual, the well-inside relations are the relational inverses of the original d-lattice's well-inside relations.*
2. *The assignment $\mathcal{L} \mapsto \mathcal{L}^\partial$ extends to a covariant involution on the category* dLat.

The lemma above will allow us to write some of the proofs more concisely, because any property of a d-lattice involving the relation con can be translated into a dual property involving tot via order-dual.

The role of covering and disjoint opens in the T_4 axiom suggest the following definition of normality.

Definition 3. A d-lattice $\mathcal{L} = (L_-, L_+, \mathsf{con}, \mathsf{tot})$ is *normal* if $\mathsf{tot}; \mathsf{con}; \mathsf{tot} = \mathsf{tot}$.

Observe that the inclusion $\mathsf{tot}; \mathsf{con}; \mathsf{tot} \subseteq \mathsf{tot}$ holds for all d-lattices by axioms (con-tot) and (tot-$\uparrow$). A crucial consequence of the normality axiom is that both $\lhd_+$ and $\lhd_-$ are interpolative. Indeed, write $\lhd_+ = \mathsf{con}; \mathsf{tot}$ and expand tot according to normality to obtain $\lhd_+ = \lhd_+; \lhd_+$. Likewise $\rhd_- = \rhd_-; \rhd_-$.

Regularity is not a first-order property, so in formalising it we need to work with complete lattices. We adopt the notion of a *d-frame* which features in Jung and Moshier's bitopological treatment of Stone duality [8].

Definition 4. The category dFrm of d-frames has as objects d-lattices $\mathcal{L}$ where the component lattices L_- and L_+ are frames and the relation con is closed under directed joins in $L_+ \times L_-$. Morphisms of d-frames are d-lattice morphisms

(h_-, h_+) where the component maps also preserve directed joins (and thereby arbitrary joins).

The additional axiom on con implies that for any $v \in L_+$ there is a largest element $u \in L_-$ which is consistent with v. For good reasons we can call this the pseudocomplement of v and write it as $\neg v$. With this notation one has $v \lhd_+ v'$ if and only if $\neg v \,\mathsf{tot}\, v'$ which is formally similar to the axiom (vii) we gave in the introduction. Furthermore, using this characterisation of $\lhd_+$ one shows that $v \lhd_+ v'$ implies $\neg v \rhd_- \neg v'$. The latter fact is formally similar to the axiom (viii) from the introduction. The operation $\neg : L_+ \to L_-$ is antitone and transforms all joins to meets.

There is a dual adjunction between the category of d-frames and the category of bitopological spaces and bi-continuous maps. In this duality, the consistency relation of a d-frame is the formal analogue of two open sets being disjoint, and the totality relation is the formal analogue of two opens covering the space.

Example 1. The real line with the topologies of upper and lower semicontinuity has the following bitopological Stone dual $\mathcal{L}\mathbb{R}$. The lower topology L_- consists of open rays $]-\infty, x[$ for x in the extended reals $[-\infty, +\infty]$. Likewise, the upper topology L_+ consists of open rays $]x, +\infty[$ where x ranges over the extended reals. Consistency and totality is defined in the obvious way. Thus both component frames of $\mathcal{L}\mathbb{R}$ are isomorphic to the extended reals, but the frame order on L_+ is the opposite of the algebraic order on $[-\infty, +\infty]$. One can recover the points of $\mathbb{R}$ from $\mathcal{L}\mathbb{R}$ as certain pairs of meet-prime elements of the component frames. Concretely, the point $x \in \mathbb{R}$ is recovered from the pair $(]-\infty, x[,]x, +\infty[)$.

Definition 5. A d-frame $(L_-, L_+, \mathsf{con}, \mathsf{tot})$ is *regular* if every element of each component frame is the (directed) join of the elements well-inside it.

Next we introduce two topologies on the component lattice L_+ of a d-lattice, and likewise two topologies on L_-.

Definition 6. Let $(L_-, L_+, \mathsf{con}, \mathsf{tot})$ be a d-lattice and $\Downarrow : L_+ \to \mathrm{Idl}\, L_+$ be the lower set operation with respect to the well-inside relation $\lhd_+$. Define an operation on subsets of L_+ as $V \mapsto \{v \in L_+ \mid {\Downarrow} v \cap V \neq \emptyset\}$. The set ΩL_+ consists of all subsets which are invariant under this operation. The collection of filters of L_+ which are elements of ΩL_+ is denoted by $\mathrm{Filt}^{\lhd}\, L_+$. Dually, let $\Uparrow : L_+ \to \mathrm{Filt}\, L_+$ be the upper set operation with respect to the well-inside relation $\lhd_+$. Define an operation on subsets of L_+ as $V \mapsto \{v \in L_+ \mid {\Uparrow} v \cap V \neq \emptyset\}$. The set $\mho L_+$ consists of all subsets which are invariant under this operation. The collection of ideals of L_+ which are elements of $\mho L_+$ is denoted by $\mathrm{Idl}^{\lhd}\, L_+$. In the same manner one defines the collections ΩL_-, $\mathrm{Filt}^{\lhd}\, L_-$, $\mho L_-$ and $\mathrm{Idl}^{\lhd}\, L_-$.

Lemma 2. *Let $\mathcal{L}$ be a d-lattice.*

1. *ΩL_+ is a topology on L_+. When restricted to upper sets, the defining operation on subsets is the interior operation with respect to this topology.*

2. *The set* $\mathrm{Idl}^{\lhd} L_+$ *is a sub-frame of the frame of ideals* $\mathrm{Idl}\, L_+$.
3. *If* $\mathcal{L}$ *is a normal d-frame (more generally, if the well-inside relations are interpolative) then* $\mathrm{Idl}^{\lhd} L_+$ *is a domain where the way-below relation is closed under finite joins on the left and finite meets on the right. All principal lower sets* $\downarrow v$ *belong to* $\mathrm{Idl}^{\lhd} L_+$ *and the way-below relation on* $\mathrm{Idl}^{\lhd} L_+$ *is characterised as* $I' \ll I$ *if and only if there exists a* $v \in I$ *such that* I' *is contained in* $\downarrow v$.
4. *Similar statements hold for* $\mho L_+$ *and* $\mathrm{Filt}^{\lhd} L_+$.

Proof. (1) Note that $\downarrow v \cap V \neq \emptyset$ is shorthand for $\exists v' \lhd_+ v. v' \in V$. With this it is easy to see that ΩL_+ is closed under arbitrary unions. For finite intersections, notice that because of $0 \lhd_+ 0$ the set L_+ is in ΩL_+. Now suppose V_1 and V_2 are elements of ΩL_+. Since $\lhd_+$ is contained in the lattice order, any element of ΩL_+ must be an upper set. If $v \in V_1 \cap V_2$ then there exist $v_i \in \downarrow v \cap V_i$ for $i \in \{1, 2\}$. Then $v_1 \sqcup v_2 \lhd_+ v$ and since both V_1 and V_2 are upper sets $v_1 \sqcup v_2 \in V_1 \cap V_2$. This finishes the proof of (1).

(2) It is well-known in lattice theory that the set of ideals of a bounded distributive lattice is a frame, where the binary meet of two ideals I_1 and I_2 is given by intersection, which by distributivity is the same as collecting all $v_1 \sqcap v_2$ with $v_1 \in I_1$ and $v_2 \in I_2$. Directed joins are given by union and the binary join of two ideals I_1 and I_2 is computed as the set of joins $v_1 \sqcup v_2$ where v_1 ranges over the elements of I_1 and v_2 ranges over the elements of I_2. By the order-dual of (1) we know that the set $\mathrm{Idl}^{\lhd} L_+$ is closed under directed joins and finite meets. Suppose $I_1, I_2 \in \mathrm{Idl}^{\lhd} L_+$ and let $v_i \in I_i$ for $i \in \{1, 2\}$. By hypothesis there exist $v_i' \in I_i$ with $v_i \lhd_+ v_i'$. Then $v_1 \sqcup v_2 \lhd_+ v_1' \sqcup v_2'$ which shows that the binary join $I_1 \vee I_2$ is again an element of $\mathrm{Idl}^{\lhd} L_+$.

(3) Now suppose $\mathcal{L}$ is a normal d-frame. From the interpolation property of $\lhd_+$ we deduce that for any $v \in L_+$ the set $\downarrow v$ is an element of $\mathrm{Idl}^{\lhd} L_+$. Indeed, $v' \lhd_+ v$ implies $v' \lhd_+ v'' \lhd_+ v$ for some v''. We claim that for any ideal $I \in \mathrm{Idl}^{\lhd} L_+$ and $v \in I$ the relation $\downarrow v \ll I$ holds. To show this, suppose $\mathcal{D}$ is a directed set of elements of $\mathrm{Idl}^{\lhd} L_+$ with $\bigcup \mathcal{D} \supseteq I$ (recall that directed joins are computed as set union). Then certainly some ideal $I_v \in \mathcal{D}$ must contain v and consequently $\downarrow v \subseteq I_v$. Furthermore by $I \in \mho L_+$ it is obvious that the ideal I is the union of all the $\downarrow v$ where v ranges over the elements of I. This union is actually directed. Therefore any ideal I' way-below I must be contained in some $\downarrow v$ already. We have shown that $\mathrm{Idl}^{\lhd} L_+$ is a domain. Observe that whenever $v \lhd_+ v$ then the ideal $\downarrow v$ is way-below itself. In particular $L_+ = \downarrow 1 \ll \downarrow 1$. In any complete lattice the way-below relation is closed under finite joins on the left. It remains to show that $\ll$ is closed under binary meets on the right. Suppose $I \ll I_i$ for $i \in \{1, 2\}$. That means that there exist $v_i \in I_i$ with $I \subseteq \downarrow v_i$. Observe that the map $v \mapsto \downarrow v$ preserves binary meets because $\lhd_+$ is closed under finite meets on the right. Thus $\downarrow(v_1 \sqcap v_2)$ still contains I and since $v_1 \sqcap v_2 \in I_1 \wedge I_2$ we know that $I \ll I_1 \wedge I_2$.

Remark 1. The arguments from the proof above appear in the works of Smyth, Vickers and Banaschewski. Vickers shows in [13] that, given any interpolative transitive relation $\prec$ on a set X, the set ΩX defined in a way similar to ours yields the Scott topology on a domain. Smyth [11] uses a relation $\prec$ on the lattice of opens $\mathcal{O}X$ of a space X which satisfies the same order-theoretic properties as our $\lhd_+$. He extends the lattice of opens to $\mathrm{Idl}^{\prec}\,\mathcal{O}X$ and shows that this is the topology of a "stable compactification" of X.

For the sake of brevity we call the elements of $\mathrm{Filt}^{\lhd}\, L_+$ the *open filters* of L_+ and elements of $\mathrm{Idl}^{\lhd}\, L_+$ the *open ideals* of L_+, and likewise for L_-. The set $\mathrm{Idl}^{\lhd}\, L_+$ is the *open ideal completion* of L_+.

The reader should be warned that in general the set of open ideals may be small. In the extreme case it consists of precisely two elements, namely $\mathop{\downarrow\!\!\!\downarrow} 0$ and $\mathop{\downarrow\!\!\!\downarrow} 1$. In contrast, regularity together with normality gives us a plentiful supply of open ideals. The following result is not needed for our later development, so we omit the proof. But we will compare it with other definitions.

Proposition 1. *The following are equivalent for a d-lattice $\mathcal{L}$.*

1. *$\mathcal{L}$ is normal.*
2. *If we define a topology $\Omega(L_- \times L_+)$ on $L_- \times L_+$ using the relation $\lhd_- \times \lhd_+$ similarly to Definition 6, then tot is open in this topology.*
3. *The map $u \mapsto \{v \in L_+ \mid u \,\mathsf{tot}\, v\}$ takes values in the open filters of L_+ and is continuous with respect to the topology ΩL_- and the Scott topology on the frame $\mathrm{Filt}^{\lhd}\, L_+$.*

3. A coreflection of regular normal d-frames

This section is concerned with a form of round ideal completion of d-lattices and its categorical properties. The auxiliary relations we use are the well-inside relations $\lhd_+$ and $\lhd_-$ we defined using con and tot. Recall that for a d-lattice $\mathcal{L}$ the open filters of its first component lattice L_- are the same as the open ideals of the second component lattice of the order-dual $\mathcal{L}^{\partial}$.

Lemma 3. *Let $\mathcal{L} = (L_-, L_+, \mathsf{con}, \mathsf{tot})$ be a d-lattice, $I \in \mathrm{Idl}^{\lhd}\, L_-$, $J \in \mathrm{Idl}^{\lhd}\, L_+$ be open ideals and $F \in \mathrm{Filt}^{\lhd}\, L_-$, $G \in \mathrm{Filt}^{\lhd}\, L_+$ be open filters. Define the following consistency and totality relations.*

$$J \,\mathsf{con}_{\diamond}\, I \quad :\Leftrightarrow \quad J \times I \subseteq \mathsf{con} \tag{1}$$

$$I \,\mathsf{tot}_{\diamond}\, J \quad :\Leftrightarrow \quad (I \times J) \cap \mathsf{tot} \neq \emptyset \tag{2}$$

$$F \,\mathsf{con}^{\diamond}\, G \quad :\Leftrightarrow \quad F \times G \subseteq \mathsf{tot} \tag{3}$$

$$G \,\mathsf{tot}^{\diamond}\, F \quad :\Leftrightarrow \quad (G \times F) \cap \mathsf{con} \neq \emptyset \tag{4}$$

1. *Both the* open ideal completion $\mathcal{L}_{\diamond} := (\mathrm{Idl}^{\lhd}\, L_-, \mathrm{Idl}^{\lhd}\, L_+, \mathsf{con}_{\diamond}, \mathsf{tot}_{\diamond})$ *and the* open filter completion $\mathcal{L}^{\diamond} := (\mathrm{Filt}^{\lhd}\, L_+, \mathrm{Filt}^{\lhd}\, L_-, \mathsf{con}^{\diamond}, \mathsf{tot}^{\diamond})$ *are d-frames.*
2. *The two d-frames defined in (1) are isomorphic. That is, the open ideal completion of $\mathcal{L}$ is isomorphic to the open ideal completion of the order dual $\mathcal{L}^{\partial}$.*

PROOF. We prove (2) first. Consider the following map on lower sets of L_+.

$$\varphi_+(J) = \{u \in L_- \mid \exists v \in J.\, u \,\mathsf{tot}\, v\}\,. \tag{5}$$

On upper sets of L_- define a map

$$\psi_+(F) = \{v \in L_+ \mid \exists u \in F.\, v \,\mathsf{con}\, u\}\,. \tag{6}$$

We claim that φ_+ and ψ_+ are mutually inverse when restricted to open ideals and filters, respectively. First observe that by the axiom (tot-$\vee$) the map φ_+ takes ideals to filters. Likewise, by (con-$\vee$) the map ψ_+ takes filters to ideals. Expanding the definitions shows that the composites $\psi_+ \circ \varphi_+$ and $\varphi_+ \circ \psi_+$ are just the interior operations of the topologies $\mho L_+$ and ΩL_-, respectively. Therefore φ_+ is the inverse of ψ_+. In the same manner one defines an isomorphism $\psi_- = (\varphi_-)^{-1}$ between the open ideals of L_- and open filters of L_+.

Next we show that a pair (J, I) of open ideals is consistent if and only if the pair of images $(\varphi_+(J), \varphi_-(I))$ is consistent. J and I are consistent ideals if and only if for all $v \in J$ and $u \in I$ the relation $v \,\mathsf{con}\, u$ holds. The images $\varphi_+(J)$ and $\varphi_-(I)$ are consistent if and only if $u' \,\mathsf{tot}\, v \in J$ and $I \ni u \,\mathsf{tot}\, v'$ implies $u' \,\mathsf{tot}\, v'$. Since $\mathsf{tot};\mathsf{con};\mathsf{tot}$ is contained in tot the implication $J \,\mathsf{con}_\diamond\, I \Rightarrow \varphi_+(J) \,\mathsf{con}^\diamond \varphi_-(I)$ holds. For the converse, use the fact that $I = \psi_-(\phi_-(I))$ and $J = \psi_+(\phi_+(J))$ and observe that the order-dual swaps the map ϕ_+ with ψ_- and the map ϕ_- with ψ_-. Thus the implication $\varphi_+(J) \,\mathsf{con}^\diamond \varphi_-(I) \Rightarrow J \,\mathsf{con}_\diamond\, I$ follows from the implication we already proved, applied to the order-dual.

The ideal I is total with J if and only if $I \times J$ intersects the relation tot. The filter $\varphi_-(I)$ is total with the filter $\varphi_+(J)$ if and only if $I \ni u \,\mathsf{tot}\, v' \,\mathsf{con}\, u' \,\mathsf{tot}\, v \in J$ holds for some u, u', v and v'. With the inclusion $\mathsf{tot};\mathsf{con};\mathsf{tot} \subseteq \mathsf{tot}$ we obtain the implication $\varphi_-(I) \,\mathsf{tot}^\diamond \varphi_+(J) \Rightarrow I \,\mathsf{tot}_\diamond\, J$. For the converse one may again use the existing implication and apply it to the order-dual.

It remains to show that $\mathcal{L}^\diamond$ and $\mathcal{L}_\diamond$ are indeed d-frames, so the axioms of Figure 1 need to be verified. We do this for $\mathcal{L}_\diamond$. The axioms (tot-$\uparrow$) and (con-$\downarrow$) are trivial. Let $J \,\mathsf{con}_\diamond\, I$ and $J' \,\mathsf{con}_\diamond\, I'$. The join $J \vee J'$ consists of elements $v \sqcup v'$ where $v \in J$ and $v' \in J'$. The meet $I \wedge I'$ consists of elements of the form $u' \sqcap u$ for $u \in I$ and $u' \in I'$. Now it is easy to see that the axiom (con-$\vee$) for the d-lattice $\mathcal{L}$ implies the axiom (con-$\vee$) for $\mathcal{L}_\diamond$. In the same manner one verifies (con-$\wedge$). Now suppose $I \ni u \,\mathsf{tot}\, v \in J$ and $I' \ni u' \,\mathsf{tot}\, v' \in J'$ are witnesses for $I \,\mathsf{tot}_\diamond\, J$ and $I' \,\mathsf{tot}_\diamond\, J'$. The axiom ($\mathsf{tot}$-$\wedge$) for $\mathcal{L}$ yields $u \sqcup u' \,\mathsf{tot}\, v \sqcap v'$ and this is a witness for $I \vee I' \,\mathsf{tot}_\diamond\, J \wedge J'$. The axiom ($\mathsf{tot}$-$\vee$) is verified in the same way. For the axiom (con-tot) suppose that $J' \,\mathsf{con}_\diamond\, I \,\mathsf{tot}_\diamond\, J$. Expanding the definitions yields that there exists some $v \in J$ with $J' \subseteq \Downarrow v$. Now by the axiom ($\mathsf{con}$-$\mathsf{tot}$) for $\mathcal{L}$ we know that $\Downarrow v$ is contained in the principal lower set $\downarrow v$, whence $J' \subseteq J$. Finally, recall that directed joins in the frame of open ideals are given by set union, whence it is easy to see that the set $\mathsf{con}_\diamond \subseteq \mathrm{Idl}^\lhd L_+ \times \mathrm{Idl}^\lhd L_-$ is closed under directed joins. Hence $\mathcal{L}_\diamond$ is a d-frame.

Remark 2. The isomorphism $\mathrm{Idl}^\lhd L_+ \cong \mathrm{Filt}^\lhd L_-$ generalises to an order-isomorphism of topologies $\Omega L_+ \cong \mho L_-$. In the context of Vickers' information systems, the isomorphism between $\mathrm{Idl}^{\lhd} L_+$ and $\mathrm{Filt}^{\lhd} L_-$ is related to Lawson duality: For any information system $(X, \prec)$, the Lawson dual of the domain of round ideals $\mathrm{Idl}^{\prec} X$ is isomorphic to the domain of round filters $\mathrm{Filt}^{\prec} X$. Similarly, Lawson [10] showed that the Scott topology of a domain D is the order-dual of the Scott topology of its Lawson dual $D^{\wedge}$. We make these similarities more precise in Proposition 2 below. Banaschewski [3] proves a result similar to the lemma above where $L_- = L_+$ and $\lhd$ is a strong inclusion.

If (X, τ_-, τ_+) is a bitopological space then one can ask when the common refinement of the topologies τ_- and τ_+ is a compact topology. Using the Alexander Subbase Lemma, compactness is equivalent to the following assertion. Whenever $\{(u_i, v_i)\}_{i \in I} \subseteq \tau_- \times \tau_+$ is a directed family of opens with the property $\left(\bigcup_{i \in I} u_i\right) \cup \left(\bigcup_{i \in I} v_i\right) = X$ then $u_i \cup v_i = X$ for some $i \in I$ already. This motivates the following definition.

Definition 7. A d-frame $(L_-, L_+, \mathsf{con}, \mathsf{tot})$ is *compact* if for every directed family $\{(u_i, v_i)\}_{i \in I}$ of the product $L_- \times L_+$ the following holds. Whenever $\bigsqcup_{i \in I} u_i$ is total with $\bigsqcup_{i \in I} v_i$ then there is some $i \in I$ such that u_i is total with v_i already. In the language of domain theory, the compact d-frames are those for which the relation tot is Scott open in the product frame $L_- \times L_+$.

Remark 3. Compare the definition of compactness with Proposition 1. The only difference between compactness and normality is the choice of topology on the product $L_- \times L_+$.

Lemma 4. *Let $\mathcal{L}$ be a d-lattice and $\mathcal{L}_\diamond$ be the d-frame defined in Lemma 3.*

1. *The d-frame $\mathcal{L}_\diamond$ is compact.*
2. *The assignment $\mathcal{L} \mapsto \mathcal{L}_\diamond$ extends to a functor from d-lattices to compact d-frames.*
3. *If $\mathcal{L}$ is normal then the d-frame $\mathcal{L}_\diamond$ is regular.*
4. *If $\mathcal{L}$ is a d-frame then the join operation of open ideals yields a d-frame homomorphism $\varepsilon_{\mathcal{L}} : \mathcal{L}_\diamond \to \mathcal{L}$.*
5. *If $\mathcal{L}$ is a regular normal d-frame then both component maps of $\varepsilon_{\mathcal{L}}$ are surjective.*

Proof. (1) Clearly, if the union of a family of sets intersects a given set, then some member of the family must intersect the given set already. Since directed joins in $\mathrm{Idl}^{\lhd} L_- \times \mathrm{Idl}^{\lhd} L_+$ are computed as set union and totality is defined by non-empty intersection with tot, the d-frame $\mathcal{L}_\diamond$ is always compact.

(2) It is a well-known fact from lattice theory that a homomorphism of bounded distributive lattices $h : M \to L$ extends to a frame homomorphism $\mathrm{Idl}(h) : \mathrm{Idl}\, M \to \mathrm{Idl}\, L$ via $\mathrm{Idl}(h)(I) = \{x \in L \mid \exists i \in I.\, x \sqsubseteq h(i)\}$. Suppose (h_-, h_+) is a d-lattice homomorphism between $\mathcal{M}$ and $\mathcal{L}$. Recall that the component lattice homomorphism h_+ preserves the auxiliary relation $\lhd_+$. Therefore, if $I \subseteq M_+$ is an open ideal then the down-closure $h_+^\diamond := \mathrm{Idl}(h_+)(I)$ of the forward image $h_+(I)$ is an open ideal of L_+. Similarly one defines a frame

homomorphism $h_-^\diamond : \mathrm{Idl}^\lhd M_- \to \mathrm{Idl}^\lhd L_-$. It is straightforward to check that the pair $(h_-^\diamond, h_+^\diamond)$ preserves the relations $\mathsf{con}_\diamond$ and $\mathsf{tot}_\diamond$.

(3) Recall that by Lemma 2 the frame $\mathrm{Idl}^\lhd L_+$ is a domain whenever $\mathcal{L}$ is normal, and moreover its way-below relation is characterised by $J' \ll J$ iff $J' \subseteq \mathord{\downarrow\!\!\!\!\!\cdot}\, v$ for some $v \in J$. In the proof of Lemma 3 verifying (con-tot) we have seen that the well-inside relation on $\mathrm{Idl}^\lhd L_+$ has the same characterisation. In any domain the way-below relation is approximating, whence the well-inside relation on $\mathrm{Idl}^\lhd L_+$ is approximating. The corresponding fact holds for the open ideals of L_-, whence the d-frame $\mathcal{L}_\diamond$ is regular in the sense of Definition 5.

(4) Define the morphism $\varepsilon_\mathcal{L} : \mathcal{L}_\diamond \to \mathcal{L}$ by mapping a pair of open ideals $(I, J) \in \mathrm{Idl}^\lhd L_- \times \mathrm{Idl}^\lhd L_+$ to the pair of joins $(\bigsqcup I, \bigsqcup J)$. It is well-known that for any frame the join operation of ideals is a frame homomorphism. Since the open ideals form a sub-frame of all ideals by Lemma 2, the join operation $\bigsqcup$ restricts to a frame homomorphism on open ideals. An ideal is in particular a directed set. If $\mathcal{L}$ is a d-frame then the relation con is closed under directed joins, whence $J \times I \subseteq \mathsf{con}$ implies $\bigsqcup J \,\mathsf{con} \bigsqcup I$. If the product $I \times J$ intersects tot, then by (tot-$\uparrow$) the pair $(\bigsqcup I, \bigsqcup J)$ is an element of tot. Thus $\varepsilon_\mathcal{L}$ is indeed a d-frame homomorphism.

(5) For a normal d-frame $\mathcal{L}$, the ideal $\mathord{\downarrow\!\!\!\!\!\cdot}\, v$ is open for every $v \in L_+$. If $\mathcal{L}$ is also regular, we know that $v = \bigsqcup \mathord{\downarrow\!\!\!\!\!\cdot}\, v$ for any $v \in L_+$, and likewise for any $u \in L_-$. Thus $\varepsilon_\mathcal{L}$ has surjective component maps.

Although not concerned with compactifications, Jung and Moshier exhibited the category of compact regular d-frames as particularly well-behaved. We cite some results from [8].

Proposition 2 (Jung and Moshier). *If $\mathcal{L} = (L_-, L_+, \mathsf{con}, \mathsf{tot})$ is a compact regular d-frame then $L_- \cong \mathrm{Filt}^\lhd L_+$ and $L_+ \cong \mathrm{Filt}^\lhd L_-$. Moreover, the d-frame $\mathcal{L}$ is isomorphic to the d-frame $\mathcal{L}^\diamond$. Every compact regular d-frame is normal. The category of compact regular d-frames is dually equivalent to the category of stably compact spaces and perfect maps, that is, maps which are bi-continuous with respect to the topology and its de-Groot dual. More precisely, if $(L_-, L_+, \mathsf{con}, \mathsf{tot})$ is a compact regular d-frame, then L_- is the topology of some stably compact space X and L_+ is isomorphic to the de-Groot dual of the topology L_-. The relation $v \,\mathsf{con}\, u$ holds precisely when u and v are disjoint as opens of X, and likewise $u \,\mathsf{tot}\, v$ holds precisely when u and v cover X. Furthermore, every compact regular d-frame arises in this way.*

Remark 4. The equivalence with stably compact spaces requires the Axiom of Choice. It relies on the Hofmann-Mislove Theorem and the assertion that points of a locale can be described equivalently by completely prime filters of opens or by meet-prime opens.

Together with Lemma 4 we conclude:

Proposition 3. *The assignment $\mathcal{L} \mapsto \mathcal{L}_\diamond$ restricts to a functor from regular normal d-frames to the subcategory of compact regular d-frames. This functor is idempotent up to isomorphism.*

Theorem 1. *The open ideal completion functor $(-)_\diamond$ is a coreflection of the category of regular normal d-frames into the category of compact regular d-frames and the homomorphism ε defined in Lemma 4 (4) is its counit.*

Proof. We show that the map ε defined in the proof of Lemma 4 (4) is a natural transformation from the open ideal completion functor to the identity. Further, if $\mathcal{L}$ is a regular normal d-frame and $\mathcal{M}$ a compact regular d-frame, we show that every d-frame homomorphism $h : \mathcal{M} \to \mathcal{L}$ factors uniquely through the map $\varepsilon_{\mathcal{L}}$. We prove every statement for the positive component only, because the negative component works analogously.

To see that ε is a natural transformation, suppose that I is an open ideal of M_+ and $h : \mathcal{M} \to \mathcal{L}$ is a d-frame homomorphism. In particular h_+ preserves directed joins, so $h_+(\bigsqcup I) = \bigsqcup \{h_+(m) \mid m \in I\}$. But then also $h_+(\bigsqcup I) = \bigsqcup h_+^\diamond(I)$ where $h_+^\diamond = \mathrm{Idl}(h_+)$ is the map defined in the proof of Lemma 4 (2). Thus ε is a natural transformation.

Next consider the following diagram.

$$\begin{tikzcd} \mathrm{Idl}^{\lhd} M_+ \arrow[r, "h_+^\diamond"] & \mathrm{Idl}^{\lhd} L_+ \arrow[d, "\bigsqcup = (\varepsilon_{\mathcal{L}})_+"] \\ M_+ \arrow[u, "\downarrow"] \arrow[ur, dashed, "\tilde{h}_+"'] \arrow[r, "h_+"'] & L_+ \end{tikzcd} \tag{7}$$

By hypothesis $\mathcal{M}$ is compact regular, so the composition $\bigsqcup \circ \downarrow$ is the identity on M_+. With this we obtain $\bigsqcup \circ h_+^\diamond \circ \downarrow = h_+ \circ \bigsqcup \circ \downarrow = h_+$ and so the square in (7) commutes. Recall from Lemma 4 (2) that $h_+^\diamond$ is a frame homomorphism, and so is $\downarrow : M_+ \to \mathrm{Idl}^{\lhd} M_+$ because for compact regular d-frames it is actually an isomorphism. Hence $\tilde{h}_+ = h_+^\diamond \circ \downarrow$ is a frame homomorphism.

An immediate consequence of the factorisation $h_+ = \bigsqcup \circ \tilde{h}_+$ is that the open ideal completion functor is faithful on regular d-frames, since $h_+^\diamond = g_+^\diamond$ implies $h_+ = \bigsqcup \circ h_+^\diamond \circ \downarrow = \bigsqcup \circ g_+^\diamond \circ \downarrow = g_+$. Faithfulness of the open ideal completion functor now implies that the factorisation of h_+ in the diagram (7) is unique. Indeed, if $f : M_+ \to \mathrm{Idl}^{\lhd} L_+$ is any map with $(\varepsilon_{\mathcal{L}})_+ \circ f = h_+$ then $(\varepsilon_{\mathcal{L}})_+^\diamond \circ f^\diamond = h_+^\diamond$. But $(\varepsilon_{\mathcal{L}})_+^\diamond$ is an isomorphism, whence there is only one such $f^\diamond$.

The careful reader might have noticed that the statement about the diagram (7) holds in more generality. We state a surprising variant of that part of the proof above.

Theorem 2. *Let $\mathcal{M}$ be a regular d-frame and $h : \mathcal{M} \to \mathcal{L}$ a d-frame homomorphism. Then the components h_- and h_+ determine each other.*

Proof. We claim that the following diagram commutes, where the maps φ_+

and ψ_+ are the maps (5) and (6) from the proof of Lemma 3.

$$\begin{tikzcd}
\mathrm{Filt}\, M_- \arrow[r, "\mathrm{Filt}(h_-)"] & \mathrm{Filt}\, L_- \arrow[d, "\psi_+"] \\
\mathrm{Idl}\, M_+ \arrow[u, "\varphi_+"] & \mathrm{Idl}\, L_+ \arrow[d, "\bigsqcup"] \\
M_+ \arrow[u, "\downarrow"] \arrow[r, "h_+"'] & L_+
\end{tikzcd} \tag{8}$$

Suppose $m \in M_+$. Expanding the definition yields

$$(\psi_+ \circ \mathrm{Filt}(h_-) \circ \varphi_+ \circ \downarrow)(m) = \{v \in L_+ \mid \exists n \in M_-.\, v \,\mathsf{con}\, h_-(n),\, n \,\mathsf{tot}\, m\}\,. \tag{9}$$

By regularity of $\mathcal{M}$ we know that $h_+(m)$ is the join of the set

$$\{h_+(m') \mid \exists n \in M_-.\, m' \,\mathsf{con}\, n \,\mathsf{tot}\, m\}\,. \tag{10}$$

Since the d-frame homomorphism h preserves con we know that the set (10) is contained in the set (9). From preservation of tot we deduce that every element v of the set (9) satisfies $v \lhd h_+(m)$. Together this yields both inequalities of the desired identity.

Bitopological Stone duality is a contravariant duality between the category of d-frames and the category of bitopological spaces. Therefore a coreflection of d-frames corresponds to a reflection of spaces. In topology, the compact regular reflection of a space is known as the Stone-Čech compactification, whence we adopt the same name for our open ideal completion functor.

Example 2. Let $Q = \mathbb{Q} \cap [0,1]$ be the lattice of rationals in the unit interval. Define a relation $\prec$ on $[0,1]$ by $q \prec p$ if $q < p$. In addition let $0 \prec 0$ and $1 \prec 1$. We turn Q into a normal d-lattice $\mathcal{Q}$ by letting $Q_- = Q$ and $Q_+ = Q^\partial$. Consistency is given by $p \,\mathsf{con}\, q$ iff $q \leq p$ in Q and totality is given by $q \,\mathsf{tot}\, p$ iff $p \prec q$. One has $\lhd_- = \prec$ and $\lhd_+ = \succ$. Then the compactification $\mathcal{Q}_\diamond$ is the bitopological Stone dual of the closed unit interval with the lower and upper order topologies. The spectrum of $\mathcal{Q}_\diamond$ is in bijective correspondence with pairs of ideals $I_x \in \mathrm{Idl}^\lhd Q_-$ and $J_x \in \mathrm{Idl}^\lhd Q_+$ where $I_x = \{q \in Q \mid q \prec x\}$ and $J_x = \{p \in Q \mid x \prec p\}$ for some $x \in [0,1]$. Thus the compactification $\mathcal{Q}_\diamond$ can be regarded as the construction of the real unit interval by Dedekind cuts on the rationals.

4. Complete regularity and proximities

We wish to characterise the largest class of d-frames where the counit morphism of the Stone-Čech compactification has surjective component maps. The standard approach to the Stone-Čech compactification is via bounded real-valued functions. Kopperman uses a bitopological version of the unit interval which carries the topologies of upper and lower semicontinuity. We adopt this concept and demonstrate that the point-free version of the bitopological unit interval arises naturally in the standard proof of the Urysohn Lemma.

Definition 8. Let $\mathcal{I}$ denote the bitopological Stone dual of the unit interval $[0,1]$ with its lower and upper topologies. Its component frame R_- is the unit interval with an additional top element. We write $R_- = [0,1'] + \{1\}$. The algebraic order $\leq$ on R_- coincides with the frame order $\sqsubseteq_-$. The component frame R_+ is again isomorphic to the unit interval with an additional top element, but we write this as $\{0\} + [0',1]$ and let the algebraic order $\leq$ be the dual of the frame order $\sqsubseteq_+$. In R_- an element t of $[0,1']$ stands for the lower open $[0,t[$, so $1'$ corresponds to $[0,1[$. Dually, an element $s \in [0',1] \subseteq R_+$ corresponds to the open $]s,1]$ and thus $0'$ corresponds to $]0,1]$. Consistency and totality for pairs $(t,s) \in [0,1'] \times [0',1]$ is characterised as $t \,\mathsf{tot}\, s$ iff $t > s$ and $s \,\mathsf{con}\, t$ iff $s \geq t$.

Definition 9. Let $\mathcal{L}$ be a d-frame and $v_0, v_1 \in L_+$. Then v_0 is *really inside* v_1, written as $v_0 \lessdot_+ v_1$ if there exists a d-frame morphism $f : \mathcal{I} \to \mathcal{L}$ such that $v_0 \,\mathsf{con}\, f_-(1')$ and $f_+(0') \sqsubseteq v_1$. We say that f separates v_0 from v_1. Likewise, $u_0 \lessdot_- u_1$ if there exists $f : \mathcal{I} \to \mathcal{L}$ such that $f_+(0') \,\mathsf{con}\, u_0$ and $f_-(1') \sqsubseteq u_1$.

Remark 5. Johnstone [7, IV 1.4] characterises the really-inside relation on the opens of a locale A in the same way: $a \lessdot b$ iff there exists a locale map $f : A \to L(\mathbb{R})$ such that $f^*(0,\infty) \wedge a = 0_A$ and $f^*(-\infty,1) \leq b$.

Notice that in the d-frame $\mathcal{I}$ the relation $1' \,\mathsf{tot}\, 0'$ holds and d-frame homomorphisms preserve totality. As an immediate consequence the really-inside relation $\lessdot_+$ is contained in the well-inside relation $\lhd_+$. As we shall see, it is the largest interpolative auxiliary relation contained in the well-inside relation.

Another immediate consequence of the definition of really-inside relation is that d-frame homomorphisms preserve it. Indeed, if $f : \mathcal{I} \to \mathcal{L}$ is a d-frame homomorphism separating v_0 from v_1 in L_+ and $h : \mathcal{L} \to \mathcal{M}$ is another d-frame homomorphism, then $h \circ f : \mathcal{I} \to \mathcal{M}$ separates $h_+(v_0)$ from $h_+(v_1)$.

The *dyadic rationals* D is the set of rationals in the unit interval whose denominator is a power of two.

Lemma 5. *Let $\mathcal{L}$ be a d-frame and $u_0 \lessdot_+ u_1$ in L_-. Then $\{u_0, u_1\}$ extends to a dyadic-indexed chain $\{u_d\}_{d \in D}$ such that $d < e$ implies $u_d \lhd_- u_e$.*

Proof. By definition there exists a d-frame morphism $f : \mathcal{I} \to \mathcal{L}$ with the property that $f_+(0') \,\mathsf{con}\, u_0$ and $f_-(1') \sqsubseteq u_1$. Restrict f to the dyadic rationals in $[0,1']$ and $[0',1]$. Then the image of $D + \{1\}$ under f_- yields the desired chain. Indeed, pick a dyadic rational $0 < d < 1$. We have $d \,\mathsf{con}\, d$ and therefore $u_1 \sqsupseteq f_-(1') \,\mathsf{tot}\, f_+(d) \,\mathsf{con}\, f_-(d)$ which shows $f_-(d) \lhd_- u_1$. To see that $u_0 \lhd_- f_-(d)$, observe $f_-(d) \,\mathsf{tot}\, f_+(0') \,\mathsf{con}\, u_0$. Finally, $0 < d < e < 1$ implies $d \lhd_- e$ in R_- whence $f_-(d) \lhd_- f_-(e)$. Thus we can define $u_d = f_-(d)$ for $0 < d < 1$ and have the desired chain.

Corollary 1. *The really-inside relation satisfies axioms (i)–(v) from the introduction.*

Proof. We showed that $\lessdot_- \subseteq \lhd_-$ which in particular implies that $\lessdot_-$ is contained in the order $\sqsubseteq_-$. The axioms (ii)–(iv) are inherited from the well-inside relation. Notice that the dyadic rationals are self-similar, as $D \cap [0, \frac{1}{2}]$ and $D \cap [\frac{1}{2}, 1]$ are both order-isomorphic to D. With this and Lemma 5 one shows that whenever $u_0 \lessdot_- u_1$ then $u_0 \lessdot_- u_{\frac{1}{2}} \lessdot_- u_1$, where $u_{\frac{1}{2}}$ is taken from the dyadic-indexed chain constructed in the proof of the Lemma.

A dyadic-indexed chain $u_0 \lhd_- \ldots \lhd_- u_d \lhd \ldots \lhd_- u_1$ is called a *scale* between u_0 and u_1. Lemma 5 has a converse, which is essentially the content of the Urysohn Lemma. Indeed, in the classical formulation of the Urysohn Lemma for normal spaces one uses the fact that the well-inside relation is interpolative and therefore coincides with the really-inside relation. Our version of the Urysohn Lemma highlights the bitopological and point-free nature of the classical proof.

Lemma 6 (Urysohn Lemma for d-frames). *Suppose $\mathcal{L}$ is a d-frame and u_0 and u_1 are elements of L_-. Suppose there exists a scale $\{u_d\}_{d\in D}$ between u_0 and u_1. Then u_0 is really inside u_1.*

Proof. We begin by building a scale on L_+ from the given scale on L_-. For each dyadic rational d define v_d to be the d-frame theoretic pseudocomplement of u_d. The pseudocomplement operation is antitone with respect to the frame orders. We extend the ascending chain $\{u_d\}_{d\in D}$ and the descending chain $\{v_d\}_{d\in D}$ to a d-frame homomorphism on $\mathcal{I}$ as follows. Set $f_-(t) = \bigsqcup_{d<t} u_d$ for $s \sqsubseteq 1'$ and $f_-(1) = 1$. Likewise, define $f_+(s) = \bigsqcup_{e>s} v_e$ for $s \sqsubseteq 0'$ and $f_+(0) = 1$. These are indeed frame homomorphisms, because they preserve the top element by construction and also the least element, since for example $f_-(0) = \bigsqcup \emptyset$. Monotonicity is enough to enforce preservation of finite meets. Preservation of arbitrary joins follows from the identity $\{d \mid d < t\} = \bigcup_{t'<t} \{d \mid d < t'\}$. By construction u_0 is consistent with every v_e, whence it is also consistent with $f_+(0')$. Further it is obvious that $f_-(1')$ is below u_1 because the u_d form an ascending chain.

It remains to show that $f = (f_-, f_+)$ preserves con and tot. For $t \in [0, 1']$ and $s \in [0', 1]$ we have $s \,\mathsf{con}\, t$ iff $t \leq s$. If d and e are dyadic rationals with $d < t \leq s < e$ then $d < e$ and therefore $u_d \,\mathsf{con}\, \neg u_d = v_d \sqsupseteq v_e$. It follows that $f_+(s) \,\mathsf{con} f_-(t)$. Likewise we have $t \,\mathsf{tot}\, s$ iff $s < t$. In that situation we can find dyadic rationals d and e with $s < e < d < t$. Then $u_e \lhd_- u_d$ and by the characterisation of the well-inside relation we know $u_d \,\mathsf{tot}\, \neg u_e = v_e$. We conclude $f_-(t) \,\mathsf{tot}\, f_+(s)$. Summing up, we have constructed a d-frame homomorphism $f : \mathcal{I} \to \mathcal{L}$ separating u_0 from u_1, so by definition u_0 is really-inside u_1.

Remark 6. In the classical proof of the Urysohn Lemma for a normal space X, one starts with two disjoint closed sets. These yield a pair of opens U_0, U_1 such that the closure of U_0 is contained in U_1. Using the interpolation property of the well-inside relation one extends this to a scale $\{U_d\}_{d\in D}$ where $e < d$ implies that the closure of U_e is contained in U_d. From this one constructs an upper semicontinuous map $X \to [0,1]$ separating U_0 from U_1, where in fact the frame homomorphism from the opens of the lower topology on the unit interval into the topology of the space X is defined first. Using the same chain of opens, one constructs a lower semicontinuous map – again via its frame homomorphism. Then one shows that these two maps are in fact the same. For the construction of the lower semicontinuous map it is crucial that the intersections of the form $\bigcap_{d>t} U_d$ are closed.

Definition 10. A d-frame $\mathcal{L}$ is *completely regular* if every element of each component frame is the join of the elements really inside it. In domain theoretic terms, completely regular d-frames are those which have approximating really-inside relations.

Corollary 2. *Regular normal d-frames are completely regular.*

Proof. For any normal d-frame, one can use countable dependent choice and the interpolation property of the well-inside relation to show that the well-inside relation and the really-inside relation agree. Regularity then implies complete regularity.

Before we relate complete regularity to compactifications of d-frames, we need to specify what we mean by a "compactification". A compactification of a topological space X is a topological embedding $X \hookrightarrow Y$ of X as a dense subspace of a compact Hausdorff space Y. In our theory the compact regular d-frames take the place of compact Hausdorff spaces, and just like in locale theory subspace embeddings are replaced with surjective frame homomorphisms. But recall from Proposition 2 that the component frames of a compact regular d-frame are stably compact topologies and thus in general far from Hausdorff. Therefore we need to modify the standard notion of density to a stronger notion which works for T_0 spaces as well. The idea is due to Smyth [11].

Every frame homomorphism $f : B \to A$ has a right adjoint $f_* : A \to B$ which is constructed as $f_*(a) = \bigsqcup\{b \in B \mid f(b) \sqsubseteq a\}$. It is implicitly defined by the equivalence $f(b) \sqsubseteq a \Leftrightarrow b \sqsubseteq f_*(a)$ for all $a \in A$ and $b \in B$. If f is surjective, then $f \circ f_*$ is the identity on A.

Definition 11. Let $f : B \to A$ be a frame homomorphism and $\prec$ be an auxiliary relation on B. We say that f is *dense with respect to* $\prec$, or $\prec$-dense for short, if $b' \prec b$ implies that there exists an $a \in A$ with $b' \prec f_*(a) \prec b$.

Observe that in case the auxiliary relation $\prec$ satisfies $0 \prec 0$ then a $\prec$-dense surjective frame homomorphism is in particular dense in the sense of locale theory, because $0 \prec f_*(a) \prec 0$ implies $a = f(f_*(a)) \sqsubseteq f(0) = 0$ and so $f_*(0) = 0$.

Lemma 7. *Let $h : B \twoheadrightarrow A$ be a surjective frame homomorphism. Suppose $\prec$ is an auxiliary relation on B and f is $\prec$-dense. The following are equivalent:*

1. $h_*(a') \prec h_*(a)$.

2. *There exist* $b' \prec b$ *with* $a' \sqsubseteq h(b')$ *and* $h(b) \sqsubseteq a$.

Proof. For surjective frame homomorphisms $h : B \to A$ the composite $h \circ h_*$ is the identity on A. Therefore, if $h_*(a') \prec h_*(a)$ then one can choose $b' = h_*(a')$ and $b = h_*(a)$ and obtain the implication (1) $\Rightarrow$ (2). For the converse implication, suppose that $b' \prec b$, $a' \sqsubseteq h(b')$ and $h(b) \sqsubseteq a$. By hypothesis there exists some $a_0 \in A$ with $b' \prec h_*(a_0) \prec b$. The auxiliary relation $\prec$ is contained in the order $\sqsubseteq$ and h preserves the order, whence $a' \sqsubseteq a_0 \sqsubseteq a$. The right adjoint h_* is monotone as well, whence $h_*(a') \sqsubseteq h_*(a_0)$. Now use the fact that h_* is the right adjoint to h and deduce $b \sqsubseteq h_*(a)$. Together we have $h_*(a') \sqsubseteq h_*(a_0) \prec b \sqsubseteq h_*(a)$ and thereby $h_*(a') \prec h_*(a)$.

Definition 12. A *compactification* of a d-frame $\mathcal{L}$ is a d-frame homomorphism $f : \mathcal{M} \to \mathcal{L}$ where $\mathcal{M}$ is compact regular and the component frame homomorphisms f_- and f_+ are surjective and dense with respect to the well-inside relations.

Proposition 4. *If* $f : \mathcal{M} \twoheadrightarrow \mathcal{L}$ *is a d-frame morphism on a compact regular d-frame and both components* f_- *and* f_+ *are surjective, then* $\mathcal{L}$ *is completely regular.*

Proof. Fix an element v_0 of L_+. Since f_+ is surjective, there exists some $m_0 \in M_+$ with $v_0 = f_+(m_0)$. Now $\mathcal{M}$ is completely regular whence $m_0 = \bigsqcup\{m \in M_+ \mid m \lessdot_+ m_0\}$. The frame homomorphism f_+ preserves all joins and the relation $\lessdot_+$, so $v_0 = \bigsqcup\{f_+(m) \in M_+ \mid m \lessdot_+ m_0\}$. Since v_0 was chosen arbitrary, $\mathcal{L}$ is completely regular.

We go on to show that any completely regular d-frame admits a largest compactification and on the way characterise all compactifications. The central tool in this endeavour is a d-frame version of *proximity*.

Definition 13. A *proximity* on a d-frame $\mathcal{L}$ consists of a pair of approximating quasi-proximities $\prec_-$ on L_- and $\prec_+$ on L_+. The quasi-proximities must be contained in $\lhd_-$ and $\lhd_+$, respectively. Moreover, the following relational identities are required to hold, where $\succ_-$ is the relational inverse of $\prec_-$.

$$(\prec_+; \mathsf{con}) = (\mathsf{con}; \succ_-) \text{ and } (\mathsf{tot}; \prec_+) = (\succ_-; \mathsf{tot}) \tag{11}$$

The relations $\prec_-$ and $\prec_+$ are called the *component quasi-proximities* of the proximity.

If $\prec_+$ is a component quasi-proximity of a proximity then $\prec_+ = \lhd_+; \prec_+$. Indeed, $\prec_+$ is contained in $\lhd_+; \prec_+$ because $\prec_+$ is interpolative and contained in $\lhd_+$. Conversely, $\lhd_+; \prec_+$ is contained in $\prec_+$ because $\prec_+$ satisfies axiom (ii) and $\lhd_+$ satisfies axiom (i) from the introduction. Given $v_0 \prec_+ v_1$ one constructs a scale between v_0 and v_1 using countable dependent choice which shows that $\prec_+$ must be contained in the really-inside relation $\lessdot_+$. Since both component quasi-proximities are approximating, any d-frame which admits a proximity on it must be completely regular. The pair $(\lessdot_-, \lessdot_+)$ is the largest proximity for every completely regular d-frame:

Lemma 8. *On any completely regular d-frame $\mathcal{L}$ the really-inside relations form a proximity.*

Proof. Most properties of Definition 13 follow from the characterisation in Lemma 5 and the algebraic properties of the well-inside relations. The only non-trivial fact are the identities (11). We show only one inclusion of each identity, since the other inclusion is dually proved by swapping the signs. Suppose $v_0 \lessdot_+ v_1 \,\mathsf{con}\, u$. Then there exists a d-frame homomorphism $f : \mathcal{I} \to \mathcal{L}$ with $v_0 \,\mathsf{con} f_-(1')$ and $f_+(0') \sqsubseteq v_1 \,\mathsf{con}\, u$. Then also $f_+(0') \,\mathsf{con}\, u$ and so $v_0 \,\mathsf{con} f_-(1') \gtrdot_- u$. Suppose $u \,\mathsf{tot}\, v_0 \lessdot_+ v_1$. Let f be as before. In the d-frame $\mathcal{I}$ we have $1' \gtrdot_- \frac{1}{2} \,\mathsf{tot}\, 0'$ and f preserves these relations, whence $u \sqsupseteq f_-(1') \gtrdot_- f_-(\frac{1}{2}) \,\mathsf{tot}\, f_+(0') \sqsubseteq v_1$ and therefore $u \gtrdot_- f_-(\frac{1}{2}) \,\mathsf{tot}\, v_1$.

As promised in the abstract, we present a "normalisation" construction for d-frames which in particular yields a regular normal coreflection of completely regular d-frames.

Lemma 9. *Let $\mathcal{L}$ be a d-frame and $(\prec_-, \prec_+)$ be a pair of relations which satisfy all axioms of a proximity except that the relations do not need to be approximating.*

1. *Define a relation $\mathsf{tot}^{\prec} = \mathsf{tot}; \prec_+$. The structure $\mathcal{L}^{\prec} = (L_-, L_+, \mathsf{con}, \mathsf{tot}^{\prec})$ is a normal d-frame and the well-inside relations agree with the component quasi-proximities.*
2. *The normal d-frame $\mathcal{L}^{\prec}$ is regular if and only if $(\prec_-, \prec_+)$ is a proximity.*

Proof. First notice that by moving from tot to $\mathsf{tot}; \prec_+$ we do not break any axioms of Figure 1. This is because $\prec_+$ has all necessary algebraic properties. Because of the second identity in (11) it does not matter whether we define $\mathsf{tot}^{\prec}$ as $\mathsf{tot}; \prec_+$ or as $\succ_-; \mathsf{tot}$. Even without $\prec_+$ being approximating, the identity $\prec_+; \lhd_+ = \prec_+$ holds. With this we write

$$
\begin{aligned}
\mathsf{tot}^{\prec}; \mathsf{con}; \mathsf{tot}^{\prec} &= \mathsf{tot}; \prec_+; \mathsf{con}; \mathsf{tot}; \prec_+ \\
&= \mathsf{tot}; \prec_+; \lhd_+; \prec_+ \\
&= \mathsf{tot}; \prec_+; \prec_+ \\
&= \mathsf{tot}; \prec_+ \\
&= \mathsf{tot}^{\prec}.
\end{aligned}
$$

The identity we just proved is precisely normality of the d-frame $\mathcal{L}^{\prec}$. To see that the well-inside relations of $\mathcal{L}^{\prec}$ coincide with $\prec_-$ and $\prec_+$, write

$$\mathsf{con}; \mathsf{tot}^{\prec} = \mathsf{con}; \mathsf{tot}; \prec_+ = \lhd_+; \prec_+ = \prec_+ .$$

Likewise,

$$\mathsf{tot}^{\prec}; \mathsf{con} = \mathsf{tot}; \prec_+; \mathsf{con} = \succ_-; \mathsf{tot}; \mathsf{con} = \succ_- .$$

Therefore regularity of the d-frame $\mathcal{L}^{\prec}$ is equivalent to $\prec_+$ and $\prec_-$ being approximating.

Theorem 3 (The normal coreflection). *The category of d-frames has a normal coreflection. This coreflection takes completely regular d-frames to regular normal d-frames.*

Proof. Instantiate the construction of Lemma 9 to the proximity $(\leqslant_-, \leqslant_+)$. We know that d-frame morphisms preserve the really-inside relations whence the assignment $\mathcal{L} \mapsto \mathcal{L}^{\leqslant}$ is functorial. The counit of this coreflection is simply the pair of identity frame homomorphisms $\mathcal{L}^{\leqslant} \to \mathcal{L}$ which trivially preserve consistency and also totality because $\mathsf{tot}; \leqslant_+$ is contained in tot. Every d-frame morphism $f : \mathcal{N} \to \mathcal{L}$ from a normal d-frame into $\mathcal{L}$ factors uniquely through $\mathcal{L}^{\leqslant}$ because on $\mathcal{N}$ the relation tot coincides with $\mathsf{tot}; \leqslant_+$ and any d-frame morphism preserves the relation $\mathsf{tot}; \leqslant_+$.

Remark 7. The category of topological spaces does not have a normal reflection. Thus the theorem above underlines our claim that the category of d-frames is more than just a reformulation of bitopology or locale theory.

Corollary 3 (Stone-Čech compactification of d-frames). *The category of completely regular d-frames coreflects into the category of compact regular d-frames.*

Given any proximity $(\prec_-, \prec_+)$ on a completely regular d-frame $\mathcal{L}$, one can form the regular normal d-frame $\mathcal{L}^{\prec}$, then its Stone-Čech compactification $(\mathcal{L}^{\prec})_\diamond$ and so obtain a compactification associated with the proximity. We show that every compactification arises in this way.

Theorem 4. *Let $\mathcal{L}$ be a completely regular d-frame. There is a bijection between compactifications of $\mathcal{L}$ and proximities on $\mathcal{L}$.*

Proof. We already know that any proximity $(\prec_-, \prec_+)$ induces a compact regular d-frame $(\mathcal{L}^{\prec})_\diamond$. The component frames of this compact regular d-frame are the round ideal completions $\mathrm{Idl}^{\prec} L_-$ and $\mathrm{Idl}^{\prec} L_+$. The corresponding surjective d-frame homomorphism is the pair of join operations $\bigsqcup : \mathrm{Idl}^{\prec} L_- \to L_-$ and $\bigsqcup : \mathrm{Idl}^{\prec} L_+ \to L_+$. It is not hard to see that their right adjoints are simply the principal round ideal maps $u \mapsto \{u' \in L_- \mid u' \prec_- u\}$ and $v \mapsto \{v' \in L_+ \mid v' \prec_+ v\}$. By the characterisation of the way-below relations in Lemma 2 it is obvious that the join operations are dense with respect to the way-below relations. Thus $(\mathcal{L}^{\prec})_\diamond$ is indeed a compactification of $\mathcal{L}$.

Given a compactification $f : \mathcal{M} \to \mathcal{L}$ we construct a proximity as follows.

$$u_0 \prec_- u_1 \quad :\Leftrightarrow \quad f_{-*}(u_0) \lhd_- f_{-*}(u_1) \tag{12}$$

$$v_0 \prec_+ v_1 \quad :\Leftrightarrow \quad f_{+*}(v_0) \lhd_+ f_{+*}(v_1) \tag{13}$$

The definitions above are by Lemma 7 equivalent to

$$u_0 \prec_- u_1 \quad \Leftrightarrow \quad \exists m_0 \lhd_- m_1 .\, u_0 \sqsubseteq f_-(m_0) \text{ and } f_-(m_1) \sqsubseteq u_1 \tag{14}$$

$$v_0 \prec_+ v_1 \quad \Leftrightarrow \quad \exists n_0 \lhd_+ n_1 .\, v_0 \sqsubseteq f_+(n_0) \text{ and } f_+(n_1) \sqsubseteq v_1 \tag{15}$$

Therefore all defining properties of a proximity except the identities (11) are obvious. Suppose $v_0 \prec_+ v_1 \operatorname{\mathsf{con}} u$. By definition there are $n_0 \lhd_+ n_1$ in M_+ with $v_0 \sqsubseteq f_+(n_0) \lessdot_+ f_+(n_1) \sqsubseteq v_1 \operatorname{\mathsf{con}} u$. Using normality of $\mathcal{M}$ we expand $n_0 \lhd_+ n_1$ to $n_0 \operatorname{\mathsf{con}} m_1 \rhd_- m_0 \operatorname{\mathsf{tot}} n_1$. We map all these elements through f and obtain $v_0 \operatorname{\mathsf{con}} f_-(m_1) \succ_- u$. Hence $\prec_+; \mathsf{con}$ is contained in $\mathsf{con}; \succ_-$. The other inclusion is proved dually. Now suppose $u \operatorname{\mathsf{tot}} v_0 \prec_+ v_1$. Again, use the definition and expand $n_0 \lhd_+ n_1$ as above. The image under f yields $u \succ_- f_-(m_0) \operatorname{\mathsf{tot}} v_1$. Thus $\mathsf{tot}; \prec_+$ is contained in $\succ_-; \mathsf{tot}$. The other inclusion is proved dually.

It remains to show that the two constructions above are mutually inverse. Given a compactification $f : \mathcal{M} \to \mathcal{L}$ and the induced proximity as defined in (12) and (13), we want to show that $\mathcal{M}$ is isomorphic to $(\mathcal{L}^{\prec})_\diamond$. Since $f_+ \circ f_{+*}$ is the identity on L_+, the right adjoint f_{+*} must be injective and thereby an order-embedding from L_+ into M_+. Regularity of M_+ and $\lhd_+$-density implies that we can regard L_+ as a basis of M_+, and the relation $\prec_+$ translates to $\lhd_+$. We deduce that $\mathrm{Idl}^{\prec} L_+$ and $\mathrm{Idl}^{\lhd} M_+$ are isomorphic. But M_+ is a domain and $\lhd_+$ its way-below relation, whence $\mathrm{Idl}^{\lhd} M_+$ is isomorphic to M_+. From Proposition 2 we know that any compact regular d-frame is completely determined by one of its component frames, whence $\mathrm{Idl}^{\prec} L_+ \cong M_+$ is enough to deduce $(\mathcal{L}^{\prec})_\diamond \cong \mathcal{M}$.

Now suppose $(<_-, <_+)$ is a proximity on $\mathcal{L}$, and let $(\prec_-, \prec_+)$ be the proximity induced by the compactification $(\mathcal{L}^{<})_\diamond$. We show $<_+ = \prec_+$. By Lemma 7 we know $v_0 \prec_+ v_1$ if and only if there exist ideals $I_0 \ll I_1$ in $\mathrm{Idl}^{<} L_+$ with $v_0 \sqsubseteq \bigsqcup I_0$ and $\bigsqcup I_1 \sqsubseteq u_1$. Observe that $I_0 \ll I_1$ implies that $\bigsqcup I_0 \in I_1$, and since I_1 is round with respect to $<_+$ there exists some $v \in I_1$ such that $v_0 \sqsubseteq \bigsqcup I_0 <_+ v \sqsubseteq v_1$. Thus $\prec_+$ is contained in $<_+$. Conversely, $v_0 <_+ v_1$ implies that the round ideal $I_0 := \{v \in L_+ \mid v <_+ v_0\}$ is way-below the round ideal $I_1 := \{v \in L_+ \mid v <_+ v_1\}$. Furthermore $v_0 = \bigsqcup I_0$ and $v_1 = \bigsqcup I_1$ because the relation $<_+$ is approximating. Hence $v_0 \prec_+ v_1$ and thus $<_+$ is contained in $\prec_+$. The same argument applies to $<_-$ and $\prec_-$.

Remark 8. The proof above essentially does Smyth's proof twice in parallel. The idea for the proof of $(\mathcal{L}^{\prec})_\diamond \cong \mathcal{M}$ is precisely the argument Smyth used in [11] to show that every stable compactification arises from a quasi-proximity. The definition of the proximity induced by a compactification appears in both the work of Smyth and Banaschewski, where Banaschewski uses the right adjoint characterisation (12) and (13), and Smyth uses the characterisation from Lemma 7 (2). If we consider d-frames $\mathcal{L}$ where $L_- = L_+$, $u \operatorname{\mathsf{tot}} v$ iff $u \sqcup v = 1$ and $v \operatorname{\mathsf{con}} u$ iff $v \sqcap u = 0$ then our theorem collapses to Banaschewski's result about compactifications of frames. But we lose the intermediate normal coreflection in doing this.

The correspondence between proximities and compactifications is more than a bijection: If a proximity $(\prec_-, \prec_+)$ is contained in another proximity $(<_-, <_+)$ then the compactification corresponding to the former factors through the compactification corresponding to the latter. Indeed, open ideals with respect to $\prec_+$ form a sub-frame of the open ideals with respect to $<_+$. Thus sub-frame inclusion provides the required factorisation. Conversely, if a compactification $f : \mathcal{M} \to \mathcal{L}$ factors through another compactification $g : \mathcal{N} \to \mathcal{L}$ then the proximity on $\mathcal{L}$ which corresponds to f is easily seen to be contained in the one generated by g.

Remark 9. The d-frame of the real line has precisely one proximity. This is because the way-below relation on a component frame coincides with the well-inside relation except for $\mathbb{R} \not\ll \mathbb{R}$. Any approximating relation $\prec$ must contain the way-below relation. Thus a component quasi-proximity $\prec_+$ has $\ll \subseteq \prec_+ \subseteq \lhd_+$ which renders $\prec_+$ unique. Consequently, the bitopological reals $\mathcal{L}\mathbb{R}$ have precisely one compactification, which is the d-frame of the extended reals $[-\infty, +\infty]$ endowed with the lower and upper topologies.

5. Classical point-free compactifications

In this section we show how to obtain classical point-free compactifications of frames using our bitopological framework. In particular, the compact regular coreflection of completely regular frames is presented by pre- and post-composing the compact regular coreflection of completely regular d-frames with suitable functors from and to frames. The constructions presented are interesting in their own right because they provide the link between the theories of frames and d-frames.

Definition 14. Let A be a frame. Then $A_=$ denotes the *symmetric d-frame* $(A, A, \mathsf{con}_=, \mathsf{tot}_=)$ where $a \,\mathsf{con}_=\, b$ iff $a \sqcap b = 0$ and $a \,\mathsf{tot}_=\, b$ iff $a \sqcup b = 1$. This definition obviously extends to a functor $(-)_= : \mathsf{Frm} \to \mathsf{dFrm}$. We call the image of the functor $(-)_=$ the subcategory of symmetric d-frames.

Proposition 5. *Let A be a frame. A is regular if and only if the symmetric d-frame $A_=$ is regular. A is compact if and only if $A_=$ is compact. A is normal if and only if $A_=$ is normal. If A is completely regular, so is $A_=$.*

Proof. Regularity and normality are straightforward to check. A frame A is compact if and only if the top element singleton $\{1\}$ is a Scott open subset of A. Clearly, if $\mathsf{tot}_=$ is a Scott open subset of $A \times A$ then $\{1\}$ is Scott open in A. The converse is also true because a directed subset of $A \times A$ translates to a directed subset of A via $\{(a_i, b_i)\}_{i \in I} \mapsto \{a_i \sqcup b_i\}_{i \in I}$. For complete regularity, let $\mathcal{O}[0,1]$ denote the opens of the Euclidean topology on the unit interval. A is completely regular iff frame homomorphisms $\mathcal{O}[0,1] \to A$ separate opens. There are subframe embeddings $e_-, e_+ : [0,1'] + \{1\} \hookrightarrow \mathcal{O}[0,1]$ corresponding to the upper and lower order topologies. Thus, if $f : \mathcal{O}[0,1] \to A$ is a frame homomorphism witnessing $a \lessdot b$ in A then $(f \circ e_-, f \circ e_+)$ is a d-frame homomorphism $\mathcal{I} \to A_=$ witnessing $a \lessdot_+ b$.

The following definition is the point-free analogue of the so-called patch topology, the common refinement of two topologies. It was used in [8] for an adjunction between d-frames and Banaschewski's *biframes* [2]. The patch frame is presented in terms of generators and relations. For a detailed account of this technique consult [12].

Definition 15. Let $\mathcal{L}$ be a d-frame. Then $\mathrm{Patch}\,\mathcal{L}$ is the frame with generators $\{\ulcorner u\urcorner^- \mid u \in L_-\} \cup \{\ulcorner v\urcorner^+ \mid v \in L_+\}$ subject to the relations that the pair $(\ulcorner - \urcorner^-, \ulcorner - \urcorner^+)$ is a d-frame homomorphism from $\mathcal{L}$ to $(\mathrm{Patch}\,\mathcal{L})_=$. The frame $\mathrm{Patch}\,\mathcal{L}$ is called the *patch frame* of $\mathcal{L}$.

The patch construction extends to a functor $\mathsf{dFrm} \to \mathsf{Frm}$.

Proposition 6. *The functor $(-)_=$ is right adjoint to the functor* Patch*. Moreover, for a regular frame A the composite* $\mathrm{Patch}(A_=)$ *is isomorphic to A.*

Proof. For the proof it is convenient to use the pseudocomplement $\neg$ of the frame A and the characterisation of the well-inside relation of a frame as $a \lhd b$ iff $\neg a \sqcup b = 1$. By definition the unit $\eta_{\mathcal{L}} = (\ulcorner - \urcorner^-, \ulcorner - \urcorner^+)$ is a d-frame homomorphism. For a frame A and a d-frame homomorphism $f : \mathcal{L} \to A_=$ define a frame homomorphism $\ulcorner f \urcorner$ by its action on the generators: $\ulcorner u\urcorner^- \mapsto f_-(u)$ and $\ulcorner v\urcorner^+ \mapsto f_+(v)$. This extends to a well-defined frame homomorphism precisely because f preserves con and tot. Clearly $\ulcorner f \urcorner_= \circ \eta_{\mathcal{L}}$ equals f. For uniqueness, observe that any frame homomorphism $h : \mathrm{Patch}\,\mathcal{L} \to A$ with $h_= \circ \eta_{\mathcal{L}} = f$ must coincide with $\ulcorner f \urcorner$ on the generators of $\mathrm{Patch}\,\mathcal{L}$.

Note that for every element a of a frame A the patch frame $\mathrm{Patch}(A_=)$ has two generators $\ulcorner a\urcorner^-$ and $\ulcorner a\urcorner^+$. Suppose A is a regular frame, and suppose $b \lhd a$ in A, that is $a \sqcup \neg b = 1$. Since $\ulcorner - \urcorner^-$ preserves binary joins and meets we obtain $\ulcorner a\urcorner^- \sqcup \ulcorner \neg b\urcorner^- = 1$ and $\ulcorner b\urcorner^- \sqcap \ulcorner \neg b\urcorner^- = 0$ whence $\ulcorner b\urcorner^- \lhd \ulcorner a\urcorner^-$. From $b \sqcap \neg b = 0$ in A it follows that $\neg b \,\mathsf{con}_=\, b$ and thereby $\ulcorner b\urcorner^- \sqcap \ulcorner \neg b\urcorner^+ = 0$. Then clearly $\ulcorner b\urcorner^- \sqcap \ulcorner \neg b\urcorner^+ \sqsubseteq \ulcorner a\urcorner^+ \sqcap \ulcorner \neg b\urcorner^+$. Also $\ulcorner a\urcorner^+ \sqcup \ulcorner \neg b\urcorner^+ = 1$ because of $\neg b \sqcup a = 1$, whence $\ulcorner b\urcorner^- \sqcup \ulcorner \neg b\urcorner^+ \sqsubseteq \ulcorner a\urcorner^+ \sqcup \ulcorner \neg b\urcorner^+$. The last two inequalities together imply $\ulcorner b\urcorner^- \sqsubseteq \ulcorner a\urcorner^+$ because $\mathrm{Patch}(A_=)$ is a distributive lattice. With regularity one obtains $\ulcorner a\urcorner^- = \bigsqcup_{b \lhd a} \ulcorner b\urcorner^- \sqsubseteq \ulcorner a\urcorner^+$ and swapping the signs shows that in fact $\ulcorner a\urcorner^- = \ulcorner a\urcorner^+$.

Corollary 4. *If a symmetric d-frame $A_=$ is completely regular in the d-frame sense then A is completely regular.*

Proof. If $A_=$ is completely regular then d-frame morphisms $\mathcal{I} \to A_=$ separate the opens. Further, the patch frame of $\mathcal{I}$ is the Euclidean topology $\mathcal{O}[0,1]$ on the unit interval. Hence a d-frame morphism $f : \mathcal{I} \to A_=$ witnessing $a \lessdot_+ b$ in the d-frame $A_=$ translates to a witness $\mathrm{Patch}(f) : \mathcal{O}[0,1] \to A$ for $a \lessdot b$ in A.

Proposition 7. *The functor $(-)_=$ is full on regular frames. The category of regular frames is equivalent to the category of symmetric regular d-frames.*

Proof. Let $(f, g) : A_= \to L_=$ be a d-frame morphism where A is a regular frame. Notice that a is well-inside b in the frame sense if and only if this relation holds in the d-frame sense in $A_=$. We mix these characterisations, saying $a \,\mathsf{con}_=\, x$ and $x \sqcup b = 1$ for some x. Using the preservation of $\mathsf{con}_=$ and

$\mathsf{tot}_=$ one deduces $g(a)\,\mathsf{con}_=\,f(x)$ and $f(x) \sqcup f(b) = 1$. But the latter fact can be expressed as $f(x)\,\mathsf{tot}_=\,f(b)$ whereby $g(a) \lhd f(b)$ in A. Now use regularity to deduce $g(b) = f(b)$.

Let π_- be the forgetful functor from d-frames to frames which sends a d-frame homomorphism $(f_-, f_+) : \mathcal{M} \to \mathcal{L}$ to its first component $f_- : M_- \to L_-$. When restricted to symmetric regular d-frames, π_- is the inverse to $(-)_=$.

Remark 10. Normality for d-frames is a much more inclusive concept than normality for frames. In fact every completely regular frame arises as the patch frame of some regular normal d-frame. To see this, observe that the second part of the proof of Proposition 6 does not use the relation $\mathsf{tot}_=$. It follows that, as long as $\mathsf{tot}' \subseteq \mathsf{tot}_=$ and $(A, A, \mathsf{con}_=, \mathsf{tot}')$ remains regular, the patch frame of this d-frame will be isomorphic to A. With $\mathsf{tot}' = \mathsf{tot}^{\leqslant}$, the identity functor on completely regular frames factors as $\mathrm{Patch} \circ (-)^{\leqslant} \circ (-)_=$.

Now we have all tools to factor the Stone-Čech compactification of frames through d-frames.

Theorem 5. *The Stone-Čech compactification of frames factors through the Stone-Čech compactification of d-frames.*

Proof. The functor $(-)_=$ restricts to an equivalence between completely regular frames and completely regular symmetric d-frames by Propositions 5 and 7. The coreflection $((-)^{\leqslant})_\diamond$ of symmetric completely regular d-frames into compact regular d-frames takes the d-frame $A_=$ to the d-frame $(\mathrm{Idl}^{\leqslant} A, \mathrm{Idl}^{\leqslant} A, \mathsf{con}_\diamond, \mathsf{tot}_\diamond)$ where $I\,\mathsf{con}_\diamond\,J$ iff $I \times J \subseteq \mathsf{con}_=$ iff $I \wedge J = \{0\}$ and $I\,\mathsf{tot}_\diamond\,J$ iff $(I \times J) \cap (\mathsf{tot}_=; \geqslant) \neq \emptyset$. If $a \in I$, $b \in J$ and $a \sqcup b' = 1$ for some $b' \leqslant b$ then clearly $a \sqcup b = 1$ and thus $I \vee J = A$. Conversely, if $I \vee J = A$ then $a \sqcup b' = 1$ for some $a \in I$ and $b' \in J$. But J is round with respect to $\leqslant$, whence $b' \leqslant b$ for some $b \in J$ and thus $I\,\mathsf{tot}_\diamond\,J$. We conclude that the compactification $((A_=)^{\leqslant})_\diamond$ is symmetric (and regular). Therefore we can post-compose the functor $((-)^{\leqslant})_\diamond$ with the equivalence between regular symmetric d-frames and regular frames.

Other compactifications, although not functorial in general, can be presented in the same manner.

Proposition 8. *Let A be a (completely regular) frame and $\prec$ be a strong inclusion on A as defined in the introduction, meaning $\prec$ satisfies axioms (i)–(viii). Then the pair $(\prec, \prec)$ is a proximity on the symmetric d-frame $A_=$ in the sense of Definition 13.*

Proof. It suffices to show that the identities $\prec; \mathsf{con}_= \;=\; \mathsf{con}_=; \succ$ and $\mathsf{tot}_=; \prec \;=\; \succ; \mathsf{tot}_=$ hold. Suppose $a \prec b'$ and $b' \sqcap b = 0$ in A. Let again $\neg$ denote the pseudocomplement operation on A. Then $b' \sqsubseteq \neg b'$ and so $a \sqcap \neg a = 0$ and $\neg a \succ \neg b' \sqsupseteq b$. Hence $\prec; \mathsf{con}_= \;\subseteq \succ; \mathsf{con}_=$. The other inclusion is shown similarly.

Now suppose $a \sqcup a' = 1$ and $a' \prec b$. Interpolate: $a' \prec b' \prec b$ for some $b' \in A$. Since $\prec$ is contained in the well-inside relation of A we know that $b' \prec b$

implies $\neg b' \sqcup b = 1$. From $a' \prec b'$ we get $\neg a' \succ \neg b'$. Now $a \sqcup a' = 1$ implies $a \sqsupseteq \neg a' \succ \neg b'$. Thus $a \succ \neg b'$ and we have shown the inclusion $\mathtt{tot}_=; \prec \subseteq \succ; \mathtt{tot}_=$. The proof for the converse inclusion is similar.

With the proposition above we can factor the compactification of A with respect to the proximity $\prec$ as follows. First form the symmetric d-frame $A_=$. Then use the proximity $\prec$ to modify the totality relation to $\mathtt{tot}_=; \prec$ which yields the regular normal d-frame $(A_=)^{\prec}$. Then the compact regular symmetric d-frame $((A_=)^{\prec})_\diamond$ has the compactification $\mathrm{Idl}^{\prec} A$ as component frames.

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