Cartesian Closed Categories of Domains

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Introduction

Consider a computer program for numerical integration. It takes as its input the algorithmic description of a real valued function f, boundary values a and b and after its calculations prints out a real number which equals the integral over f from a to b. So this program could be viewed as a function from $[\mathbf{R} \longrightarrow \mathbf{R}] \times \mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$. But this is not quite right, as we cannot input arbitrary real numbers, in fact, only a finite subset of the rational numbers is admissible. Also, not every integrable real valued function can be given an algorithmic description (take e^{x^2}, for example) and even if it can, the result will normally not exactly equal the integral.

We could switch to the other extreme and say that our program — as any computer program — accepts strings of 0's and 1's and outputs a string of 0's and 1's. But this is not at all helpful, as we want to compare different integration routines.

What we do need is a description of the input domain which is at the same time idealistic and realistic. 'Idealistic', because it should contain the ideal infinite object (e.g. the real numbers) or should at least indicate how it comes into it. 'Realistic', because it should contain finite realizable models of the ideal object and because it should allow to compare the finite models and suggest ways to improve accuracy.

The framework, which to a high extend satisfies these requirements, is that of algebraic (or continuous) directed-complete partial orders and continuous functions. We will not repeat the whole story of why this is a good framework nor how one arrives at this concept by necessity if one accepts certain basic decisions. There are enough good sources available for this, most notably [21, 22, 23, 20, 10].

So, generally speaking, a domain is a description of a set of data which satisfies the two requirements mentioned above and, specificially, we here take Scott's standpoint and equate 'domain' with (algebraic or continuous) directed-complete partial order.

Prime examples of objects, which have no finite description, are functions on infinite sets. So if we have managed to describe two sets of data D and E as domains then we should demand that the space $[D \longrightarrow E]$ of functions from D to E is also a domain. For what subclasses of the class of all domains is this always the case? This is the leading theme of this work.

A first step towards answering this question was taken by Michael B.Smyth in 1983. Under the additional assumptions that a domain should have a least element, should be algebraic, and should have only countably many compact elements, he could confirm a conjecture of Gordon Plotkin, namely, that any such domain must be representable as a limit of finite posets, that is, must be a bifinite domain. (See Section 1.4 for a precise definition.) In [14] we proved that this is still true if we pass to domains without least element. In the present work we answer the question completely for all algebraic dcpo's, whether they have a least element (Chapter 2) or not (Chapter 3), and we give half of the answer for continuous directed-complete partial orders.

The course of the work is as follows:

In the first chapter we mainly collect basic results of the theory of directed-complete partial orders; much of it can also be found in [1, 20, 9, 26]. Section 1.5, however, is original and the results therein are used again and again in the following chapters.

In the first section of Chapter 2 we discuss Smyth's result and give a proof. It turns out that in the uncountable case the bifinite domains are not the only possible class of domains. A new class of domains comes into play, which is introduced in

Section 2.2. We suggest the name L-domain for them, which alludes to the fact that every principal ideal is a complete lattice in these dcpo's. We give several equivalent characterizations for them and prove that they form a cartesian closed category inside the category $\mathbf{ALG}_\perp$ of all algebraic dcpo's with least element.

Section 2.3 contains the heart of this work. With the help of the crucial Lemma 2.13 we are able to divide all algebraic pointed domains with well-behaved function space into two classes: those which are L-domains and those which are bifinite.

This complete overview over the possible classes of algebraic domains allows us to prove an interesting connection between the space $[D \longrightarrow D]$ of all continuous functions on D and the space $[D \stackrel{s}{\longrightarrow} D]$ of strict (preserving the least element) continuous functions, namely, the first is an algebraic domain if and only if the latter is an algebraic domain. There seems to be no direct proof for this, so it is a true application of the classification proved before.

Finally removing the requirement that a domain should have a least element, we treat general algebraic dcpo's in Chapter 3. Surprisingly, there is is a complete answer for this general case also: each of the two classes discussed in Section 2.3 splits into two larger classes, so there are four maximal cartesian closed full subcategories in **ALG**. One extension — by taking disjoint unions — is discussed in the first section of that chapter. The other extension needs a careful study of the set of minimal elements in such dcpo's. This is done in Section 3.2. There is a lemma (3.15) which discriminates between the two extensions for both L-domains and bifinite domains. This allows us to prove the classification in Section 3.3.

As a byproduct, we can easily derive from the general result our earlier theorem about countably based algebraic dcpo's, see Section 3.4.

In the last chapter we turn to retracts of algebraic dcpo's which gives us the larger category **CONT** of continuous dcpo's to work in. Retracts of bifinite domains (we suggest the name continuous B-domain for these structures) had been of little interest so far (see [16, 8]) which is largely due to the fact that there was a good internal description of them only in the countably based case. We have proved this characterization to hold for all continuous B-domains which allows us to derive several results about this class in Section 4.1. Most notably, there is a simple proof now that codirected limits of continuous B-domains are again in the same category **cB**.

However, we are not completely satisfied with this characterization and have to admit that there is still no description on the element level for them. We try our hand at such an internal characterization in Section 4.2. Its usefulness, however, should be tested by proving or disproving a classification theorem paralleling the one for algebraic domains. So all we can do in the moment is to prove maximality for the class of continuous L-domains. This is carried out in Section 4.4, after a closer analysis of continuous L-domains in Section 4.3.

In recent years, a new branch of Domain Theory — now commonly referred to as 'Stable Domain Theory' — has evolved, starting with the work of G.Berry (cf. [2, 7, 3]). The domains studied are rather special: they are isomorphic to a Scott-closed subset of a powerset. This implies that they have more structure than general algebraic dcpo's, for example, infima of nonempty sets exist. The mappings — called 'stable functions' — connecting these domains respect not only directed suprema but also infima of bounded pairs of elements. It was found that one can still get cartesian closed categories this way. They have the additional advantage that some functors, which are designed to model polymorphism, have small descriptions.

There is some inconsequence in this approach, though, as the domains have all nonempty infima but the morphisms respect only infima for bounded sets. By Theorem 2.9 one wonders whether L-domains do not provide a better framework for this. Paul Taylor (cf. [27, 28]) has strongly taken this point. He has proved two remarkable results about continuous L-domains: they can be viewed as the algebras of a monad over the category of locally connected topological spaces, and, secondly, they form a cartesian closed category with stable functions (preserving filtered meets and directed joins) as arrows.

Thierry Coquand (cf. [4]) discovered L-domains in a general categorical form independently of us and proved cartesian closedness for both Scott-continuous functions and stable functions. It was Carl Gunter who saw that his definition yields L-domains if restricted to the poset case.

As we have the two maximal categories of L-domains and bifinite domains inside $\mathbf{ALG}_\perp$, it seems reasonable to study their intersection. What we get is a cartesian closed category properly containing Scott's bounded-complete dcpo's. In [12] it is proved that a universal domain exists for this class. We should mention here that these posets appear as 'short domains' in [9] already.

The definition of 'L-domain' is 'local', that is, we require a property of the principal ideals. One wonders whether the 'global' requirement that an L-domain is a dcpo can't be dispensed with. So, what about local dcpo's (each principal ideal is a dcpo), local bifinite domains and local L-domains?

Finally, I should like to thank all those people who have helped me in the course of writing this thesis: Dana Scott, who introduced me to the theory of domains through a series of enlightening lectures and who helped with many things during my visit to Carnegie-Mellon University in the academic year 1984/85. Carl Gunter, who invited me into his own field of studies, who tirelessly explained and helped by stating research goals. Paul Taylor, whose deep interest in the theory of L-domains encouraged me to tackle the (at first seemingly unsolvable) general case of algebraic dcpo's without least element. But foremost I express my deep feeling of gratitude to my doctoral advisor, Prof. Klaus Keimel, whose gentle guidance gave me the necessary amount of orientation whilst leaving me considerable freedom in conducting my research.

Chapter 1

Basic Concepts

In this chapter we collect some of the standard results about partially ordered sets, dcpo's and continuous functions. Interspersed are several important original contributions, most notably Proposition 1.6, Corollary 1.7, Proposition 1.10, Corollary 1.13, Proposition 1.25, and all of Section 1.5. Proposition 1.1, due to M.Krasner (cf. [17]), should be better known in the domain theory community. It allows to base an induction proof on any directed set.

We have also included several new examples and counterexamples, which should help the reader to understand the concept of bifiniteness.

1.1 Ordered sets, directed sets, and directed-complete partial orders

Definition. A set D with a binary relation $\leq$ is called an ordered set if the following holds for all $x, y, z \in D$:

(i) $x \leq x$ (Reflexivity)

(ii) $x \leq y \wedge y \leq z \Longrightarrow x \leq z$ (Transitivity)

(iii) $x \leq y \wedge y \leq x \Longrightarrow x = y$ (Antisymmetry)

Ordered sets are also called partially ordered sets or posets in the literature. Small finite ordered sets can be drawn as line diagrams (Hasse diagrams). We will also allow ourselves to draw infinite posets by showing a finite part which illustrates the building principle.

The word 'partially ordered set' indicates that there are also 'totally ordered sets'. Indeed, if $x \leq y$ or $y \leq x$ holds for any pair x, y of elements in a nonempty poset D then we call D a chain, or totally ordered.

Given an ordered set $(D, \leq)$ we can define the dual order $\leq'$ on D by setting $x \leq' y \Leftrightarrow y \leq x$. The set D together with the dual order is denoted by D^{op}.

In the following definition we develop some of the standard notation connected with the theory of ordered sets.

Definition. Let $(D, \leq)$ be an ordered set.

(i) A subset A of D is an upper (lower) set if $x \in A$ implies $y \in A$ for all $y \geq x$ ($y \leq x$). We denote by $\uparrow A$ ($\downarrow A$) the smallest upper (lower) set which contains the subset A of D. The interval $\uparrow x \cap \downarrow y$ we denote by $[x, y]$.

(ii) An element $x \in D$ is called an upper (lower) bound for a subset $A \subseteq D$, if $A \subseteq \downarrow x$ ($A \subseteq \uparrow x$). We denote by $ub(A)$ ($lb(A)$) the set of all upper (lower) bounds of A.

(iii) An element $x \in D$ is maximal (minimal) if there is no other element of D above (below) it: $\uparrow x \cap D = \{x\}$ ($\downarrow x \cap D = \{x\}$). For a subset $A \subseteq D$ the minimal elements of $ub(A)$ are called minimal upper bounds of A. The set of all minimal upper bounds of A is denoted by $mub(A)$.

(iv) If all elements of D are below (above) one element $x \in D$, then x is said to be the largest (least) element. The least element of a poset is also called bottom and is commonly denoted by $\perp$. Posets with a least element we will call pointed.

(v) If for a subset $A \subseteq D$ the set of upper (lower) bounds has a least (largest) element x, then x is called the supremum (infimum) of A. We write $x = \bigvee A$ ($x = \bigwedge A$) in this case.

(vi) The ordered set D is a $\vee$-semilattice ($\wedge$-semilattice) if the supremum (infimum) for each pair of elements exists. If D is both a $\vee$- and a $\wedge$-semilattice then D is called a lattice. A lattice D is complete if suprema and infima exist for all subsets $A \subseteq D$.

The first structures used in denotational semantics were lattices. It was soon recognized, however, that in the situations, which denotational semantics tries to model, suprema of arbitrary subsets do not necessarily exists. On the other hand, the constructions and methods developed by Dana Scott for lattices can also be carried out for weaker structures. In some sense, Smyth's work in [24] and also this dissertation can be read as a search for the weakest possible definition of 'semantic domain'.

Definition. Let D be a poset.

(i) A subset $A \subseteq D$ is directed (filtered) if it is nonempty and each pair of elements of A has an upper (lower) bound in A.

(ii) A lower (upper) subset of D is called an ideal (filter) if it is directed (filtered). Ideals (filters), which contain a largest element, are called principal. They are of the form $\downarrow x$ ($\uparrow x$), $x \in D$.

(iii) If all directed sets in D have a supremum, then we say that D is a directed-complete partial order or dcpo for short. If, in addition, D^{op} is also a dcpo, then we call the poset bicomplete.

Most posets considered in this work are in fact bicomplete, but this will be a theorem and hence needs not to be included in the definition.

We use the notation $x = \bigvee^{\uparrow} A$ when we want to express that A is a directed set with supremum x.

Directed sets are interesting objects in themselves. We reserve the remainder of this section to a closer analysis of this concept.

Definition. (i) A monotone net in a poset D is a monotone function α from a directed set I into D. The set I is called the index set of the net.

(ii) Let $\alpha: I \to D$ be a monotone net. A subnet of α is a monotone net $\beta: J \to I$ such that for all $i \in I$ there is $j \in J$ with $\beta(j) \geq i$.

(iii) A monotone net $\alpha: I \to D$ has a supremum in D, if the set $\{\alpha(i) \mid i \in I\}$ has a supremum in D.

Every directed set can be viewed as a monotone net: let the set itself be the index set. On the other hand, the image of a monotone net $\alpha: I \to D$ is a directed set in D. So what are nets good for? The answer to this question is given in the following proposition.

Proposition 1.1 *Let D be a poset and let $\alpha: I \to D$ be a monotone net. Then α has a subnet $\beta: J \to I$, whose index set J is a lattice in which every principal ideal is finite.*

Proof. Let J be the set of finite subsets of I. Clearly, J is a lattice in which every principal ideal is finite. We define the mapping $\beta: J \to I$ by induction on the cardinality of the elements of J:

$$\begin{aligned} \beta(\phi) &= \text{any element of } I; \\ \beta(A) &= \text{any upper bound of the set } A \cup \{\beta(B) \mid B \subset A\}, A \neq \phi. \end{aligned}$$

It is obvious that β is monotone and defines a subnet. ▮

The preceding proposition appears as 'Théorème 1' in [17].

We will make crucial use of nets with lattice ordered index set in the Characterization Theorem for retracts of bifinite domains in Section 4.1.

Not every net has a subnet with a totally ordered index set. An example is the set of finite subsets of the real numbers. The following theorem is therefore just the more surprising.

Theorem 1.2 *A partially ordered set D is a dcpo if and only if each chain in D has a supremum.*

The proof, which uses the Axiom of Choice, goes back to a lemma of Iwamura [13] and can be found in [18].

Corollary 1.3 *A partially ordered set D is a dcpo if and only if each monotone injective net $\alpha\colon I \to D$, with I an ordinal number, has a supremum in D.*

1.2 Algebraic and continuous posets

In the last section we have introduced dcpo's as structures, in which a directed collection of elements describes a new element: order theoretically the supremum of the collection. We will now restrict our attention to such dcpo's, in which every element can be represented as a directed collection of approximating elements. In the following definition we will make precise what is meant by 'one element approximating another element'.

Definition. Let D be a dcpo.

(i) For elements $x, y \in D$ we say that x is way-below y ($x \ll y$), if for all directed sets $A \subseteq D$, $\bigvee^{\uparrow} A \geq y$ implies $a \geq x$ for some $a \in A$.

(ii) For an element $x \in D$ we define the following subsets:

$$\begin{aligned} \Uparrow x &= \{y \in D \mid x \ll y\} \\ \Downarrow x &= \{y \in D \mid y \ll x\}. \end{aligned}$$

(iii) For A a subset of D we define $\Uparrow A = \bigcup_{a \in A} \Uparrow a$ and $\Downarrow A = \bigcup_{a \in A} \Downarrow a$.

(iv) An element $x \in D$ is said to be compact, if it is way-below itself.

(v) The set of compact elements is denoted by $K(D)$.

Proposition 1.4 *Let D be a dcpo. Then the following is true for all $x, x', y, y' \in D$:*

(i) $x \ll y \Longrightarrow x \le y$;

(ii) $x' \le x \ll y \le y' \Longrightarrow x' \ll y'$.

Proof. (i) Let A be the directed set consisting of the single element y.

(ii) It suffices to note that if the supremum of a directed set A is above y' then it is also above y, and if there is an element $a \in A$ above x then also $a \ge x'$. ∎

Definition. We say that a dcpo D is continuou, if for all $x \in D$ the set $\Downarrow x$ is directed and $\bigvee^{\uparrow} \Downarrow x = x$. It is algebraic, if the set of all compact elements below x is directed with x as the supremum of this set.

It is time for some examples. Algebraic lattices were studied long before the advent of electronic computers. They arise as the lattices of substructures or as the lattices of congruence relations for general algebraic structures.

For finite ordered sets the way-below relation coincides with the order relation and therefore every finite ordered set is an algebraic dcpo.

Every algebraic dcpo is also a continuous dcpo but the converse does not hold. The unit interval, for example, is a continuous lattice but contains only a single compact element: 0. Ordered by inclusion, the open subsets of a compact topological space form a continuous lattice. Here a set O_1 is way-below a set O_2 if $\overline{O_1}$ is contained in O_2.

It is instructive to give an example of a dcpo which is not continuous. Figure 1.1 shows such a poset. There the element b is not compact, because the limit of the

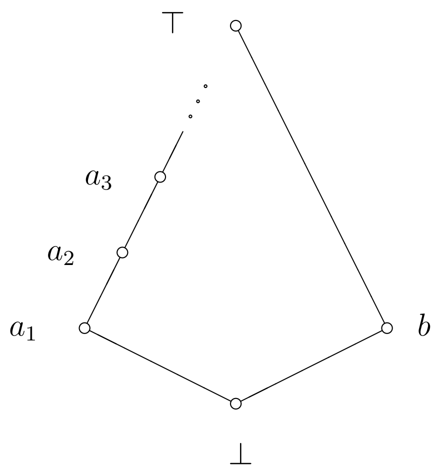

Figure 1.1: A non-continuous dcpo

sequence $(a_n)_{n\in\mathbf{N}}$ is above b although no element of the sequence is greater than b. Therefore b is not the supremum of the elements way-below it.

The following trivial observation will be of some help in the following sections:

Proposition 1.5 (i) *If D is a continuous dcpo and if x, y are elements in D then x is way-below y if and only if for all directed subsets A of D with $\bigvee^{\uparrow} A = y$ there is $a \in A$ such that $a \geq x$.*

(ii) *Let D be an algebraic dcpo. An element x of D is compact if and only if for all directed sets $A \subseteq D$ with $\bigvee^{\uparrow} A = x$ there is $a \in A$ such that $a = x$.*

Proof. In both cases only the 'if'-part is interesting.

(i) We take for A the set $\mathord{\downarrow\!\!\downarrow} y$ and get an element $z \ll y$ which is above x. By Proposition 1.4, x is also way-below y.

(ii) Take for A the set of compact elements below x. ▮

Proposition 1.6 *Let D be a dcpo in which every principal ideal is a continuous dcpo. Then the following holds:*

(i) *If $y \ll y'$ holds in $\downarrow x$ then $y \ll y'$ holds in D.*

(ii) $K(D) = \bigcup_{x \in D} K(\downarrow x)$.

Proof. Assume $y \ll y'$ in $\downarrow x$. Let $(z_i)_{i\in I}$ be a directed set with $z = \bigvee^{\uparrow}_{i\in I} z_i \geq y'$. Inside the continuous dcpo $\downarrow z$ we can represent y' as the directed supremum of elements $(a_j)_{j\in J}$, all of which are way-below y' in $\downarrow z$. All elements a_j belong also to $\downarrow x$ and because y is way-below y' here, there is some a_{j_0} which is above y. Going back to $\downarrow z$ we see that y is below a_{j_0} which is way-below y' and hence y is way-below y' in $\downarrow z$ by Proposition 1.4. This implies in particular that some z_{i_0} must be above y.

The second part follows directly from (i). ▮

Corollary 1.7 *Let D be a dcpo.*

(i) *D is continuous if and only if $\downarrow x$ is a continuous dcpo for all $x \in D$.*

(ii) *D is algebraic if and only if $\downarrow x$ is an algebraic dcpo for all $x \in D$.*

For continuous dcpo's the way-below relation has the following important interpolation property.

Proposition 1.8 *Let D be a continuous dcpo and let x, y be elements of D. If x is way-below y then there is $z \in D$ such that $x \ll z \ll y$ holds.*

Proof. Given an element x way-below some element y we define the set

$$A = \{a \in D \mid \exists a' \in D : a \ll a' \ll y\}.$$

The set A is directed because if $a \ll a' \ll y$ and $b \ll b' \ll y$ then by the directedness of $\mathop{\downarrow\!\!\!\downarrow} y$ there is $c' \in D$ such that $a' \leq c' \ll y$ and $b' \leq c' \ll y$ and again by the directedness of $\mathop{\downarrow\!\!\!\downarrow} c'$ there is $c \in D$ with $a \leq c \ll c'$ and $b \leq c \ll c'$. We calculate the supremum of A: let y' be any element way-below y. Since $\mathop{\downarrow\!\!\!\downarrow} y' \subseteq A$ we have that $\bigvee^{\uparrow} A \geq \bigvee^{\uparrow} \mathop{\downarrow\!\!\!\downarrow} y' = y'$. This holds for all $y' \ll y$ so by continuity $y = \bigvee^{\uparrow} \mathop{\downarrow\!\!\!\downarrow} y \leq \bigvee^{\uparrow} A$. All elements of A are less than y, so in fact equality holds: $\bigvee^{\uparrow} \mathop{\downarrow\!\!\!\downarrow} y = \bigvee^{\uparrow} A$. Remember that we started out with an element x way-below y. By definition there is $a \in A$ with $a \geq x$ and hence x belongs to A. That was to be proved. ▌

Proposition 1.9 *In a continuous dcpo minimal upper bounds of finite sets of compact elements are again compact.*

Proof. Let x be a minimal upper bound of the finite set A of compact elements. We have $A \subseteq \mathop{\downarrow\!\!\!\downarrow} x$ and since the latter set is directed there is an upper bound x' of A in $\mathop{\downarrow\!\!\!\downarrow} x$. Because x is a minimal upper bound of A we must have $x = x'$ which is tantamount to saying that x is compact. ▌

In order to represent a continuous dcpo D it is not necessary to give all elements explicitely. It is sufficient to know about a dense subset of D. Each element of D can then be represented as a directed collection of elements from this dense subset. The following definition presents this idea in a precise form.

Definition. A subset B of a continuous dcpo D is called a <u>base</u> of D if for each element $x \in D$ the set $\mathop{\downarrow\!\!\!\downarrow} x \cap B$ is directed with x as the supremum.

A continuous dcpo may have many different bases and none of these may be minimal: if D is the unit interval we may take for a base all rational numbers between 0 and 1. We may also take only those rational numbers which have a denominator divisible by 2, or divisible by 6, and so on. If D is algebraic, however, each base must contain the compact elements. Conversely, the definition of 'algebraic dcpo' tells us that each element of D can be expressed as the directed supremum of compact elements. This explains why we can speak of the base $K(D)$ of an algebraic dcpo.

Definition. A dcpo D is called ω-continuous (ω-algebraic) if D is continuous (algebraic) and contains a countable base.

Proposition 1.10 *If D is algebraic then $K(D) = \bigcup_{x \in D} K(\downarrow x)$.*

Proof. This follows directly from Proposition 1.6 and Corollary 1.7. ❙

1.3 Scott-topology and continuous functions

We have said before that for us the interpretation of the order relation on a dcpo is that of one element approximating another. It is therefore not surprising that we choose as homomorphisms between dcpo's those functions, which allow us to calculate the value of an element x from the values of the approximations to x.

Definition. Let D and E be dcpo's. A function $f\colon D \to E$ is continuous if for each directed subset A of D the equality $f(\bigvee^{\uparrow} A) = \bigvee^{\uparrow}_{a \in A} f(a)$ holds. We denote the set of all continuous functions from D to E by $[D \longrightarrow E]$. The functions in $[D \longrightarrow E]$ are ordered pointwise, that is: $f \leq g \Leftrightarrow \forall x \in D : f(x) \leq g(x)$. The identity function on a poset D is denoted by id_D, the constant function with image x is denoted by c_x.

Proposition 1.11 *Let D and E be dcpo's.*

(i) *Each continuous function from D to E is monotone.*

(ii) *The composition of two continuous functions is continuous.*

(iii) *The function space $[D \longrightarrow E]$ is a dcpo.*

Proof. (i) If $x \le x'$ are elements of D and if $f\colon D \to E$ is a continuous function then we consider the directed set $\{x, x'\}$. By definition we have that $f(x') = f(x \vee x') = f(x) \vee f(x')$. This says that $f(x)$ is below $f(x')$.

(ii) This is trivial.

(iii) Let F be a directed collection of functions from D to E. Let $g\colon D \to E$ be the function, which is defined by $g(x) = \bigvee^{\uparrow}_{f\in F} f(x)$. Let $A \subseteq D$ be directed.

$$\begin{aligned} g(\textstyle\bigvee^{\uparrow} A) &= \bigvee^{\uparrow}_{f\in F} f(\textstyle\bigvee^{\uparrow} A) \\ &= \bigvee^{\uparrow}_{f\in F} \bigvee^{\uparrow}_{a\in A} f(a) \\ &= \bigvee^{\uparrow}_{a\in A} \bigvee^{\uparrow}_{f\in F} f(a) \\ &= \bigvee^{\uparrow}_{a\in A} g(a). \end{aligned}$$

This shows that g is continuous. ▮

Not every monotone function is continuous but for continuous dcpo's we have the following

Proposition 1.12 *Let D be a continuous dcpo, E a dcpo, and let $f\colon D \to E$ be a monotone function. Then*

$$f^c(x) = \bigvee^{\uparrow}_{y \ll x} f(y)$$

is the largest continuous function below f.

Proof. Let $A \subseteq D$ be directed. First note that by the interpolation property an element y is way-below $\bigvee^{\uparrow} A$ if and only if it is way-below some element of A. Thus we can calculate

$$\begin{aligned} f^c(\textstyle\bigvee^{\uparrow} A) &= \bigvee^{\uparrow}_{y \ll \bigvee^{\uparrow} A} f(y) \\ &= \bigvee^{\uparrow}_{a\in A} \bigvee^{\uparrow}_{y \ll a} f(y) \\ &= \bigvee^{\uparrow}_{a\in A} f^c(a). \end{aligned}$$

If $g\colon D \to E$ is any continuous function below f then for all $x \in D : g(x) = \bigvee^{\uparrow}_{y \ll x} g(y) \leq \bigvee^{\uparrow}_{y \ll x} f(y) = f^c(x)$. ∎

Corollary 1.13 *The function space $[D \longrightarrow D]$ is bicomplete whenever D is a continuous and bicomplete partial order.*

Proof. Let A be a filtered collection of functions from D into D. We define $f(x) = \bigwedge_{a \in A} a(x)$. Clearly, f is a monotone function on D. By Proposition 1.12 f^c is the infimum of A in $[D \longrightarrow D]$. ∎

In the light of Proposition 1.12 it is clear that we may define a continuous function on a continuous dcpo by assigning values to the elements of a base only. If the domain D is even algebraic, then there is a 1-1 correspondence between monotone functions on the base and continuous functions on D.

The term 'continuous function' is justified by the observation that each dcpo carries a topology which makes continuous functions into topologically continuous ones.

Definition. For a dcpo D we define the Scott-topology $\sigma(D)$ on D as follows: a set $O \subseteq D$ is Scott-open if it is an upper set and if for each directed set $A \subseteq D$, $\bigvee^{\uparrow} A \in O$ implies the existence of an $a \in A \cap O$.

First examples for Scott-open sets in a dcpo D are sets of the form $D \setminus {\downarrow} x$ for x an arbitrary element of D. The following propositions illustrate the connection between Scott-topology and the order theoretic concepts developed so far.

Proposition 1.14 *Let D be a continuous dcpo.*

(i) *If x is an element of a Scott-open set O then there is a $y \in O$ with $y \ll x$.*

(ii) *The sets $\Uparrow x$ form a basis of $\sigma(D)$.*

(iii) *The Scott-topology is* T_0.

Proof. Let O be a Scott-open subset of D and let x be an element of O. We represent x as a directed supremum: $x = \bigvee^{\uparrow}_{y \ll x} y$ and by the definition of $\sigma(D)$ there is some $y \ll x$ which belongs to O.

(ii) follows immediately from (i).

As for (iii), suppose x and x' are distinct elements of D. By the antisymmetry of the order relation we must either have that x is not below x' or that x' is not below x. Assume $x \not\leq x'$. The set $\downarrow x'$ is Scott-closed and hence $D \setminus \downarrow x'$ is an open neighborhood of x not containing x'. ▮

Proposition 1.15 *For dcpo's D and E, a function f from D to E is continuous if and only if it is topologically continuous with respect to $\sigma(D)$ and $\sigma(E)$.*

Proof. Let f be a continuous function from D to E and let O be an open subset of E. It is clear that $f^{-1}(O)$ is an upper set because continuous functions are monotone. If f maps the element $x = \bigvee^{\uparrow}_{i \in I} x_i \in D$ into O then we have $f(x) = f(\bigvee^{\uparrow}_{i \in I} x_i) = \bigvee^{\uparrow}_{i \in I} f(x_i) \in O$ and by definition there must be some x_i which is mapped into O. Hence $f^{-1}(O)$ is open in D. For the converse assume that $f \colon D \to E$ is topologically continuous. We first show that f must be monotone: let $x \leq x'$ be elements of D. If $f(x) \not\leq f(x')$ then $O = E \setminus \downarrow f(x')$ is an open neighborhood of $f(x)$ not containing $f(x')$. The inverse image of O contains x but not x'. This contradicts our assumption as open sets are always upper sets. Now let $A \subseteq D$ be directed. If the supremum of the directed set $f(A)$ is not above $f(\bigvee^{\uparrow} A)$ then arguing as before we have the Scott-open set $O = E \setminus \downarrow \bigvee^{\uparrow}_{a \in A} f(a)$ which is a neighborhood of $f(\bigvee^{\uparrow} A)$ but not of $\bigvee^{\uparrow}_{a \in A} f(a)$. The inverse image of O is open and contains $\bigvee^{\uparrow} A$, hence some element $a \in A$. Since O is an upper set and since $f(a)$ is in O we also must have $\bigvee^{\uparrow}_{a \in A} f(a) \in O$. This contradiction finishes our proof. ▮

The image under a continuous mapping is not necessarily a dcpo again. For this to hold true we need to impose further restrictions.

Definition. Let D and E be dcpo's. A continuous function $r\colon D \to D$ is called a retraction if $r \circ r = r$. It is called a projection if also $r \leq id_D$ holds.

Continuous functions $r\colon D \to E$ and $e\colon E \to D$ are said to form a retraction-embedding pair, if $r \circ e$ equals the identity function on E. The pair (r, e) is a projection-embedding pair if $e \circ r$ is a projection on D and $r \circ e = id_E$.

If there is a retraction-embedding pair between D and E, the retraction mapping D onto E, we say that E is a retract of D.

Typical retracts, which we will use throughout this work, are those of the form $\downarrow d$ for d an arbitrary element of a dcpo D. In this case the retraction $r\colon D \to D$ is given by

$$r(x) = \begin{cases} x, & \text{if } x \leq d; \\ d, & \text{otherwise.} \end{cases}$$

By dualizing this definition we have an idempotent function onto the principal filter $\uparrow c$. In order to get a continuous mapping we have to require that c is a compact element.

Proposition 1.16 *Let D be a dcpo and let r be a retraction on D. Then $im(r)$ is a dcpo and the supremum of a directed subset of $im(r)$ formed in $im(r)$ coincides with the supremum formed in D. If D is continuous then so is $im(r)$.*

Proof. Let A be a directed subset of $im(r)$. Applying the retraction to the supremum of A in D we get: $r(\bigvee^{\uparrow} A) = \bigvee^{\uparrow}_{a \in A} r(a) = \bigvee^{\uparrow}_{a \in A} a = \bigvee^{\uparrow} A$. So $\bigvee^{\uparrow} A$ belongs to $im(r)$. This shows the first part of the proposition.

For x an element of $im(r)$ and x' any element of D way-below x we show that $r(x')$ is way-below x in $im(r)$. Let $A \subseteq im(r)$ be directed with $\bigvee^{\uparrow} A \geq x$. Since we can calculate directed suprema either in D or in $im(r)$, A must contain some element a which is above x'. For this element we also have $a = r(a) \geq r(x')$. From this it is clear that a retraction preserves continuity of the domain. ∎

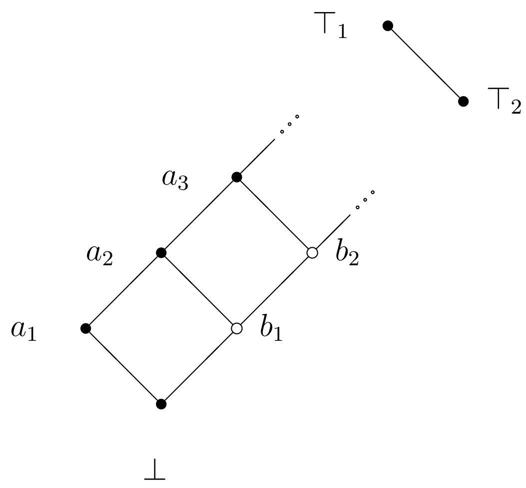

Figure 1.2: A non-continuous dcpo as the image of a monotone idempotent function on D.

We would like to note that the image of a monotone idempotent function on a dcpo is again a dcpo, but that continuity is not necessarily preserved. Figure 1.2 shows an example. The function on D is given by

$$f(x) = \begin{cases} \bot, & \text{if } x = b_1, b_2, \ldots; \\ x, & \text{otherwise.} \end{cases}$$

The image is isomorphic to the non-continuous example shown in Figure 1.1.

Retracts of algebraic dcpo's may not be algebraic again, but any continuous dcpo may be gotten as a retract from an algebraic domain. This is the content of the following

Proposition 1.17 (i) *Let D be a poset. Then the set $I(D)$ of all ideals in D ordered by inclusion, is an algebraic dcpo.*

(ii) *If D is a continuous dcpo then there is a projection from $I(D)$ onto D.*

Proof. It is easy to check that the directed union of directed sets is again a directed set and that principal ideals are compact elements in $I(D)$. This proves part (i).

The retraction $r\colon I(D) \to D$ is given by $A \mapsto \bigvee^{\uparrow} A$, the embedding by $x \mapsto {\downarrow}x$. Again, the details are easy to check. ▮

Proposition 1.18 *Let D be a dcpo.*

(i) *The set $[D \xrightarrow{r} D]$ of retractions on D is a dcpo.*

(ii) *The set $[D \xrightarrow{p} D]$ of projections on D is a dcpo.*

(iii) *If p is a projection on D then for all $x \in D : p(x) = max\,\{y \in im(p) \mid y \leq x\}$.*

(iv) *For projections $p, p'\colon D \to D$ we have the equivalence: $p \leq p'$ if and only if $im(p) \subseteq im(p')$.*

Proof. (i) Let $(r_i)_{i\in I}$ be a directed family of retractions. For any $x \in D$ we can calculate

$$\begin{aligned}
(\bigvee_{i\in I}^{\uparrow} r_i) \circ (\bigvee_{i\in I}^{\uparrow} r_i)(x) &= \bigvee_{i\in I}^{\uparrow} r_i(\bigvee_{i\in I}^{\uparrow} r_i(x)) \\
&= \bigvee_{i\in I}^{\uparrow} \bigvee_{j\in I}^{\uparrow} r_i(r_j(x)) \\
&= \bigvee_{i\in I}^{\uparrow} r_i(r_i(x)) \\
&= \bigvee_{i\in I}^{\uparrow} r_i(x).
\end{aligned}$$

Hence the supremum of retractions is again an idempotent function. We have proved in Proposition 1.11 that it is also continuous.

(ii) Projections are retractions below the identity function. The supremum of such functions is again below id_D.

(iii) Clearly, $x \geq p(x) \in im(p)$ holds, so $p(x) \leq \bigvee^{\uparrow}\{y \in im(p) \mid y \leq x\}$. On the other hand, for each $y \in im(p)$ below x we have $y = p(y) \leq p(x)$.

(iv) If $p \leq p'$ and x is in $im(p)$ then we have $p'(x) \leq x = p(x) \leq p'(x)$ and x is in $im(p')$. The other implication follows directly from (iii). ▮

As we have now exhibited the morphisms for dcpo's we may define the following categories:

DCPO: Directed-complete partial orders with continuous functions.

CONT: Continuous dcpo's with continuous functions.

ALG: Algebraic dcpo's with continuous functions.

We use the subscript $\perp$ to denote the respective full subcategory consisting of objects with a least element.

Definition. Let $\mathbf{C}$ be any category. We say that $\mathbf{C}$ is cartesian closed if the following three conditions are satisfied:

(i) There is a terminal object T in $\mathbf{C}$ such that for any object $A \in \mathbf{C}$ there is exactly one morphism $\alpha: A \to T$.

(ii) For any two objects $A, B \in \mathbf{C}$ there exists an object $A \times B$ in $\mathbf{C}$ and morphisms $pr_1: A \times B \to A$, $pr_2: A \times B \to B$ such that for any object C and morphisms $f: C \to A$, $g: C \to B$ there is a unique morphism $f \times g: C \to A \times B$ such that $pr_1 \circ (f \times g) = f$ and $pr_2 \circ (f \times g) = g$. The object $A \times B$ is called the product of A and B.

(iii) For any two objects $A, B \in \mathbf{C}$ there exists an object A^B in $\mathbf{C}$ and a morphism $ev: A^B \times B \to A$ such that for each $f: C \times B \to A$ there exists a unique morphism $\Lambda_f: C \to A^B$ such that $ev \circ (\Lambda_f \times id_B) = f$. The object A^B is called the exponential object for A and B.

Lemma 1.19 *Let D, E, and F be dcpo's and let $f: D \times E \to F$ be a function of two variables. Then f is continuous if and only if f is continuous in each variable separately.*

Proof. Let f be separately continuous and let A be a directed subset of $D \times E$. We calculate

$$\begin{aligned} f(\bigvee\nolimits^{\uparrow} A) &= f(\bigvee_{d \in pr_1(A)}^{\uparrow} \bigvee_{e \in pr_2(A)}^{\uparrow} (d,e)) \\ &= \bigvee_{d \in pr_1(A)}^{\uparrow} f(\bigvee_{e \in pr_2(A)}^{\uparrow} (d,e)) \\ &= \bigvee_{d \in pr_1(A)}^{\uparrow} \bigvee_{e \in pr_2(A)}^{\uparrow} f(d,e) \\ &= \bigvee_{(d,e) \in A}^{\uparrow} f(d,e). \end{aligned}$$

The converse is immediate. ∎

Proposition 1.20 *The categories* **DCPO** *and* $\mathbf{DCPO}_{\perp}$ *are cartesian closed.*

Proof. The one-point domain serves as the terminal object in both categories. For the categorical product we take the set-theoretic product together with the pointwise order. It is trivial to check that the projections are continuous and satisfy the required equations.

We have already proved (Proposition 1.11) that the space $[D \longrightarrow E]$ of all Scott-continuous functions is again a dcpo. It is the natural choice for the exponential object of dcpo's D and E. We prove that the evaluation function $ev: [D \longrightarrow E] \times D \to E$ is continuous: by Lemma 1.19 we can check continuity for both variables separately, so let first F be a directed collection of functions from D to E.

$$\begin{aligned} ev(\bigvee\nolimits^{\uparrow} F, d) &= (\bigvee\nolimits^{\uparrow} F)(d) \\ &= \bigvee_{f \in F}^{\uparrow} f(d) \\ &= \bigvee_{f \in F}^{\uparrow} ev(f,d) \end{aligned}$$

Assume now that A is a directed set in D:

$$\begin{aligned} ev(f, \bigvee^{\uparrow} A) &= f(\bigvee^{\uparrow} A) \\ &= \bigvee^{\uparrow}_{a \in A} f(a) \\ &= \bigvee^{\uparrow}_{a \in A} ev(f, a). \end{aligned}$$

Given a morphism $f\colon F \times E \to D$ we define the function $\Lambda_f\colon F \to D^E$ elementwise: $x \mapsto f(x, \cdot)$. It is again trivial to check that this a continuous mapping and that $ev \circ (\Lambda_f \times id_E) = f$. ∎

We note that any full subcategory **C** of **DCPO** or of $\mathbf{DCPO}_\perp$, which contains the one-point domain, the cartesian product, and the space $[A \longrightarrow B]$ of Scott-continuous functions for any two objects $A, B \in \mathbf{C}$, is itself cartesian closed since we have defined cartesian closedness in terms of equations.

On the other hand, there is not much choice for these constructs in a cartesian closed full subcategory of **DCPO**. This can be seen from the following lemma which essentially appears in [24] already.

Lemma 1.21 *Let* **C** *be a cartesian closed full subcategory of* **DCPO**. *Then the following holds for any two objects* $A, B \in \mathbf{C}$.

(i) *The terminal object* T *of* **C** *is isomorphic to the one-point domain.*

(ii) *The categorical product of* A *and* B *is isomorphic to the cartesian product* $A \times B$.

(iii) *The exponential object* A^B *is isomorphic to* $[B \longrightarrow A]$.

Proof. (i) Suppose T has two distinct elements x and x'. Then there are two continuous functions from T into itself: the constant functions with image x and with image x', respectively.

(ii) We denote the categorical product of A and B in $\mathbf{C}$ by $A \cdot B$ and show that it is isomorphic to $A \times B$. For each pair of elements $a \in A, b \in B$ there are functions $\bar{a}: T \to A$ and $\bar{b}: T \to B$ which map the one element of T onto a and b, respectively. By the universal property of the categorical product there is a unique function $\bar{a} \times \bar{b}: T \to A \cdot B$ whose image is thus the unique element (a, b) of $A \cdot B$ which is projected onto a and b, respectively. This proves that there is a bijection between the elements of $A \times B$ and $A \cdot B$.

We still have to show that $A \cdot B$ carries the right order. Since the projections pr_1 and pr_2 must be monotone, the order on $A \cdot B$ is contained (via the bijection) in the order of $A \times B$. For the converse we distinguish two cases: if no two elements of any object of $\mathbf{C}$ are comparable, then $A \cdot B$ is also totally unordered. If we have $d < d'$ in some object D and if $(a, b) \leq (a', b')$ in $A \times B$ then there are continuous mappings $\bar{a}: D \to A$ and $\bar{b}: D \to B$ defined by, e.g.,

$$\bar{a}(x) = \begin{cases} a, & \text{if } x \leq d; \\ a', & \text{otherwise.} \end{cases}$$

The map $\bar{a} \times \bar{b}$ maps d onto (a, b) and d' onto (a', b') and by continuity of this map $(a, b) \leq (a', b')$ holds in $A \cdot B$.

(iii) Given objects $A, B \in \mathbf{C}$ we show that A^B is isomorphic to $[B \longrightarrow A]$. Given an object $C \in \mathbf{C}$ and a morphism $f: B \to A$ we have the arrow $f': T \times B \cong B \to A$ and by the universal property of A^B there is exactly one element $\bar{f} = im(\Lambda_{f'})$ of A^B corresponding to f. Thus there is a bijection between the elements of A^B and $[B \longrightarrow A]$. As for the product one can easily show that this bijection is an order isomorphism. ∎

Neither the category **ALG** nor the category **CONT** are cartesian closed: consider the set $\mathbf{Z}^-$ of negative integers with their usual ordering. We show that no function $g \in [\mathbf{Z}^- \longrightarrow \mathbf{Z}^-]$ is way-below a second function $f \in [\mathbf{Z}^- \longrightarrow \mathbf{Z}^-]$. For each $n \in \mathbf{N}$ define a function $f_n: \mathbf{Z}^- \to \mathbf{Z}^-$ by setting

$$f_n(x) = \begin{cases} f(x), & \text{if } x \geq -n; \\ g(x) - 1, & \text{otherwise.} \end{cases}$$

Since we may assume that $g \leq f$ holds, this is a continuous mapping. The supremum of all f_n equals f but no f_n is above g.

Proposition 1.22 *Let D be a dcpo with a continuous function space $[D \longrightarrow D]$ and let E be a retract of D. Then $[E \longrightarrow E]$ is a retract of $[D \longrightarrow D]$ and hence a continuous dcpo.*

Proof. Let $r\colon D \to E$ be the retraction onto E and let $i\colon E \to D$ be the corresponding embedding. For f an element of $[D \longrightarrow D]$, g an element of $[E \longrightarrow E]$ we define a continuous mapping $R\colon [D \longrightarrow D] \to [E \longrightarrow E]$ by $R(f) = r \circ f \circ i$ and a continuous mapping $I\colon [E \longrightarrow E] \to [D \longrightarrow D]$ by $I(g) = i \circ g \circ r$. We have $R \circ I(g) = r \circ i \circ g \circ r \circ i = g$, so (R, I) is a retraction-embedding pair. ❙

Theorem 1.23 *If* **C** *is a cartesian closed full subcategory of* **ALG** *then* **cC***, the category of retracts of objects in* **C** *(with Scott-continuous functions as arrows) is cartesian closed.*

Proof. We still have the terminal object in **cC** and it is clear that the product of two retracts is a retract of the corresponding product. For the function space we have proved this in the preceding lemma. All the necessary equations hold since we are inside the cartesian closed category **DCPO**. ❙

If we are considering dcpo's with a bottom element then there are good reasons to look only at functions which preserve this element. We call such functions strict and denote the space of all strict functions from a dcpo D to a dcpo E by $[D \stackrel{s}{\longrightarrow} E]$. However, the category $\mathbf{DCPO}_{\perp}^{s}$ of dcpo's with strict functions as arrows is not cartesian closed, although it is closed with respect to a different product, which is not the categorical product. This construction, frequently called 'smash'-product, can be described as the cartesian product with all elements of the form $(\perp, y)$ or $(x, \perp)$ identified with the bottom element. It is in accordance with one possible philosophy about the least element, namely, that a function of several variables should be undefined whenever at least one of the arguments is undefined.

1.4 Bifinite domains

At several places we have already alluded to the idea of one element approximating another. In the last section, in particular, we exhibited the distinction between 'ideal elements' and 'finite (=compact) elements' and stipulated that the former are always representable as limits of finite elements. We now wish to extend this idea to the level of domains themselves, that is, we will define domains which are representable as limits (in fact: bilimits) of finite posets. The resulting structures we call bifinite domains.

It was Gordon Plotkin who first started the study of these structures in 1976 (see [19]), when he tried to define a 'powerset' for domains. He found that his construction led him out of the categories of lattices and semilattices but worked fine on his class SFP (= Sequences of Finite Posets). Our definition is slightly more general, allowing arbitrary directed index sets but all theorems in this section are essentially due to Plotkin. We have enriched the subject with a couple of examples (Figures 1.6 to 1.11), which illustrate several aspects of bifiniteness and show that the hypotheses in some central propositions cannot be weakened.

We begin with the following general

Definition. A codirected system over a category $\mathbf{C}$ is a family $(D_i)_{i\in I}$, with I a directed set, of objects from $\mathbf{C}$ together with a set of arrows $(d_{ij})_{i\leq j, i,j\in I}$ such that the following holds for all $i,j,k\in I$:

(i) $d_{ij}\colon D_j \to D_i$,

(ii) $d_{ii} = id_{D_i}$,

(iii) $i \leq j \leq k \Longrightarrow d_{ik} = d_{ij}\circ d_{jk}$.

We say that D^* is a limit of the codirected system $((D_i)_{i\in I}, (d_{ij})_{i\leq j})$ in $\mathbf{C}$ if there is a collection $(d_i)_{i\in I}$ of mappings with $d_i\colon D^* \to D_i$ and $d_i = d_{ij}\circ d_j$ for all $i\leq j$ in I such that for any object E and mappings $e_i\colon E\to D_i$, commuting with the connecting morphisms d_{ij}, there is a unique arrow $f: E \to D$ satisfying $e_i = d_i \circ f$ for all $i \in I$.

From the general theory of limits in categories we know that D^* is unique up to isomorphism.

We have already mentioned that we wish to form limits of finite posets but we haven't said what the connecting morphisms should be. We will not use arbitrary monotone functions, since we want view the objects D_i as approximations to the limit object D^*, D_j being a better approximation than D_i whenever $i \leq j$. It is hard to compare posets D_i and D_j when there is nothing else between them than a monotone function. If we use the projection part of projection-embedding pairs instead then we can indeed speak of D_i approximating D_j: D_i is embedded in D_j and for each element x of D_j there is a largest element x' of D_i below x. This motivates our study of codirected systems in categories $\mathbf{C}^p$, where the objects are taken from a particular subclass of directed-complete partial orders and the morphisms are projections.

A projection $d_{ij}: D_j \to D_i$ uniquely determines the corresponding embedding $e_{ji}: D_i \to D_j$. So any codirected system $((D_i)_{i\in I}, (d_{ij})_{i\leq j})$ in $\mathbf{C}^p$ gives rise to a directed system $((D_i)_{i\in I}, (e_{ji})_{i\leq j})$ in the dual category $\mathbf{C}^e$. It is obvious that the limit of the former is isomorphic to the colimit of the latter. This limit-colimit coincidence is the reason why we speak of the bilimit of the system $((D_i)_{i\in I}, (d_{ij})_{i\leq j})$ and why we call the bilimits of finite posets bifinite domains. This terminology is due to Paul Taylor (cf. [26]) and we adopt it in this work.

Theorem 1.24 *Any codirected system (D_i, d_{ij}) in $\mathbf{DCPO}^p$ has a bilimit D^*.*

Proof. We define the limit object

$$D^* = \{(a_i)_{i\in I} \in \prod_{i\in I} D_i \mid \forall i \leq j : d_{ij}(a_j) = a_i\}$$

and the limiting morphisms

$$d_j((a_i)_{i\in I}) = a_j, \forall j \in I.$$

It is clear that D^* is a dcpo since the connecting morphisms d_{ij} are continuous. It remains to show that the limiting morphisms are projections. This is done most easily by giving the corresponding embeddings e_i (the embedding e_{ji} corresponds to the projection d_{ij}):

$$e_i(a) = (d_{jk} \circ e_{ki}(a))_{j\in I}, k \text{ any upper bound of } \{i, j\}.$$

First of all, e_i is well-defined: if k, k' are upper bounds for $\{i, j\}$ then there is an upper bound l of $\{k, k'\}$ in I. We calculate:

$$\begin{aligned} d_{jk} \circ e_{ki}(a) &= d_{jk} \circ d_{kl} \circ e_{lk} \circ e_{ki}(a) \\ &= d_{jl} \circ e_{li}(a) \\ &= d_{jk'} \circ d_{k'l} \circ e_{lk'} \circ e_{k'i}(a) \\ &= d_{jk'} \circ e_{k'i}(a). \end{aligned}$$

Secondly, $e_i(a)$ is an element of D^*:

$$\begin{aligned} d_{lj}(d_{jk} \circ e_{ki}(a)) &= d_{lj} \circ d_{jk} \circ e_{ki}(a) \\ &= d_{lk} \circ e_{ki}(a). \end{aligned}$$

It remains to show that (e_i, d_i) is an embedding-projection pair. The proof consists again of two simple calculations:

$$\begin{aligned} e_i \circ d_i((a_j)_{j\in I}) &= e_i(a_i) \\ &= (d_{jk} \circ e_{ki}(a_i))_{j\in I} \\ &= (d_{jk} \circ e_{ki} \circ d_{ik}(a_k))_{j\in I} \\ &\leq (d_{jk}(a_k))_{j\in I} \\ &= (a_j)_{j\in I} \end{aligned}$$

and

$$\begin{aligned} d_i \circ e_i(a) &= d_i((d_{jk} \circ e_{ik}(a))_{j \in I}) \\ &= d_{ik} \circ e_{ki}(a) \\ &= a. \end{aligned}$$

It is obvious that all functions e_i are continuous since we have defined them in terms of the connecting morphisms. ▮

Definition. A dcpo D is a bifinite domain if it is isomorphic to the limit of a codirected system of finite posets with least element in $\mathbf{DCPO}_{\perp}^{p}$. We denote the category of bifinite domains with Scott-continuous functions by $\mathbf{B}$.

Note that we require a least element for bifinite domains, although the definition works for arbitrary finite posets as well. The doctoral thesis of Carl Gunter ([9]) studies bifinite domains defined this way. However, we will exhibit a general method of passing from pointed domains to domains without least element in Chapter 3, so it seems to us the right way first to restrict our attention to pointed domains.

The limiting projection d_i from a bifinite domain D onto the finite factor D_i is, composed with the corresponding embedding, a projection on D with finite image. Such functions play a prominent rôle throughout this work, so we introduce a name for them:

Definition. Let D be a dcpo. A continuous function $f\colon D \to D$, which is smaller than the identity on D and which has a finite image, is called a deflation.

Note that a deflation is a projection if and only if it is idempotent.

Proposition 1.25 *Let D be a dcpo and let $f\colon D \to D$ be a deflation on D. Then the following statements are true:*

(i) $\forall x \in D : f(x) \ll x$.

(ii) $f^2 \ll id_D$ *in* $[D \longrightarrow D]$.

(iii) $f^3 \ll f$ *in* $[D \longrightarrow D]$.

If f is an idempotent deflation then all elements in the image of f are compact and f is a compact element of $[D \longrightarrow D]$.

Proof. Let $A \subseteq D$ be a directed family such that $x \leq \bigvee^{\uparrow} A$. Applying f we get: $f(x) \leq f(\bigvee^{\uparrow} A) = \bigvee^{\uparrow}_{a \in A} f(a)$. Since the image of f is finite the latter set has a largest element $f(a_0)$. Hence we have $f(x) \leq f(a_0) \leq a_0$.

For the second part assume that $(g_i)_{i \in I}$ is a directed family of functions from D to D such that $\bigvee^{\uparrow}_{i \in I} g_i \geq id_D$. Then for all elements in the (finite) image of f there is some g_i such that $g_i(x) \geq f(x)$ holds. From directedness we get an index i_0 such that $g_{i_0}(x) \geq f(x)$ holds for all $x \in im(f)$. Thus we have for all $x \in D : g_{i_0}(x) \geq g_{i_0}(f(x)) \geq f(f(x)) = f^2(x)$.

The proof for part (iii) is similar. Let $(g_i)_{i \in I}$ be a directed family of functions with a supremum above f. By part (i) there is $i_0 \in I$ such that $g_{i_0}(x) \geq f^2(x)$ holds for all $x \in im(f)$. This implies $g_{i_0}(x) \geq g_{i_0}(f(x)) \geq f^2(f(x)) = f^3(x)$ for all $x \in D$.

The conclusions for idempotent deflations follow immediately. ▌

Theorem 1.26 *The following are equivalent for any dcpo D with least element.*

(i) *D is a bifinite domain.*

(ii) *The set of idempotent deflations on D is directed and has id_D as its supremum in $[D \longrightarrow D]$.*

(iii) *There exists some directed set $(d_i)_{i \in I}$ of idempotent deflations on D, the supremum of which is id_D.*

Proof. (i) $\Longrightarrow$ (ii) Let D be the limit of finite posets D_i. The mappings $e_i \circ d_i$ are idempotent deflations on D. For any $x \in D$ we show that $x = \bigvee^{\uparrow}_{i \in I} e_i \circ d_i(x)$.

The element $x \in D$ can be thought as a sequence $(x_i)_{i \in I}$. For $j \geq i_0$ we have $d_{i_0} \circ e_j \circ d_j((x_i)_{i \in I}) = x_{i_0}$ and hence $e_j \circ d_j$ leaves all components x_{i_0} of the sequence $(x_i)_{i \in I}$ with $i_0 \in \downarrow j$ fixed. I is directed and this proves our claim.

Now, if f and f' are any two idempotent deflations on D then we know by Proposition 1.25 that they are compact elements of the function space and therefore some $e_i \circ d_i$ must lie above both of them.

(ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (i) We show that D is isomorphic to the limit of the finite posets $im(d_i)$. The connecting morphism d_{ij} for $i \leq j$ is given by $f_i \big|_{im(f_j)}$. If we denote by D^* the limit of this system then we have the map s from D to D^* which maps each element $x \in D$ onto the sequence $(f_i(x))_{i \in I}$. The inverse mapping is given by $(x_i)_{i \in I} \mapsto \bigvee^{\uparrow}_{i \in I} x_i$. The details are easy to check. ▮

Corollary 1.27 *A bifinite domain is algebraic.*

Proof. This follows directly from Theorem 1.26 and Proposition 1.25. ▮

Theorem 1.28 *The category* **B** *of bifinite domains is cartesian closed and allows the formation of arbitrary products.*

Proof. We use the characterization given by Theorem 1.26. If D and E are bifinite domains and if $f_D \colon D \to D$ and $f_E \colon E \to E$ are idempotent deflations then $f_D \times f_E$ is a deflation on $D \times E$. This proves that $D \times E$ is again bifinite. On the function space $[D \longrightarrow E]$ we get the idempotent deflation F defined by $F(g) = f_E \circ g \circ f_D$.

For $(D_i)_{i \in I}$ an arbitrary collection of bifinite domains we construct idempotent deflations as follows: let J be a finite subset of I and fix an idempotent deflation f_j for each $j \in J$. Then $\prod_{i \in I} g_i$, where

$$g_i = \begin{cases} f_i, & \text{if } i \in J; \\ c_\perp, & \text{otherwise,} \end{cases}$$

is an idempotent deflation on $\prod_{i \in I} D_i$.

In all three cases the set of idempotent deflations constructed this way is directed and yields the identity function. ▮

How can we see that a given dcpo D is indeed a bifinite domain? By the preceding corollary we know that D must be algebraic. Also, for any finite set A of compact elements there must be an idempotent deflation f on D which fixes these elements. Any minimal upper bound of A must also be contained in the image of f since f is below id_D. By induction we find that minimal upper bounds of minimal upper bounds of ... of minimal upper bounds of A are kept fixed under f. We will now explore this idea in more detail since it will yield an internal characterization of bifinite domains.

Definition. Let D be a partially ordered set. We say that D has property m if for each finite set $A \subseteq D$ the set $mub(A)$ is complete, that is, for all $x \geq A$ there is a minimal upper bound y of A which lies below x.

If D has property m and if for each finite set of elements the set of minimal upper bounds is finite then D has property M.

Given a poset D with property m we define for any subset A of D

$$\begin{aligned} U^0(A) &= A, \\ U^{n+1}(A) &= \{x \in D \mid x \text{ is a minimal upper bound for} \\ &\qquad\qquad \text{some finite subset of } U^n(A)\,\}, \\ U^\infty(A) &= \bigcup_{n \in \mathbf{N}} U^n(A). \end{aligned}$$

Figure 1.3 shows a dcpo which does not have property m, Figure 1.4 shows a dcpo with property m but not property M. The dcpo in Figure 1.5 has property M but the set $U^\infty(\{a, b\})$ is infinite. These are the standard examples of posets which are not bifinite and we will show below that an algebraic dcpo, which does not contain copies of these, is indeed bifinite.

Lemma 1.29 *A poset D with property m has property M if and only if the empty set and each pair of elements have a finite set of minimal upper bounds.*

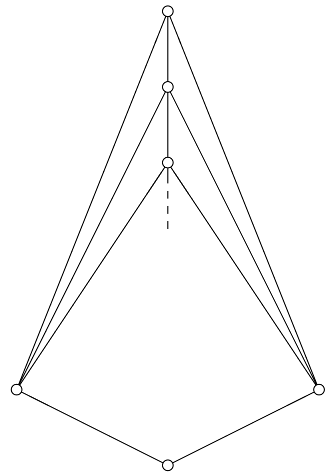

Figure 1.3: An algebraic dcpo which does not have property m.

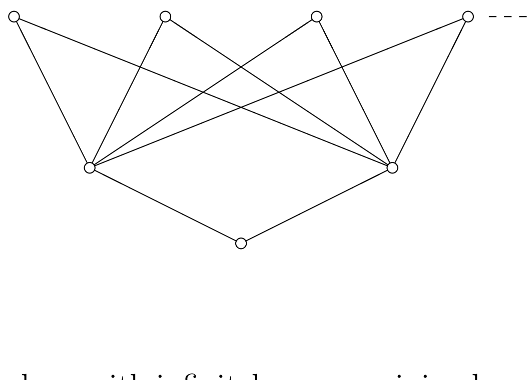

Figure 1.4: An algebraic dcpo with infinitely many minimal upper bounds for a pair of compact elements.

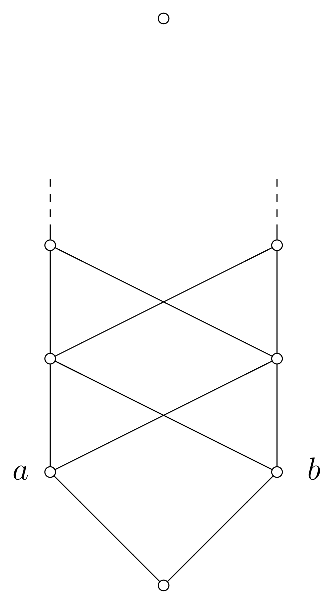

Figure 1.5: An algebraic dcpo, for which $U^{\infty}(\{a, b\})$ is infinite.

Proof. Suppose D has property M for pairs of elements. Let $A = \{a_1, a_2, \dots, a_n\}$ be a finite subset of D. We construct the set $mub(A)$ inductively:

$$\begin{aligned} M_2 &= mub(\{a_1, a_2\}), \\ M_{i+1} &= \bigcup_{x \in M_i} mub(\{x, a_{i+1}\}), 2 \leq i \leq n-1. \end{aligned}$$

The set M_n contains $mub(A)$: if x is any upper bound of A then it is above some element of M_2 and by induction it is above some element of M_n. All elements of M_n are upper bounds for A, so if x is minimal in $ub(A)$ then it must belong to M_n. It is clear that the set M_n is finite. ▌

The analogous statement for 'property m' is false, see Figure 1.6. Similarly, $U^\infty(A)$ may be finite for two element sets, but infinite for a triple of elements, see Figure 1.7.

It is also not true in general that property m for the base $K(D)$ of an algebraic dcpo D implies property m for D itself. In Figure 1.8 we give a counterexample. However, the following is true

Proposition 1.30 *If D is an algebraic dcpo and if $K(D)$ has property M then D is bicomplete.*

Proof. Let J be a filtered subset of D and let B be the set of compact lower bounds of J. We show that B is directed. If M is any finite subset of B then $mub(M)$ is finite and for each $j \in J$ there is some $x \in mub(M)$ which is below j. Hence the sets $(\downarrow j \cap mub(M))_{j \in J}$ form a filtered collection of finite nonempty sets and so their intersection is nonempty. This says that there is a minimal upper bound for M which is below all elements of J. Since it is compact by Proposition 1.9, it belongs to B.

Obviously, the directed supremum of B yields an infimum for J. ▌

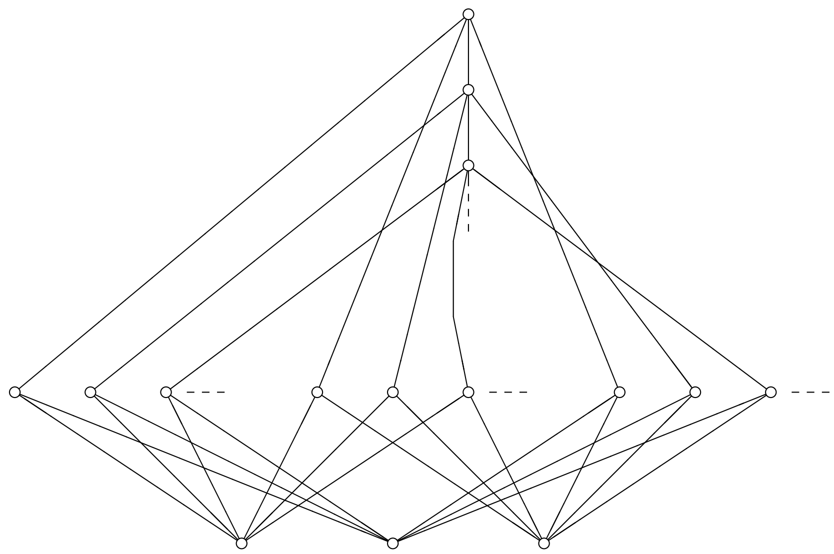

Figure 1.6: A poset in which every pair of elements has a complete set of minimal upper bounds but which does not have property m.

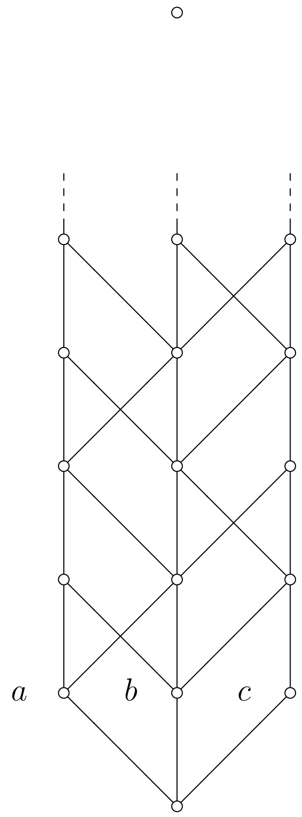

Figure 1.7: A poset in which every pair x, x' of elements yields a finite set $U^\infty(\{x, x'\})$ but in which there is a triple ($\{a, b, c\}$, for example), which generates an infinite set.

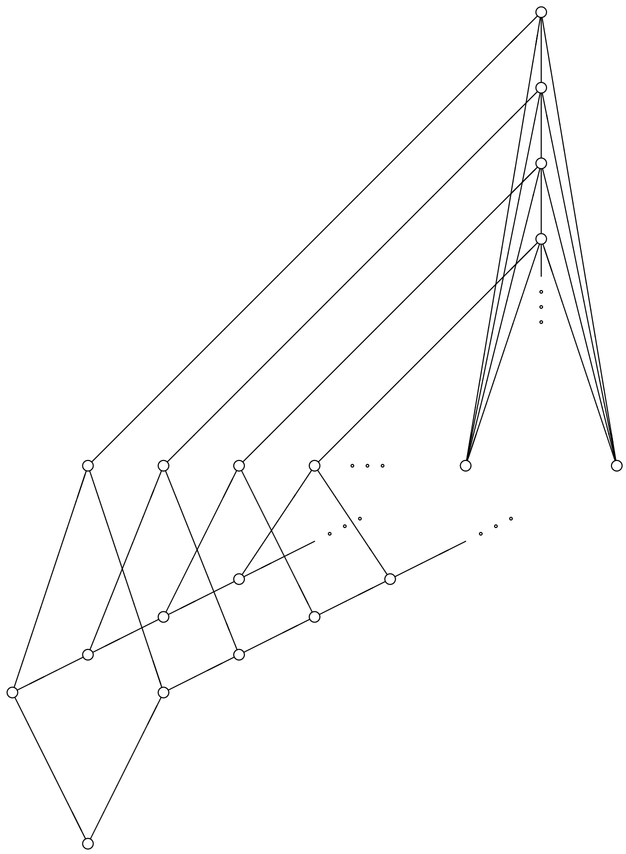

Figure 1.8: An algebraic dcpo, in which the base has property m but the dcpo itself doesn't.

Proposition 1.31 *Let D be a dcpo with property m and let A be a subset of D. The function $f\colon D \to D$ defined by*

$$f(x) = \bigvee^{\uparrow}\{e \in U^{\infty}(A) \mid e \leq x\}$$

is monotone, idempotent and below the identity function on D. If $U^{\infty}(A)$ consists of compact elements then f is continuous and therefore a projection.

Proof. Clearly, the set $\{e \in U^{\infty}(A) \mid e \leq x\}$ is directed since we assume property m. If x belongs to $U^{\infty}(A)$ then it is kept fixed by f and hence f is idempotent. The two other claims are trivial.

If $U^{\infty}(A)$ consists of compact elements only then let $(x_i)_{i\in I}$ be a directed family of elements. By compactness, any element of $U^{\infty}(A)$ which is below $\bigvee^{\uparrow}_{i\in I} x_i$ is already below some x_{i_0}. This proves continuity. ▮

We note that $U^{\infty}(A)$ consists of compact elements if $A \subseteq B(D)$ and D is a continuous dcpo. This is a consequence of Proposition 1.9.

Theorem 1.32 (G.Plotkin [19]) *An algebraic dcpo D with least element is bifinite if and only if $B(D)$ has property m and $U^{\infty}(A)$ is finite for all finite sets $A \subseteq B(D)$.*

Proof. For the 'if'-part let A be a finite subset of $B(D)$. The set $U^{\infty}(A)$ defines an idempotent deflation on D by Proposition 1.31. Given two idempotent deflations f, f' we construct the finite set $U^{\infty}(im(f) \cup im(f'))$ which contains the images of f and f' and defines an upper bound for them by Proposition 1.18(iv). Hence the set of idempotent deflations on D is directed and since every compact element is contained in the image of some idempotent deflation the supremum of all idempotent deflations equals the identity function on D. Theorem 1.26 asserts that D must be bifinite.

For the converse we also apply Theorem 1.26 and get that the set of idempotent deflations on D is directed with supremum id_D. For any finite set A of compact elements we can find an idempotent deflation f which contains A in its image. Since an idempotent deflation is smaller than the identity the image must contain all of $U^\infty(A)$ which is therefore a finite set. As for property m, note that any upper bound x of A is mapped onto an upper bound by f. The image of f is finite and so it contains a minimal upper bound of $A = f(A)$. ∎

Corollary 1.33 *A bifinite domain is bicomplete.*

Proof. Follows from Theorem 1.32 and Proposition 1.30. ∎

By the preceding corollary we know that a bifinite domain has a complete set of minimal upper bounds for arbitrary subsets. In general, the base of an algebraic dcpo may have property m for finite subsets but not for infinite subsets. Figure 1.9 shows an example of this.

Theorem 1.32 shows in particular that the base of a bifinite domain has property M. However, even in a bifinite domain the set of minimal upper bounds of a finite set of noncompact elements may be infinite. An example of this is given in Figure 1.10. The same effect we have for the U^∞-operator, see Figure 1.11.

In Chapter 4 we will study retracts of bifinite domains. These are continuous dcpo's and hence contain no distinguished base. The examples in Figure 1.10 and Figure 1.11 illustrate the difficulty in characterizing these domains internally.

1.5 Directed-complete partial orders with a continuous function space

Proposition 1.34 *Let D be a dcpo with a continuous function space and let $f\colon D \to D$ be way-below id_D. Then for all $d \in D$, $f(d)$ is way-below d.*

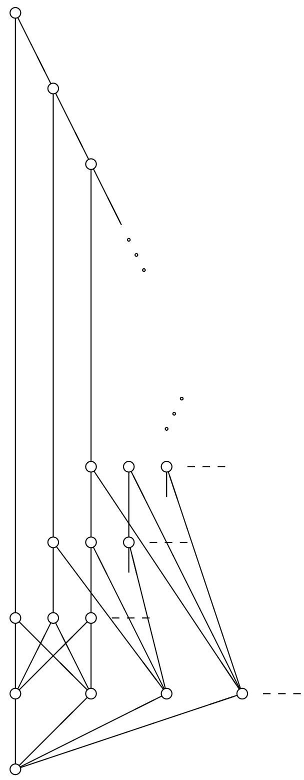

Figure 1.9: An algebraic dcpo with property m in which an infinite subset does not have a complete set of minimal upper bounds.

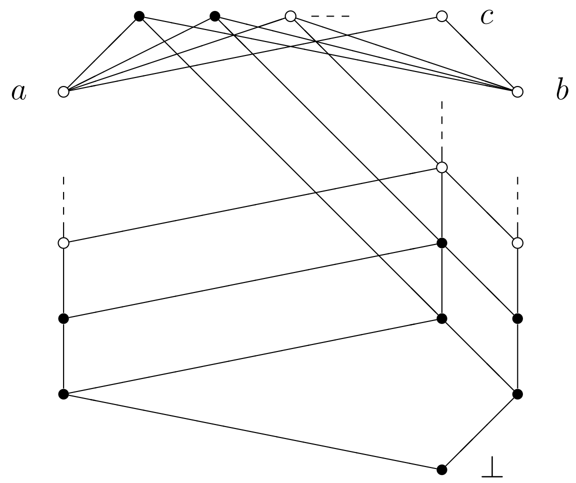

Figure 1.10: A bifinite domain with an infinite *mub*-set. (The filled dots indicate the image of an idempotent deflation on the domain.)

Proof. Let d be an element of D and let $(e_j)_{j\in J}$ be a directed family of elements with $\bigvee^{\uparrow} e_j = e \geq d$. By Proposition 1.22 the function space of $\downarrow e$ is also continuous. We use Proposition 1.5 in order to show that $f' = f\,|_{\downarrow e}$ is way-below $id_{\downarrow e}$. Let $({g'}_i)_{i\in I}$ be a directed family of functions on $\downarrow e$ such that $\bigvee^{\uparrow}_{i\in I} {g'}_i = id_{\downarrow e}$. We can extend each ${g'}_i$ to a function g_i on D by setting

$$g_i(x) = \begin{cases} {g'}_i(x), & \text{if } x \leq e; \\ x, & \text{otherwise.} \end{cases}$$

Clearly we have $\bigvee^{\uparrow}_{i\in I} g_i = id_D$. By assumption there is $i_0 \in I$ such that $g_i \geq f$ and therefore also ${g'}_i \geq f'$.

The collection $(e_j)_{j\in J}$ defines a directed family of constant functions $(c_{e_j})_{j\in J}$ on $\downarrow e$, the supremum of which is c_e. This is the largest function on $\downarrow e$ and hence is above $id_{\downarrow e}$. Therefore there is some function c_{e_j} which is above f' and this implies

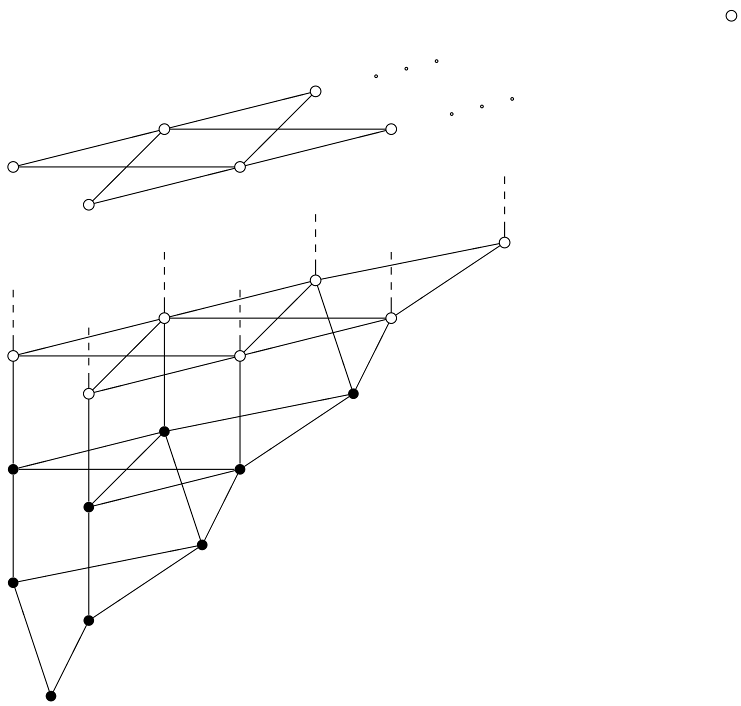

Figure 1.11: A bifinite domain in which the U^∞-operator yields an infinite set. (The filled dots show the image of an idempotent deflation.)

that $e_j = c_{e_j}(d) \geq f'(d) = f(d)$. ▌

Theorem 1.35 *A dcpo with continuous function space is itself continuous.*

Proof. From Proposition 1.34 it follows directly that each principal ideal in D is a continuous dcpo. Proposition 1.6 tells us that this implies that D as a whole is continuous. ▌

Corollary 1.36 *If D is a dcpo with a continuous function space and if for $f, g \in [D \longrightarrow D]$, f is way-below g, then $f(d)$ is way-below $g(d)$ for all $d \in D$.*

Proof. Let $f \ll g$ for arbitrary continuous functions $f, g \colon D \to D$. By the continuity of the function space we get $g = \bigvee^{\uparrow}_{h \ll id_D} h \circ g$ and hence there is a function $h \ll id_D$ such that $h \circ g \geq f$ holds. Together with Proposition 1.34 this gives us: $f(d) \leq h \circ g(d) \ll g(d)$. ▌

Theorem 1.37 *A dcpo with continuous function space is bicomplete.*

Proof. By Corollary 1.3 we have to find infima only for monotone injective nets $s \colon \alpha^{op} \to D$ where α is an ordinal number. To simplify notation let us identify the ordinal with its image in D. Denote by A the (possibly empty) set of lower bounds for α^{op} in D. We define a retraction onto $A \cup \alpha^{op}$:

$$r(x) = \begin{cases} x, & \text{if } x \in A; \\ \bigwedge\{\gamma \in \alpha^{op} \mid \gamma \geq x\}, & \text{otherwise.} \end{cases}$$

Since α is an ordinal there exists no strictly increasing infinite sequence in α^{op} and so the retraction is continuous. We apply Proposition 1.22 and get that the function space of $D' = A \cup \alpha^{op}$ is again continuous.

Assume now that the infimum of α^{op} does not exist, that is, the set A does not have a largest element. Then the set $\downarrow A$ cannot be directed. If A is not empty then we find $x'' \ll x \in A$ and $y'' \ll y \in A$ such that there is no upper bound for

$\{x'', y''\}$ in $\downarrow A$. By interpolating we find elements x', y' such that $x'' \ll x' \ll x$ and $y'' \ll y' \ll y$. For $\{x', y'\}$ there cannot be an upper bound even in A. By continuity of the function space of D' there is a function f on D' which is way-below $id_{D'}$ and which maps x above x' and y above y'. All elements of α^{op} are upper bounds for $\{x', y'\}$ so by construction α^{op} is mapped into itself under f. If A is empty this is trivially the case.

We proceed by showing that a function f which maps α^{op} into itself cannot be way-below $id_{D'}$. This contradiction will finish our proof. Consider the successor function τ on α^{op}, defined by $\tau(\gamma) = \gamma + 1$. The functions

$$g_\beta(x) = \begin{cases} \tau \circ f(x), & \text{if } x \in \alpha^{op}, x \le \beta, \\ x, & \text{otherwise.} \end{cases}$$

approximate $id_{D'}$ but none of them dominates f. ▌

Corollary 1.38 *If the function space of a dcpo is continuous then it is also bicomplete.*

Proof. This follows directly from the preceding theorem and Corollary 1.13. ▌

Proposition 1.39 *Let d be a compact element of a dcpo D and e be a compact element of a dcpo E with least element $\bot$. Then the following is a compact element of the function space $[D \longrightarrow E]$:*

$$d \searrow e(x) = \begin{cases} e, & \textit{if } x \ge d; \\ \bot, & \textit{otherwise.} \end{cases}$$

Proof. The function $d \searrow e$ is continuous because $\uparrow d$ is a Scott-open set in D. Any directed family of functions from D to E, whose supremum is above $d \searrow e$, must contain a member which maps d above e by the compactness of e. This function is then already above $d \searrow e$. ▌

Proposition 1.40 *Let D be an algebraic dcpo with least element and continuous function space. Then $[D \longrightarrow D]$ is algebraic.*

Proof. Given a continuous function $f\colon D \to D$ we have to show that the compact functions below f form a directed set with supremum f. If d and e are compact elements of D such that $e \leq f(d)$ then the function $d \searrow e$ is compact and below f. The supremum of all these functions below f is clearly equal to f. It remains to show that the set of all compact functions contained in $\downarrow f$ is directed.

The function space $[D \longrightarrow D]$ is bicomplete by Corollary 1.38, so given two compact functions g, g' below f, there is a minimal upper bound h of $\{g, g'\}$ below f. The function h must be compact by Proposition 1.9. ▌

Proposition 1.41 *Let D be a dcpo with an algebraic function space and let f be a compact element of $[D \longrightarrow D]$. Then for all $d \in D$ the element $f(d)$ is compact.*

Proof. This is an immediate consequence of Corollary 1.36. ▌

Theorem 1.42 *A dcpo with algebraic function space is itself algebraic.*

Proof. For d an element of the dcpo D the set $(f(d))_{f \ll f \in [D \to D]}$ is directed, consists of compact elements by Proposition 1.41, and has supremum d. ▌

Chapter 2

Domains with a least element

2.1 The theorem of Smyth

Gordon Plotkin, who defined bifinite domains ("SFP-objects") in 1976, conjectured that indeed this was the largest cartesian closed category which could be formed of ω-algebraic dcpo's. In 1983 this conjecture was confirmed by M.B.Smyth. The proof given by Smyth proceeds in two stages. He first shows that an ω-algebraic dcpo D with an ω-algebraic function space $[D \longrightarrow D]$ must be bifinite. He then proves that in a cartesian closed full subcategory of $\mathbf{DCPO}_\perp$ the exponential object must be isomorphic to the space of Scott-continuous functions. We have proved this part in greater generality in Section 1.3.

Smyth's proof of the first part utilizes three lemmas. In the first he shows that the base of an algebraic dcpo with least element and algebraic function space must have property m. This follows from Theorem 1.37. In a second lemma Smyth proves that if the base of an algebraic dcpo D has property m and if there is a pair of compact elements, for which there are infinitely many minimal upper bounds, then the function space cannot be countably based. This says that subposets looking like the one in Figure 1.4 will not occur. Finally he shows that with the properties guaranteed by the first two lemmas each finite subset A of $B(D)$ must yield a finite set $U^\infty(A)$. Otherwise the function space cannot be algebraic.

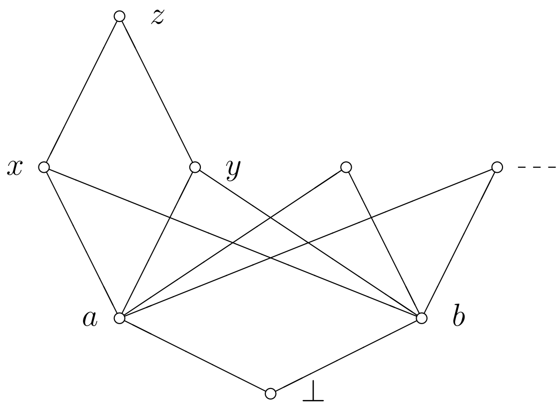

Figure 2.1: An algebraic dcpo, for which the function space is not algebraic.

Smyth's second lemma differs from the other two in the sense that it does not say, the absence of property M implies that the function space is not algebraic. It may be algebraic but its base is of the wrong size. The central result of this work emerged from a closer analysis of example 1.4. By actually calculating the function space we found that it is indeed an algebraic dcpo. But adding just one more element as in Figure 2.1 destroys algebraicity of the function space. We will explore the difference between Example 1.4 and Example 2.1 in the following sections and we will give a proof of Smyth's second lemma there (Lemma 2.17) but let us now finish stating the theorem.

We will need the following selection principle which is known as Rado's Compactness Theorem.

Theorem 2.1 *Let I be any set and for each $i \in I$ let A_i be a finite nonempty set. Given a selection function s_J for each finite subset J of I, that is, a function from J to $\bigcup_{i \in J} A_i$ such that $s_J(i) \in A_i$ for all $i \in J$, there is a selection function s_I on*

I with the following property: Given a finite set $J \subseteq I$ there is a finite set $J' \supseteq J$ with $s|_J = s_{J'}|_J$.

Proof. Each set A_i is a compact space in the discrete topology. By Tychonoff's Compactness Theorem the product $A = \prod_{i \in I} A_i$ is compact in the product topology. The set $P_F(I)$ of all finite subsets of I is directed. We construct a (topological) net $\alpha : P_F(I) \to A$. Let $(x_i)_{i \in I}$ be a fixed element of A. We define $\alpha(J) = (y_i)_{i \in I}$ where

$$y_i = \begin{cases} s_J(i), & \text{if } i \in J; \\ x_i, & \text{otherwise.} \end{cases}$$

By compactness there exists a convergent subnet β of α with limit $(z_i)_{i \in I}$. The global selection function $s : I \to \bigcup_{i \in I} A_i$ can now be defined as $s(i) = z_i$. It follows directly from the definition of the product topology that s coincides locally with one of the given local selection functions. ∎

Lemma 2.2 *If D is a dcpo with algebraic function space and if $B(D)$ has property M then $U^\infty(A)$ is finite for each finite set A of compact elements.*

Proof. By contradiction: assume that $U^\infty(A)$ is infinite. Since the base of D has property M each set $U^n(A)$ is finite and contains elements which are not in $U^{n-1}(A)$ already. So for each $n \in \mathbf{N}$ we have the finite nonempty set $B_n = U^n(A) \setminus U^{n-1}(A)$. Each element of B_n is above some element of B_{n-1} because otherwise it would belong to $U^{n-1}(A)$ already.

Given a natural number n we choose a selection function

$$s_n : \{1, \ldots, n\} \to \bigcup_{m \leq n} B_m$$

by first assigning a value to n out of the set B_n then to $n-1$ out of $B_{n-1} \cap {\downarrow} s_n(n)$ and so on. By Rado's Compactness Theorem we find a global selection function $s : \mathbf{N} \to \bigcup_{m \in \mathbf{N}} B_n$ which coincides locally with one of the selection functions s_n. In particular, s is injective, monotone, and $C = \{s(n) \mid n \in \mathbf{N}\}$ is a chain in $U^\infty(A)$.

Using the fact that the function space is algebraic we find a continuous mapping $f: D \to D$ approximating id_D which fixes all elements of A. Since f is below the identity it must also fix minimal upper bounds of subsets of A and by induction we see that in fact it keeps all elements of $U^\infty(A)$ fixed. We apply this to the chain C: it is fixed by f and hence its supremum $c = \bigvee^\uparrow C$ is mapped onto itself. But this contradicts Corollary 1.36 where we proved that f should map each element of D way-below itself. ❙

Theorem 2.3 (M.B.Smyth 1983) *If D is an algebraic dcpo with least element and if $[D \longrightarrow D]$ is ω-algebraic then D is bifinite.*

Proof. We have proved in Theorem 1.37 that a dcpo with algebraic function space is bicomplete, hence D has property m. In Section 2.3, Lemma 2.17, we will show that D must have property M or the function space has uncountably many compact elements. The preceding lemma then tells us that the U^∞ operator maps finite sets of compact elements onto finite sets. By Theorem 1.32 this implies that D is bifinite. ❙

Corollary 2.4 *The category ω-**B** of countably based bifinite domain is the largest cartesian closed full subcategory of ω-$\mathbf{ALG}_\perp$.*

Proof. Let **C** be any cartesian closed full subcategory of ω-$\mathbf{ALG}_\perp$ and let D be any object of **C**. By Lemma 1.21 the exponential object D^D is isomorphic to the space $[D \longrightarrow D]$ of Scott-continuous functions which therefore is itself ω-algebraic. From Smyth's Theorem 2.3 we infer that D must be bifinite, hence the whole category **C** is contained in **B**. ❙

Theorem 1.42 of Section 1.5 shows that we can weaken the hypothesis in Smyth's Theorem:

Corollary 2.5 *If D is a dcpo with least element and if $[D \longrightarrow D]$ is ω-algebraic then D is bifinite.*

All proofs concerning the last corollary employed only strict functions or the space $[D \stackrel{s}{\longrightarrow} D]$ of strict functions so we have the following:

Corollary 2.6 *If D is a dcpo with least element and if $[D \stackrel{s}{\longrightarrow} D]$ is ω-algebraic then D is bifinite.*

This gives us the following interesting equivalence:

Corollary 2.7 *For a dcpo D the following are equivalent:*

(i) *D is bifinite and countably based.*

(ii) *$[D \longrightarrow D]$ is ω-algebraic.*

(iii) *$[D \stackrel{s}{\longrightarrow} D]$ is ω-algebraic.*

This is remarkable as — so far — all attempts to prove (ii) from (iii) directly have failed. It constitutes the first application of Smyth's Theorem apart from his own maximality result (Corollary 2.4).

2.2 L-domains

Definition. A dcpo D with least element is called an L-domain if for all $x \in D$ the principal ideal $\downarrow x$ is a complete lattice. It is called algebraic L-domain (continuous L-domain) if it is also an algebraic (continuous) dcpo. (By Proposition 1.6 this is the case if and only if each principal ideal is an algebraic (continuous) lattice.)

The corresponding categories are denoted **Ldom** for L-domains, **L** for algebraic L-domains, and **cL** for continuous L-domains.

Figure 1.4 shows an L-domain. The posets in Figure 2.1 and 2.2 are not L-domains. The following example of L-domains occuring in General Topology was pointed out to me by K.Keimel and J.Lawson:

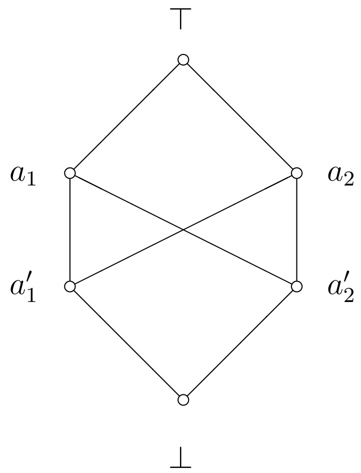

Figure 2.2: $\mathbf{X}_\bot^\top$: The smallest pointed poset which is not an L-domain.

Example. Let X be a connected and locally connected compact space and let D be the collection of all connected closed subsets of X ordered by superset inclusion. We claim that D is a continuous L-domain.

First of all, X is an element of D and serves as a least element. If $(A_i)_{i\in I}$ is a filtered collection of elements of D then the intersection $A = \bigcap_{i\in I} A_i$ is again a closed subset. It is also connected: suppose $A = B \cup B'$ with $B \cap B' = \phi$ and both B and B' are closed. Then B and B' are compact and there are open subsets O, O' of X such that $B \subseteq O, B' \subseteq O'$, and $O \cap O' = \phi$. Again by the compactness of X we get that there is some A_i which is contained in the open neighborhood $O \cup O'$ of B. This contradicts the connectedness of A_i as $A_i \cap O$ and $A_i \cap O'$ is a disjoint open covering of A_i. Hence D is a dcpo.

Let now $(A_s)_{s\in S}$ be any collection of elements of D and let B be a compact connected set contained in $A = \bigcap_{s\in S} A_s$. Let A_B be the connected component of B in A. Then A_B is the supremum of the A_s in the principal ideal $\downarrow B$. If B' is a closed connected neighborhood of the connected compact set B then B' is way-below B in the lattice $\downarrow B$. This completes the proof that D is a continuous L-domain. ❙

Proposition 2.8 *Let D be an L-domain.*

(i) *If A is a subset of D and if x and y are comparable elements above A, then the supremum of A formed in $\downarrow x$ is the same as the supremum formed in $\downarrow y$.*

(ii) *Every ideal in D is a lattice.*

Proof. (i) Let $A \leq x \leq y$ and let a be the supremum of A in $\downarrow y$. Each element of $\downarrow x$ belongs also to $\downarrow y$ and so an upper bound of A in $\downarrow x$ must be greater than or equal to a. Hence a is also the supremum of A in $\downarrow x$.

(ii) In an ideal every pair a, b of elements is bounded. So by definition their supremum and infimum exist locally. The first part tells us that both supremum and infimum (consider $lb(\{a, b\})$) do not depend on the choice of the upper bound. ❙

For the following recall that a poset D is said to be connected if every two elements x and y can be connected by a zigzag in D, i.e. there is $n \in \mathbf{N}$ and there are $x_0, \ldots, x_n, y_1, \ldots, y_n \in D$ such that $x = x_0, y = x_n$ and $x_i \leq y_j$ whenever $0 \leq j - i \leq 1$.

Theorem 2.9 *For a dcpo D with least element the following are equivalent:*

(i) *D is an L-domain.*

(ii) *For every bounded nonempty subset of D the infimum exists.*

(iii) *For every connected nonempty subset of D the infimum exists.*

Proof. (i) $\Longrightarrow$ (ii) Let A be bounded by $x \in D$ and let a be the infimum of A formed in $\downarrow x$. Since A is nonempty every lower bound of A belongs to $\downarrow x$ and is thus below a.

(ii) $\Longrightarrow$ (i) Given $x \in D$ we can form infima of nonempty sets in $\downarrow x$. The infimum for the empty set is (locally) given by x itself. Thus $\downarrow x$ is a complete lattice.

(ii) $\Longrightarrow$ (iii) First let $A \subseteq D$ be a zigzag, that is, $A = \{x_0, \ldots, x_n, y_1, \ldots, y_n\}$ and $x_i \leq y_j$ for $0 \leq j - i \leq 1$. We define the infimum of A inductively:

$$\bigwedge A = (\ldots((x_0 \wedge x_1) \wedge x_2) \wedge \ldots \wedge a_{n-1}) \wedge a_n.$$

All pairwise infima in this definition exist by (ii). They are global and therefore $\bigwedge A$ is the global infimum of A. For the general case observe that any connected set is the directed union of finite zigzags. This gives us a filtered collection of partial meets whose infimum exists by (ii).

The implication (iii) $\Longrightarrow$ (ii) is trivial. ▌

We can also recognize an L-domain by studying minimal upper bounds:

Theorem 2.10 *For a dcpo D with least element, the following are equivalent:*

(i) *D is an L-domain.*

(ii) *For each upper bound x of a subset A of D there is a unique minimal upper bound of A below x.*

(iii) *D has property m and for all subsets A of D, $U^\infty(A) = U^1(A)$.*

If D is algebraic, then the following statements are equivalent:

(iv) *D is an algebraic L-domain.*

(v) *For each $x \in D$ the set $\downarrow x \cap B(D)$ is a $\vee$-semilattice with smallest element.*

(vi) *For each upper bound x of a finite subset A of $B(D)$ there is a unique minimal upper bound of A below x.*

(vii) *For each upper bound x of a pair of compact elements there is a unique minimal upper bound below x.*

(viii) *The base of D has property m and for all finite subsets A of $B(D)$, $U^\infty(A) = U^1(A)$.*

Proof. (i) $\Longrightarrow$ (ii) For $x \geq A$ form the supremum of A in the complete lattice $\downarrow x$.

(ii) $\Longrightarrow$ (iii) Let x be a minimal upper bound of a finite subset B of $U^1(A)$. Each element b of B is a minimal upper bound for some finite subset A_b of A. Since $x \geq b$ the element b is the unique minimal upper bound $min_{A_b}(x)$ of A_b below x. Observe that $min_{A_1}(x) \geq min_{A_2}(x)$ whenever A_2 is contained in A_1. So for $A' = \bigcup_{b \in B} A_b$ the element $min_{A'}(x)$ is above all elements of B. Since x is a minimal upper bound of B it equals $min_{A'}(x)$ and so it is contained in $U^1(A)$ already.

(iii) $\Longrightarrow$ (i) Let x be an element of D and A be a finite subset of $\downarrow x$. By property m there is a minimal upper bound a of A below x. Suppose there is a second minimal upper bound a' of A below x. Again property m tells us that there must be a minimal upper bound b of $\{a, a'\}$ below x. This element b cannot belong to $U^1(A)$, so $U^2(A) \setminus U^1(A)$ is nonempty, contradicting (iii). Thus we have shown that inside $\downarrow x$ suprema of finite sets exist. We also have suprema for directed sets, so $\downarrow x$ is a complete lattice.

Now assume that D is an algebraic dcpo. The implications (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) $\Longrightarrow$ (vii) $\Longrightarrow$ (viii) $\Longrightarrow$ (v) are proved as in the nonalgebraic case. (Recall that in an algebraic dcpo a minimal upper bound of a finite set of compact elements is again compact by Proposition 1.9.)

We prove (v) $\Longrightarrow$ (iv). Let x be an upper bound of an arbitrary subset A of D. By (v) the set $\bigcup_{a \in A} B(\downarrow a)$ generates a $\vee$-subsemilattice B_A in the $\vee$-semilattice $\downarrow x \cap B(D)$. In particular, B_A is directed and the supremum $a = \bigvee^{\uparrow} B_A$ is the supremum for A in $\downarrow x$. ▮

Theorem 2.11 *The categories* **Ldom** *and* **L** *are cartesian closed. In addition, infinite products exist in both categories.*

Proof. Clearly the one-point domain is an algebraic L-domain and serves as a terminal object in both **Ldom** and **L**. It is also easy to see that the cartesian product of a set of (algebraic) L-domains is again an (algebraic) L-domain. (Note that compact elements in an infinite product are those vectors, for which all components are compact and almost all components are equal to the respective bottom element.)

As for the proof that $[D \longrightarrow E]$ is again an L-domain, let A be any collection of functions from D to E bounded by a function f. We define:

$$g(x) = \bigvee_{a \in A} a(x),$$

where the supremum on the right is taken in the complete lattice $\downarrow f(x)$. It is clear that g is the supremum of A inside $\downarrow f$ provided g is continuous.

$$\begin{aligned} g(\bigvee_{i\in I}^{\uparrow} x_i) &= \bigvee_{a\in A} a(\bigvee_{i\in I}^{\uparrow} x_i) \\ &= \bigvee_{a\in A} \bigvee_{i\in I}^{\uparrow} a(x_i) \\ &= \bigvee_{i\in I}^{\uparrow} \bigvee_{a\in A} a(x_i) \\ &= \bigvee_{i\in I}^{\uparrow} g(x_i). \end{aligned}$$

(Note that by Proposition 2.8 all suprema may be taken in the complete lattice $\downarrow f(\bigvee^{\uparrow}_{i\in I} x_i)$.) Now let D and E be algebraic. Using Proposition 1.39 we find that every function f from D to E is the supremum of compact functions of the form $a \searrow b$. It remains to show that the set of compact functions below f is directed. But this is also clear since $\downarrow f$ is a complete lattice and the supremum of finitely many compact elements in a lattice is again compact. Proposition 1.10 tells us that compact elements of $\downarrow f$ are also globally compact. ▌

In Section 4.3 we show that continuous L-domains are exactly the retracts of algebraic L-domains. By Theorem 1.23, this implies that the class **cL** is cartesian closed, too.

Theorem 2.12 *Limits of codirected systems exist in* $\mathbf{Ldom}^p$ *and* $\mathbf{L}^p$.

Proof. Let D^* be the limit of the system $((D_i)_{i\in I},(d_{ij})_{i\leq j})$ in $\mathbf{DCPO}^p_\bot$ as constructed in Theorem 1.24. Let $x=(x_i)_{i\in I}$ be an element of D^*. The set $\downarrow x\subseteq D^*$ is the limit of the codirected system of (algebraic) lattices $\downarrow x_i\subseteq D_i$. Our theorem then follows from the corresponding result about (algebraic) lattices. ▌

2.3 The two maximal cartesian closed categories of algebraic directed-complete partial orders with a least element

Lemma 2.13 *Let D and E be algebraic dcpo's with least element and with property m. If E is not an L-domain and if $B(D)$ does not have property M then $[D\longrightarrow E]$ is not continuous.*

Proof. If E is not an L-domain then by Theorem 2.10 (vii) there exists c in E and a pair $\{a_1,a_2\}$ of compact elements such that there are at least two minimal upper bounds b_1,b_2 of $\{a_1,a_2\}$ below c. Furthermore, let $\{x_1,x_2\}\subseteq B(D)$ be a pair of elements such that the set $mub\{x_1,x_2\}$ is infinite.

Assume that $[D\longrightarrow E]$ is continuous. Then we can define $g\colon D\to E$ by

$$g(d)=\begin{cases}\bot, & \text{if } d\not\geq x_1, d\not\geq x_2;\\ a_1, & \text{if } d\geq x_1, d\not\geq x_2;\\ a_2, & \text{if } d\not\geq x_1, d\geq x_2;\\ b_1, & \text{if } d\geq x_1, d\geq x_2.\end{cases}$$

Since $[D\longrightarrow E]$ is continuous, g — as a minimal upper bound of the compact functions $x_1\searrow a_1$ and $x_2\searrow a_2$ — should be compact (Proposition 1.9). On the other hand, for each finite subset A of $mub(\{x_1,x_2\})$ we have a function $f_A\colon D\to E$ defined as follows:

$$f_A(d)=\begin{cases}\bot, & \text{if } d\not\geq x_1, d\not\geq x_2;\\ a_1, & \text{if } d\geq x_1, d\not\geq x_2;\\ a_2, & \text{if } d\not\geq x_1, d\geq x_2;\\ b_2, & \text{if } d\in mub(\{x_1,x_2\})\setminus A;\\ c, & \text{otherwise.}\end{cases}$$

The supremum of the directed family (f_A), A a finite subset of $mub(\{x_1, x_2\})$, maps all of $mub(\{x_1, x_2\})$ onto c and is therefore above g. But no member of this collection is above g and this contradicts compactness. ❙

Theorem 2.14 *If D is a pointed dcpo with an algebraic function space then D is a bifinite domain or D is an algebraic L-domain.*

Proof. By Theorem 1.42 we know that D is algebraic and Theorem 1.37 tells us that D is bicomplete. Hence $B(D)$ has property m. From the preceding lemma we infer by contraposition that either D is an L-domain or $B(D)$ has property M. In the latter case we can apply Lemma 2.2 and Theorem 1.32 and get that D must be a bifinite domain. ❙

Corollary 2.15 *The category* $\mathbf{ALG}_\perp$ *contains exactly two maximal full subcategories which are cartesian closed:* $\mathbf{B}$ *and* $\mathbf{L}$.

Proof. Let $\mathbf{C}$ be a cartesian closed full subcategory of $\mathbf{ALG}_\perp$ and let D, E be objects in $\mathbf{C}$. We have shown in Lemma 1.21 that the exponential objects D^D, E^E, D^E, and E^D are isomorphic to the respective sets of Scott-continuous functions. So these function spaces are algebraic and by Theorem 2.14 we know that both D and E must belong to $\mathbf{B} \cup \mathbf{L}$. But we cannot have $D \in \mathbf{B} \setminus \mathbf{L}$ and $E \in \mathbf{L} \setminus \mathbf{B}$ by Lemma 2.13. Therefore $\mathbf{C}$ is completely contained either in $\mathbf{B}$ or in $\mathbf{L}$. These two categories are also not contained in each other: Figure 1.4 shows a dcpo belonging to $\mathbf{L} \setminus \mathbf{B}$, Figure 2.2 a dcpo belonging to $\mathbf{B} \setminus \mathbf{L}$. ❙

Figure 2.3 shows the two maximal cartesian closed full subcategories of $\mathbf{ALG}_\perp$ as well as their intersection, the bifinite L-domains. We should mention here that bifinite L-domains have been described earlier by C.Gunter ([9]). Gunter used the characterization (viii) of Theorem 2.10 and proved cartesian closedness for this class.

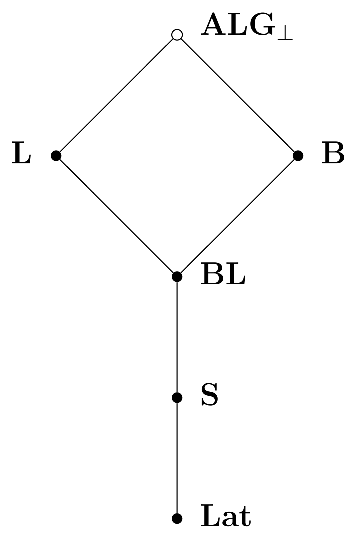

Figure 2.3: The hierarchy of cartesian closed categories of algebraic domains with least element.

In the picture are also the classes **S** of Scott-domains (also called bounded-complete dcpo's or algebraic semilattices) and **Lat** of algebraic lattices.

We note that both Lemma 2.13 and Theorem 2.14 hold true if we replace the function spaces by the corresponding strict function spaces. This gives us the following stronger version of Corollary 2.7:

Corollary 2.16 *For a dcpo D with least element the following are equivalent:*

(i) $[D \longrightarrow D]$ *is algebraic.*

(ii) $[D \stackrel{s}{\longrightarrow} D]$ *is algebraic.*

We still have to fill the gap in the proof of Theorem 2.3 which leads to Smyth's maximality result for the class of countably based algebraic dcpo's. Using Lemma 2.13 this can be done rather smoothly.

Lemma 2.17 *Let D be a pointed dcpo with an ω-algebraic function space. Then the base of D has property M.*

Proof. By Theorem 1.37 we already know that $B(D)$ must have property m. So assume that there is a finite set A of compact elements in D for which $mub(A)$ is infinite. Using Lemma 1.29 we may restrict ourselves to the case that A contains only the two elements a_1 and a_2. The functions $a_1 \searrow a_1$ and $a_2 \searrow a_2$ are compact. We will construct uncountably many minimal upper bounds for them — which are all compact by Proposition 1.9 — and this will contradict the assumption that $[D \longrightarrow D]$ is ω-algebraic.

Let b_1, b_2 be any two distinct elements of $mub(A)$. For any subset S of $mub(A)$ we define the function $f_S: D \to D$ by

$$f_S(x) = \begin{cases} \bot, & \text{if } x \not\geq a_1, x \not\geq a_2; \\ a_1, & \text{if } x \geq a_1, x \not\geq a_2; \\ a_2, & \text{if } x \not\geq a_1, x \geq a_2; \\ b_1, & \text{if } \exists s \in S: x \geq s; \\ b_2, & \text{otherwise.} \end{cases}$$

By Lemma 2.13 we have that D is an algebraic L-domain, so any element above both a_1 and a_2 is above exactly one element of $mub(A)$. This implies that f_S is monoton. It is easy to see that each f_S is also continuous and a minimal upper bound for $\{a_1 \searrow a_1, a_2 \searrow a_2\}$ and that $f_S \neq f_{S'}$ if $S \neq S'$. This completes our proof. ❙

Chapter 3

Domains without least element

3.1 Disjoint unions of domains

One way to pass from domains with a least element to those without a least element is to take disjoint unions.

Definition. A dcpo D for which every connected component belongs to **Ldom** is called a UL-domain. (Union of L-domains.)

A dcpo D for which every component belongs to **B** is called a UB-domain. (Union of bifinite domains.)

The corresponding categories are denoted by **ULdom** and **UB**. The category of algebraic UL-domains is denoted by **UL**.

Theorem 3.1 *The categories* **ULdom***,* **UL***, and* **UB** *are cartesian closed.*

Proof. The one-point domain is contained in all three categories under consideration and serves as a terminal object. If D, E are UL-domains and consist of components $(D_j)_{j\in J}$ and $(E_i)_{i\in I}$ then their product has the components $(D_j \times E_i)_{(j,i)\in J\times I}$ and thus belongs to **ULdom**. The function space $[D \longrightarrow E]$ can be written as a disjoint union:

$$\bigcup_{\alpha:J\to I} \prod_{j\in J} [D_j \longrightarrow E_{\alpha(j)}]$$

and from Theorem 2.11 we know that this is again a UL-domain. The same holds for **UB** and **UL**. ∎

We will prove below that **UL** and **UB** indeed form maximal cartesian closed full subcategories of **ALG**. But this cannot be the whole story as the class of all finite partially ordered sets is clearly cartesian closed but not contained in one of the U-categories just defined.

Lemma 3.2 *Let D and E be algebraic dcpo's with property m. If there is a principal Scott-open filter $\uparrow d$ in D for which $B(\uparrow d)$ does not have property M and if there is a principal Scott-open filter $\uparrow e$ in E which is not an L-domain, then $[D \longrightarrow E]$ is not continuous.*

Proof. The filters $\uparrow d$ and $\uparrow e$ are retracts of D and E, respectively. Hence the function space $[\uparrow d \longrightarrow \uparrow e]$ is a continuous retract of $[D \longrightarrow E]$. Applying Lemma 2.13 finishes our proof. ∎

Corollary 3.3 *Let D be a dcpo with an algebraic function space $[D \longrightarrow D]$. Then either all principal Scott-open filters are bifinite or they are all L-domains.*

Every element of an algebraic dcpo lies in some principal Scott-open filter but even if all such filters are algebraic lattices, as in the two examples in Figure 3.1, the dcpo is not necessarily contained in a cartesian closed full subcategory of **ALG**. In the next section we explore what additional conditions we must impose in order to get a well behaved domain.

3.2 The root of an ordered set

Given a dcpo D and two functions f and f' below the identity on D we may form the composition $f \circ f'$ and get a lower bound for $\{f, f'\}$ in $[D \longrightarrow D]$. This shows that the set $\downarrow id_D$ in $[D \longrightarrow D]$ is filtered. If D is bicomplete then we can form

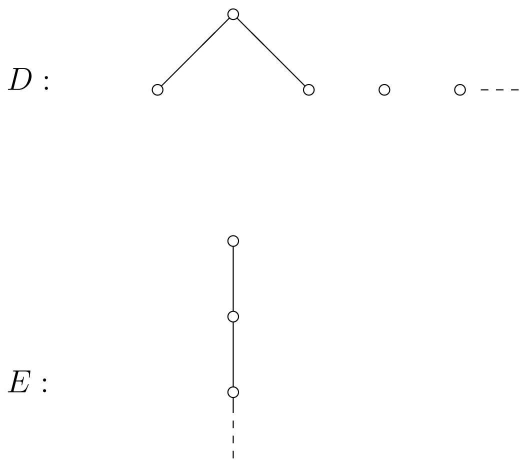

Figure 3.1: Algebraic dcpo's in which every filter is a lattice, but which do not have an algebraic function space.

the pointwise infimum of the functions in $\downarrow id_D$. In this case we get an idempotent monotone function below the identity which we call the root function r_D. If D is bicomplete and continuous then we know from Corollary 1.13 that r_D is continuous and hence a projection on D.

Projections are completely determined by their image and so we can try to characterize the image of the root function directly. If x is a minimal element in a dcpo D then every function below id_D must keep x fixed. Similarly, any minimal upper bound of a set of minimal elements must be kept fixed. By induction we see that all elements of $U^\infty(\phi)$ must belong to the image of the root function. But there is more: any element, which can be expressed as the supremum of a directed collection of elements of $U^\infty(\phi)$, also belongs to $im(r_D)$. Let us denote the collection of all such elements by $\widetilde{U^\infty(\phi)}$ for the moment. (Note that $\widetilde{U^\infty(\phi)}$ is not the Scott-closure of $U^\infty(\phi)$.)

Proposition 3.4 *Let D be continuous and bicomplete. Then $im(r_D) = \widetilde{U^\infty(\phi)}$.*

Proof. We have already argued that $\widetilde{U^\infty(\phi)}$ must belong to the image of r_D. For the converse we construct a projection p onto $\widetilde{U^\infty(\phi)}$:

$$p(x) = \bigvee^{\uparrow}\{y \in U^\infty(\phi) \mid y \leq x\}.$$

Proposition 1.31 tells us that $p: D \to D$ has the required properties.

From Proposition 1.18 we get that the image of r_D is contained in the image of p. ∎

If we do not ask for bicompleteness then the root function does not necessarily exist: an example is given by the negative numbers with their natural ordering. If we remove continuity from the hypothesis of Proposition 3.4 then the root function is not necessarily continuous: an example is the poset shown in Figure 3.2.

Definition. For a dcpo D we define the root of D ($rt(D)$) to be the set $U^\infty(\phi)$. We call D well rooted if the root of D is finite, consists of compact elements and if below each element of D there is a largest root element.

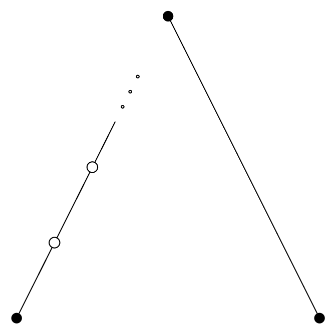

Figure 3.2: A dcpo in which $\widetilde{U^\infty(\phi)}$ (filled dots) is not the image of a projection.

Proposition 3.5 *Let D be a continuous dcpo.*

(i) *The root of D is contained in the set $B(D)$ of compact elements.*

(ii) *If D has property m then D is well rooted if and only if $rt(D)$ is finite.*

Proof. (i) As noted before, this follows directly from Proposition 1.9.

(ii) If D has property m then the set $r_x = {\downarrow}x \cap U^\infty(\phi)$ is directed, so if $rt(D)$ is finite, r_x has a largest element. ▮

Proposition 3.6 *For a well rooted dcpo the root function exists and is continuous.*

Proof. Mapping each element of D onto the largest root element below it is a continuous operation. ▮

Lemma 3.7 *A dcpo D is well rooted if and only if there is a deflation on D.*

Proof. If D is well rooted then the root function exists and is a deflation on D.

Assume that there is a deflation $f: D \to D$ on D. By iterating f we get an idempotent deflation. For the sake of simplicity we call it f again. An element of $rt(D)$ must be mapped onto itself by f so the root is finite. By Proposition 1.25 all elements in the image of f are compact. It remains to show that each element of D is above a largest root element. For this we prove that the root of D and the root of $im(f)$ coincide: we have already argued that the root of D is contained in $im(f)$ and since this is a subset of D it must belong to $rt(im(f))$. Conversely, a minimal upper bound of some finite subset of $im(f)$ in $im(f)$ is also a minimal upper bound with respect to D as f is idempotent and below the identity. Given any element $x \in D$, we find that it is above $f(x)$ which in turn is above a largest element of $rt(im(f)) = rt(D)$. ❚

Proposition 3.8 *If D and E are well rooted dcpo's then so is $[D \longrightarrow E]$.*

Proof. Given the root functions r_D and r_E we can construct an idempotent deflation F on $[D \longrightarrow E]$: $F(f) = r_E \circ f \circ r_D$. ❚

Lemma 3.9 *If the root of a bicomplete continuous dcpo D is infinite then the function space $[D \longrightarrow D]$ has infinitely many minimal elements.*

Proof. Given an element $d \in rt(D)$ we have the canonical retraction r_d onto $\downarrow d$ as defined in Section 1.3. On the ideal $\downarrow d$ this retraction equals the identity, so if $f: D \to D$ is any mapping below r_d it must still map d onto itself.

If $d \neq d'$ we have $d \not\leq d'$ or $d' \not\leq d$. Without loss of generality assume $d \not\leq d'$. For any function f below r_d we then get $f(d) = r_d(d) = d \not\leq d' = r_{d'}(d)$ and therefore $f \not\leq r_{d'}$. This proves that two retractions $r_d, r_{d'}$ with $d \neq d'$ have no common lower bound in $[D \longrightarrow D]$, hence the set of minimal elements in $[D \longrightarrow D]$ is infinite. (Note that we have a minimal element below each function because of Corollary 1.13.) ❚

3.3 The four maximal cartesian closed categories of algebraic directed-complete partial orders

As we don't have a least element we will study the Scott-open principal filters of a domain. We start off with the following observation which is dual to Proposition 1.6.

Proposition 3.10 *A well rooted dcpo D is algebraic (continuous) if and only if each principal Scott-open filter in D is algebraic (continuous).*

Proof. Let c be a compact element of D and let d be a (locally) compact element of $\uparrow c$. If $(x_i)_{i\in I}$ is a directed family of elements in D such that $\bigvee^{\uparrow}_{i\in I} x_i \geq d$ then some x_{i_0} belongs to $\uparrow c$ already and from the compactness of d in $\uparrow c$ we get that some x_{i_1} is above d. Hence any locally compact element is also globally compact. ∎

The poset in Figure 3.2 has algebraic principal filters but is not an algebraic dcpo itself. This illustrates that we have to confine our considerations to Scott-open filters. On the other hand, each element of a well rooted dcpo is contained in some Scott-open principal filter so we don't miss elements in these domains.

Definition. A dcpo which is well rooted and in which every principal Scott-open filter is bifinite, is called an FB-domain. A dcpo which is well rooted and in which every principal Scott-open filter is an L-domain, is called an FL-domain.

The corresponding categories are denoted by **FB** and **FLdom**, respectively. The category of algebraic FL-domains is denoted by **FL**.

The category **FB** can be described alternatively:

Theorem 3.11 *A dcpo D is an FB-domain if and only if the set G of idempotent deflations on D is directed and its supremum equals the identity function id_D.*

Proof. For the 'if'-part note that in particular G is nonempty and so D is well rooted by Lemma 3.7. Let c be a compact element in D and consider the Scott-open filter $\uparrow c$. By assumption there is an idempotent deflation $g \in G$ which fixes c and hence maps $\uparrow c$ into itself. The idempotent deflations above g form a directed set of idempotent deflations if restricted to $\uparrow c$. Therefore $\uparrow c$ is bifinite.

We prove the other direction by giving an idempotent deflation f_A on D which fixes an arbitrary finite set A of compact elements. First of all, we may assume that A contains all root elements since an idempotent deflation must fix these anyway.

For m a minimal element of D there is an idempotent deflation f_m onto $U^\infty(A \cap \uparrow m)$. If m is a minimal upper bound of the minimal elements m_1, m_2 then the functions f_{m_1} and f_{m_2} restricted to $\uparrow m$ are still idempotent deflations since $m \in A$. Let f_m be a deflation on $\uparrow m$ above $\{f_{m_1}|_{\uparrow m}, f_{m_2}|_{\uparrow m}\}$. It is clear that we can proceed in this fashion for all $m \in U^\infty(\phi)$. The required idempotent deflation f is then given by $f(x) = f_{r_D(x)}(x)$ (r_D being the root function). ❙

The preceding theorem shows that our FB-domains are exactly the 'profinite domains' in the sense of [9]. It seems that the fact that FB-domains have the same characterization as bifinite domains (Theorem 1.26) obscured the general method of passing from domains with a least element to those without.

Theorem 3.12 *A dcpo D is an FB-domain if and only if it is isomorphic to the bilimit of a codirected system of finite posets in* $\mathbf{DCPO}^p$.

Proof. The proof is the same as for Theorem 1.26. ❙

Theorem 3.13 *The category* **FB** *is cartesian closed.*

Proof. Like the proof of Theorem 1.28. ❙

Theorem 3.14 *The categories* **FLdom** *and* **FL** *are cartesian closed.*

Proof. There is no difficulty with the terminal object and cartesian products. For the function space of two FL-domains D and E first note that $[D \longrightarrow E]$ is again well rooted by Proposition 3.8.

Let g be a minimal, hence compact element of $[D \longrightarrow E]$. We show that $\uparrow g$ is an L-domain. Let f be an element of $\uparrow g$ and let $(h_a)_{a\in A}$ be any collection of functions above g and below f. We define $h\colon D \to E$ by

$$h(x) = \bigvee_{a\in A} h_a(x),$$

where the supremum is taken in the complete lattice $[g(x), f(x)]$. It is a continuous function:

$$\begin{aligned} h(\bigvee_{i\in I}^{\uparrow} x_i) &= \bigvee_{a\in A} h_a(\bigvee_{i\in I}^{\uparrow} x_i) \\ &= \bigvee_{a\in A} \bigvee_{i\in I}^{\uparrow} h_a(x_i) \\ &= \bigvee_{i\in I}^{\uparrow} \bigvee_{a\in A} h_a(x_i) \\ &= \bigvee_{i\in I}^{\uparrow} h(x_i). \end{aligned}$$

We use the associativity of the infinite join operation in the complete lattice $[g(x_{i_0}), f(\bigvee^{\uparrow}_{i\in I} x_i)]$, where i_0 is any index in I. This proves cartesian closedness for **FLdom**.

If D and E are also algebraic then we show that $\uparrow g$ is also algebraic. By Proposition 3.10, this ensures the algebraicity of $[D \longrightarrow E]$. Let f be any function in $\uparrow g$ and let d be compact in D. We have shown in Proposition 1.41 that $g(d)$ is compact in E. By the algebraicity of E, there exist enough compact elements $e \in E$, such that $g(d) \le e \le f(d)$ holds, that is, $f(d) = \bigvee^{\uparrow} \downarrow f(d) \cap \uparrow g(d) \cap B(E)$. For each such e we define a function $d \searrow e\colon D \to E$ by

$$d \searrow e(x) = \begin{cases} g(x) \vee e, & \text{if } x \ge d; \\ g(x), & \text{otherwise.} \end{cases}$$

Here we take the supremum in the complete lattice $[g(d), f(x)]$. Again it is an easy calculation to see that $d \searrow e$ is continuous. It is also a compact element of $\uparrow g \cap \downarrow f$: if

$(f_i)_{i\in I}$ is a directed family of functions belonging to $\uparrow g \cap \downarrow f$ such that $\bigvee^{\uparrow}_{i\in I} f_i \geq d \searrow e$ then in particular $\bigvee^{\uparrow}_{i\in I} f_i(d) \geq d \searrow e(d) = e$ and from the compactness of e we get that there is $i_0 \in I$ with $f_{i_0}(d) \geq e$. It follows directly from the definition of $d \searrow e$ that $f_{i_0}(x) \geq d \searrow e(x)$ holds for all $x \in D$. (Note that for dcpo's with a least element this definition of $d \searrow e$ coincides with the one given in Proposition 1.39.)

It is clear that f is the supremum of all functions of the form $d \searrow e$ and in a lattice this shows algebraicity as the supremum of a finite set of compact elements is again compact. ∎

We are now ready to explore the maximal cartesian closed full subcategories of **ALG**. Analogously to Lemma 3.2 we need a lemma which discriminates between the **U**-categories defined in Chapter 3.1 and the **F**-categories of this section.

Lemma 3.15 *Let D and E be continuous dcpo's with property m. If $[D \longrightarrow E]$ is continuous then D has finitely many minimal elements or the root of E is discrete.*

Proof. By contradiction: suppose there are minimal elements e_1, e_2 in E which have a minimal upper bound e and suppose D has infinitely many minimal elements. The constant function c_{e_1} is a minimal element of the function space $[D \longrightarrow E]$. By continuity, c_{e_1} is compact.

For A a finite set of minimal elements in D we define the function $f_A \colon D \to E$ by

$$f_A(x) = \begin{cases} e, & \text{if } x \in \uparrow A; \\ e_2, & \text{otherwise.} \end{cases}$$

None of the functions f_A lies above c_{e_1} but their supremum equals $c_e \geq c_{e_1}$, contradicting the compactness of c_{e_1}. ∎

Theorem 3.16 *Let D be a dcpo with a continuous second-order function space $\big[[D \to D] \longrightarrow [D \to D]\big]$. Then D is well rooted or D is the disjoint union of pointed dcpo's.*

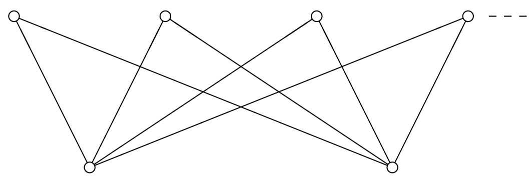

Figure 3.3: A dcpo with an algebraic function space which is not contained in any cartesian closed full subcategory of **ALG**.

Proof. From the results in Section 1.5 we learn that $[D \longrightarrow D]$ and D are continuous and bicomplete. Suppose that D is not a disjoint union of dcpo's with least element. Then the same is true for $[D \longrightarrow D]$ and from Lemma 3.15 we infer that $[D \longrightarrow D]$ has only finitely many minimal elements. This implies that the root of D is finite because otherwise we would have a contradiction to Lemma 3.9. By Proposition 3.5 this is enough to ensure that D is well rooted. ▌

Theorem 3.17 *If D is a dcpo and if $\big[[D \to D] \longrightarrow [D \to D]\big]$ is algebraic then D belongs to* $\mathbf{UB} \cup \mathbf{FB} \cup \mathbf{UL} \cup \mathbf{FL}$.

Proof. From Theorem 3.16 we get that either D is well rooted or consists of a disjoint union of dcpo's with a least element. Corollary 3.3 tells us that in any case either every principal Scott-open filter is a bifinite domain or every principal Scott-open filter is an L-domain. If D is well rooted this implies that D belongs to $\mathbf{FB} \cup \mathbf{FL}$, in the other case, D belongs to $\mathbf{UB} \cup \mathbf{UL}$. ▌

Note that in the preceding theorem we can not replace the second-order function space by the ordinary function space. Figure 3.3 shows a dcpo D for which $[D \longrightarrow D]$

is algebraic but $\big[[D \to D] \longrightarrow [D \to D]\big]$ is not. Consequently, D is not contained in any of the categories **UB**, **FB**, **UL**, or **FL**.

Theorem 3.18 (The Classification Theorem For Algebraic Domains) *The category* **ALG** *of algebraic dcpo's contains exactly four maximal cartesian closed full subcategories:* **UB**, **FB**, **UL**, *and* **FL**. *Every cartesian closed full subcategory of* **ALG** *is contained in one of these.*

Proof. From Lemma 1.21 we read that the exponential objects in a cartesian closed full subcategory **C** of **ALG** are isomorphic to the respective spaces of Scott-continuous functions, so we can apply the results of this chapter.

Let D and E be objects in **C**. By Theorem 3.17, both D and E are contained in $\mathbf{UB}\cup\mathbf{FB}\cup\mathbf{UL}\cup\mathbf{FL}$. By Lemma 3.15 they are both in $\mathbf{UB}\cup\mathbf{UL}$ or in $\mathbf{FB}\cup\mathbf{FL}$ and by Lemma 3.2 they are both in $\mathbf{UB}\cup\mathbf{FB}$ or in $\mathbf{UL}\cup\mathbf{FL}$. Together this says that **C** is contained in one of the four categories.

Separating examples are given in Figure 3.4, which show that none of the categories is contained in the union of the other three. ∎

Intersecting pairs of the maximal subcategories we get the diagram shown in Figure 3.5. (In naming the nodes we used the letter **E** to denote dcpo's consisting of finitely many components, each with a least element.)

We finish this section with two applications of the Classification Theorem. For the sake of simplicity let us just call any dcpo, which is contained in some cartesian closed full subcategory of **ALG**, an <u>algebraic domain</u>. Then we can formulate

Corollary 3.19 *A Scott-closed subset of an algebraic domain is an algebraic domain.*

Proof. A Scott-closed subset of an algebraic dcpo is again an algebraic dcpo. For L-domains, Scott-closed subsets are again L-domains, for bifinite domains, Scott-closed subsets are again bifinite (restrict the idempotent deflations to the subset).

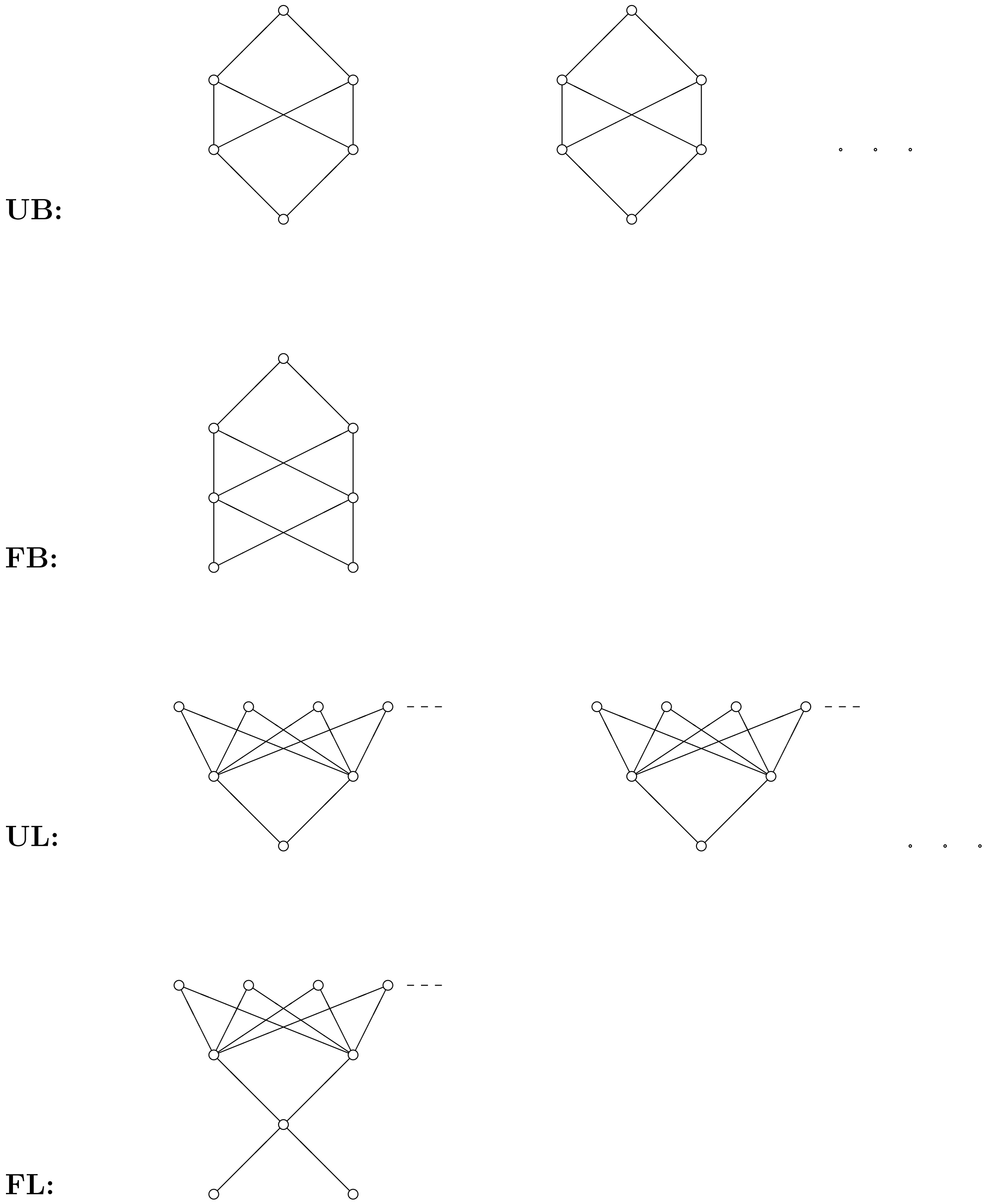

Figure 3.4: Domains which are contained in exactly one of the categories **UB**, **FB**, **UL**, and **FL**.

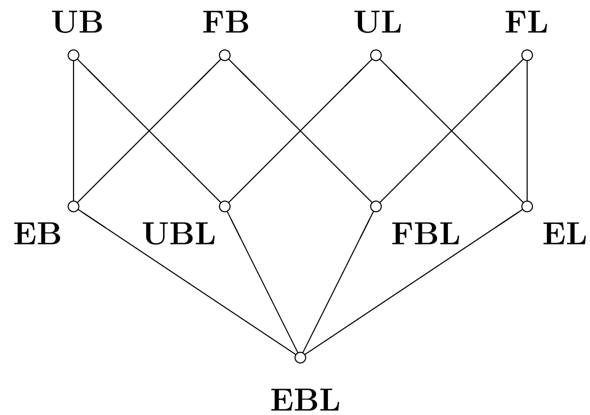

Figure 3.5: The maximal cartesian closed full subcategories of **ALG** and their intersections.

Similarly, well rootedness is preserved by the passage to a Scott-closed subset. By the Classification Theorem this exhausts all possible cases. ▎

Corollary 3.20 *An algebraic domain with top element is an FB-domain.*

Proof. A domain with top element is connected, hence contained in **FB** or **FL**. But FL-domains with top elements are lattices which in turn are bifinite. ▎

3.4 Countably based domains

In analogy to Smyth's Theorem (Theorem 2.3) we will now try to find the maximal cartesian closed subcategories of ω-**ALG**. As in the pointed case it turns out that there is a largest cartesian closed category, namely the category ω-**FB** of countably based FB-domains.

Theorem 3.21 *A dcpo D, for which $\big[[D \to D] \longrightarrow [D \to D]\big]$ is ω-algebraic, is a countably based FB-domain.*

Proof. First of all, D is algebraic and has property m by the results of Section 1.5. It must also be well rooted: if the root of D is infinite then $[D \longrightarrow D]$ has infinitely many minimal elements (Lemma 3.9 and Lemma 3.15) and hence the second order function space has uncountably many minimal elements, contradicting our assumption. By Proposition 3.5, a finite root implies well rootedness in an algebraic dcpo with property m.

A Scott-open principal filter $\uparrow c$ in D is a pointed algebraic dcpo. It is a retract of D and so its function space is a retract of $[D \longrightarrow D]$. By Proposition 1.40 $[\uparrow c \longrightarrow \uparrow c]$ is indeed ω-algebraic, so we can apply Smyth's Theorem and get that $\uparrow c$ is a bifinite domain. Hence D belongs to ω-**FB**. ∎

Corollary 3.22 *Any cartesian closed full subcategory of* ω-**ALG** *is contained in* ω-**FB**.

Chapter 4

Continuous domains

4.1 Retracts of bifinite domains

We have proved in Section 1.3 (Theorem 1.23) that any cartesian closed category **C** of algebraic dcpo's can be extended to a cartesian closed category **cC** of continuous domains by adding retracts. Thus we have the categories **cB**, **cL**, **cUB**, **cUL**, **cFB**, **cFL**, and so on, which we will study in more detail in this chapter.

The objects of **cB**, i.e. the retracts of bifinite domains, we will call continuous B-domains. Our first task is to give a characterization of them which does not employ bifinite domains.

Recall the definition of a deflation from Section 1.4. They play a similar rôle for continuous B-domains as idempotent deflations do for bifinite domains.

Theorem 4.1 *A dcpo D with least element is a retract of a bifinite domain if and only if there is a directed collection $(f_i)_{i\in I}$ of deflations on D such that $\bigvee^{\uparrow}_{i\in I} f_i = id_D$.*

Proof. (if) With the help of Proposition 1.1 we can assume that I is a lattice in which every principal ideal is a finite set. We construct a bifinite domain $\widetilde{D}$ as follows:

$$\widetilde{D} = \{(x_i)_{i\in I} \in D^I \mid \forall i \leq j : x_i \leq x_j; \forall i : x_i \in \bigcup_{k\leq i} im(f_k)\}$$

The elements of $\widetilde{D}$ are ordered componentwise. Note that for each $i \in I$ there are only finitely many values which the component x_i can take. This proves that $\widetilde{D}$ is a dcpo. Given an index $i_0 \in I$ we have the idempotent deflation d_{i_0} on $\widetilde{D}$ which sends a sequence $(x_i)_{i\in I}$ onto the sequence $(y_i)_{i\in I}$ defined by the equation $y_i = x_{i\wedge i_0}$.

It is clear that the set $(d_i)_{i\in I}$ of idempotent deflations thus defined is directed and that its supremum equals the identity on $\widetilde{D}$. By Theorem 1.26, $\widetilde{D}$ is a bifinite domain.

A retraction $r\colon \widetilde{D} \to D$ is given by $(x_i)_{i\in I} \mapsto \bigvee^{\uparrow}_{i\in I} x_i$, the corresponding embedding by $x \mapsto (f_i(x))_{i\in I}$.

(only if) Let (r, e) be a retraction-embedding pair from a bifinite domain $\widetilde{D}$ to a continuous B-domain D. If d is an idempotent deflation on $\widetilde{D}$ then the function $r \circ d \circ e$ is a deflation on D and it is clear from Theorem 1.26 that we get a directed collection of deflations with supremum id_D this way. ▮

Our proof didn't use the least element, so by Theorem 3.11 we also have:

Theorem 4.2 *A dcpo D is a retract of an FP-domain if and only if there is a directed collection $(f_i)_{i\in I}$ of deflations on D such that $\bigvee^{\uparrow}_{i\in I} f_i = id_D$.*

Corollary 4.3 *An algebraic object of* **cB** *(***cFB***) is a bifinite domain (an FB-domain).*

Proof. Let $D \in$ **cB** be algebraic and let $A \subseteq D$ be a finite set of compact elements. Recall that D has property m by Theorem 1.37 and that $U^\infty(A)$ consists of compact elements by Proposition 1.9. Therefore we can apply Proposition 1.31 and get that there is a projection onto $U^\infty(A)$. This mapping is compact as it is a minimal upper bound of the compact functions $a \searrow a, a \in A$. Let $(f_i)_{i\in I}$ be a directed set of deflations on D which generates id_D. We get that $p = \bigvee^{\uparrow}_{i\in I} f_i \circ p$ and by compactness there is an index $i_0 \in I$ such that $p = f_{i_0} \circ p$. Hence p has a finite image.

Idempotent deflations are completely determined by their image and so it is clear how we construct an upper bound p for two idempotent deflations p_1, p_2: we let p

be the idempotent deflation onto $U^\infty(im(p_1) \cup im(p_2))$. Applying Theorem 1.26 finishes our proof for **cB**.

For the case of **cFB** we first observe that by Lemma 3.7 a retract of an FB-domain is again well rooted. We then study the principal Scott-open filters in D, generated by root elements. Under the assumption that D is algebraic, principal Scott-open filters are both algebraic domains with least element and retracts of D. We can therefore apply the considerations of the first part of the proof to our situation (in particular, functions $a \searrow a$ are available) and find that each principal Scott-open filter is bifinite. So D is an FB-domain. ∎

It is natural to ask whether there is an analogue to Theorem 1.32 for continuous B-domains. Unfortunately, no such characterization is known up to now. One reason for this unsatisfactory state of affairs is that the ideal completion $I(D)$ of a continuous B-domain is not necessarily a bifinite domain: Figure 1.10 shows a bifinite domain for which the base of the ideal completion does not have property M; the U^∞-operator produces infinite sets on the ideal completion of the bifinite domain depicted in Figure 1.11. We will come back to this in Section 4.2. In the remainder of this section we will work with the characterization given in Theorem 4.1.

We note that there is a slight difference in the formulation of Theorem 4.1 and Theorem 1.26: in a bifinite domain the set of all idempotent deflations is directed whereas in a continuous B-domain there is some directed set of deflations. Indeed, the set of all deflations is not necessarily directed. Consider the poset in Figure 1.10. The function f_a, which maps $\downarrow c$ onto $\perp$ and everything else onto a, is clearly an deflation. Similarly, the function f_b, which maps $\downarrow c$ onto $\perp$ and $D \setminus \downarrow c$ onto b. An upper bound for these two functions below id_D must keep $D \setminus \downarrow c$ fixed and hence has an infinite image.

If we restrict our attention to those deflations which are way-below the identity function then we get directedness: If D is a continuous FP-domain then there is a directed collection $(f_i)_{i\in I}$ of deflations with $\bigvee^{\uparrow}_{i\in I} f_i = id_D$. The interpolation property allows us to conclude that also $\bigvee^{\uparrow}_{i\in I} f_i^2$ equals the identity function. All these squared deflations are way-below id_D by Proposition 1.25. If g_1, g_2 are two deflations way-below id_D then we find some f_i^2 which is above both of them. This proves directedness and also gives us the following

Corollary 4.4 *A dcpo D is a continuous FB-domain if and only if the set G of deflations way-below id_D is directed and $\bigvee^{\uparrow} G = id_D$.*

Definition. Let D be a dcpo and $G \subseteq [D \longrightarrow D]$ be a set of functions below id_D. We say that G is finitely separating if given a finite sequence $(x_1, \ldots, x_n) \in D^n$ and a corresponding approximating sequence $(y_1, \ldots, y_n), y_i \ll x_i, i = 1, \ldots, n$, there is an element f of G which satisfies $y_i \leq f(x_i) \leq x_i$ for all $i \in \{1, \ldots, n\}$.

Theorem 4.5 *A poset D is a continuous FB-domain if and only if it is a continuous dcpo and the set G of deflations is finitely separating.*

Proof. For the 'if'-part we show that $H = \{f \circ f \mid f \in G\}$ is a directed set of deflations yielding the identity function on D.

Given $f \in G$ we know by Proposition 1.25 that $f(x) \ll x$ holds for all $x \in D$. Let g be a second function in G. The image of a deflation is finite, so we may form sequences $(x_1, \ldots, x_n)$ containing all elements of $im(f)$ and $(y_1, \ldots, y_m)$ containing all elements of $im(g)$. By the interpolation property let $x'_1, \ldots, x'_n$ be elements which satisfy $f(x_i) \ll x'_i \ll x_i, i = 1, \ldots, n$ and $y'_1, \ldots, y'_m$ be elements which satisfy $g(y_i) \ll y'_i \ll y_i, i = 1, \ldots, m$. By assumption there is a deflation h which separates the sequence $(x_1, \ldots, x_n, y_1, \ldots, y_m, x'_1, \ldots, x'_n, y'_1, \ldots, y'_m)$ from the approximating sequence $(x'_1, \ldots, x'_n, y'_1, \ldots, y'_m, f(x_1), \ldots, f(x_n), g(y_1), \ldots, g(y_m))$. We claim that $h \circ h$ is an upper bound for $\{f \circ f, g \circ g\}$: given $x \in D$ we can calculate

$$\begin{aligned} f \circ f(x) &\leq h(x'_i) \qquad \text{for } x_i = f(x) \\ &\leq h(h(x_i)) \\ &\leq h \circ h(x) \end{aligned}$$

and for g

$$\begin{aligned} g \circ g(x) &\leq h(y_j') \qquad \text{for } y_j = g(x) \\ &\leq h(h(y_j)) \\ &\leq h \circ h(x). \end{aligned}$$

Since we have the interpolation property it is immediate that the supremum of H is the identity on D.

The converse is a trivial consequence of Theorem 4.1. ❙

Theorem 4.6 *The bilimit of a codirected system of continuous B-domains (continuous FB-domains) is again a continuous B-domain (continuous FB-domain).*

Proof. Let $((D_i)_{i\in I}, (d_{ij})_{i\leq j})$ be a codirected system in $\mathbf{cB}^p \subseteq \mathbf{DCPO}^p_\perp$. Let D^* be the bilimit in $\mathbf{DCPO}^p_\perp$. We have to prove that it is again a continuous B-domain. D^* is a dcpo by Theorem 1.24. Let x be an element of D^* and let x_i be way-below $d_i(x)$ in D_i. We claim that $e_i(x_i)$ is way-below x. Let $(y^j)_{j\in J}$ be a directed collection of elements in D^* such that $\bigvee^{\uparrow}_{j\in J} y^j \geq x$. Then $\bigvee^{\uparrow}_{j\in J} d_i(y^j) = d_i(\bigvee^{\uparrow}_{j\in J} y^j) \geq d_i(x)$ and there is an index $j_0 \in J$ such that $d_i(y^{j_0}) \geq x_i$. This implies that we also have $y^{j_0} \geq e_i \circ d_i(y^{j_0}) \geq e_i(x_i)$.

It is easy to check that the set $\bigcup_{i\in I} e_i(\Downarrow d_i(x))$ is directed. Its supremum is clearly equal to x. We check the condition in Theorem 4.5 for a finite sequence $(x^1, \ldots, x^n) \in D^{*n}$ and an approximating sequence $(y^1, \ldots, y^n)$. By what we just proved there is an index $i \in I$ and elements $(x_i^1, \ldots, x_i^n) \in D_i^n$ such that $y^k \leq e_i(x_i^k)$ and $x_i^k \ll d_i(x^k)$ for $k = 1, \ldots, n$. On D_i there is a deflation f separating x_i^k and $d_i(x^k)$. The lifted function $e_i \circ f \circ d_i$ is a deflation on D^* and we have $e_i \circ f \circ d_i(x^k) = e_i(f(d_i(x^k))) \geq e_i(x_i^k) \geq y^k$, hence $e_i \circ f \circ d_i$ is a separating deflation for the pairs (y^k, x^k) we started with. ❙

We remark that Theorem 4.6 is the basic result which makes continuous B-domains usable for solving recursive domain equations in the style of [25] or [9].

Although one would not call Theorem 4.1 a true intrinsic characterization of continuous B-domains, we can still work quite well with the set of deflations. We illustrate this by proving that continuous B-domains carry an intrinsic compact Hausdorff topology which refines the Scott-topology.

Definition. Given a dcpo D we define the Lawson-topology $\lambda(D)$ to be the topology generated by the Scott-topology $\sigma(D)$ and sets of the form $D \setminus \uparrow x, x \in D$.

Theorem 4.7 (i) *The Lawson-topology on a continuous dcpo is Hausdorff.*

(ii) *The Lawson-topology on a continuous FB-domain is compact.*

Proof. (i) Let x and y be two distinct elements of D. We may assume that $x \not\leq y$ holds. Because D is continuous we find $z \in D$ with $z \ll x$ and $z \not\leq y$. We have that $\twoheaduparrow z$ is a Lawson-open neigborhood of x and $D \setminus \uparrow z$ is an open neighborhood of y.

(ii) Let $\mathcal{F}$ be an ultrafilter on D and let G be a directed set of deflations with supremum id_D. For each $f \in G$ define $f(\mathcal{F})$ to be the direct image of the ultrafilter. It is an ultrafilter on the finite set $im(f)$ and therefore has a unique limit point x_f. If f is below g in G then we have that $x_f = f(f^{-1}(x_f) = f(f^{-1}(x_f) \cap g^{-1}(x_g)) \leq g(f^{-1}(x_f) \cap g^{-1}(x_g)) = g(g^{-1}(x_g)) = x_g$ because the sets $f^{-1}(x_f)$ belong to $\mathcal{F}$. We claim that $x = \bigvee^{\uparrow}_{f \in G} x_f$ is the limit point of $\mathcal{F}$.

If O is a Scott-open neighborhood of x then some x_f is in O already and, because $f \leq id_D$, $f^{-1}(x_f) \subseteq \uparrow x_f \subseteq O$. If $D \setminus \uparrow y$ is a Lawson-open neighborhood of x, we find $f \in G$ such that $f(y) \not\leq x$ holds and for this deflation we have $f(y) \not\leq x_f \leq x$ and therefore $f^{-1}(x_f) \cap \uparrow y = \phi$. ❚

Each continuous lattice L (and similarly each continuous L-domain, see Section 4.3) can be represented as the image of a projection on an algebraic lattice (algebraic L-domain). The proof is simple: L is embedded in its ideal completion $I(D)$ via the mapping $x \mapsto \twoheaddownarrow x$. The corresponding retraction $I \mapsto \bigvee^{\uparrow} I$ can easily be proved to be a projection. The proof of Theorem 4.1 was much more involved and still yielded a retraction only. For countably based continuous B-domains we can do more:

Proposition 4.8 *A dcpo D is a countably based continuous FB-domain if and only if there is an ω-chain $(f_n)_{n\in\mathbf{N}}$ of deflations such that $\bigvee^{\uparrow}_{n\in\mathbf{N}} f_n = id_D$.*

Proof. If we have an ω-chain of deflations then $B = \bigcup_{n\in\mathbf{N}} im(f_n)$ is countable. We show that it is a base for D. Given an element $x \in D$ and approximating elements $y_1, y_2 \in B \cap \twoheaddownarrow x$ we have $n_0 \in \mathbf{N}$ such that $f_{n_0}(x) \geq y_1, y_2$. We proved in Proposition 1.25 that $f_{n_0}(x)$ is way-below x.

For the converse assume that B is a countable base of D. let G be a directed family of deflations as given by Theorem 4.1. We can enumerate all those pairs (b, b') in $B \times B$ for which $b \ll b'$ holds and this allows us to define recursively an ω-chain of deflations: let $f_1 \in G$ be such that $b_1 \leq f(b'_1)$, let $f_2 \in G$ be greater or equal to f_1 and such that $b_2 \leq f_2(b'_2)$, and so on. The supremum of this chain must yield the identity function since for any $x \in D$ and any $y \ll x$ there are — by the interpolation property — base elements b, b' with $y \leq b \ll b' \ll x$ and hence there is some f_n such that $y \leq b \leq f_n(b') \leq f_n(x)$ holds. ▌

Theorem 4.9 *Any countably based continuous B-domain (continuous FB-domain) is isomorphic to the image of a projection on some countably based bifinite domain (FB-domain).*

Proof. Let $(f_n)_{n\in\mathbf{N}}$ be an ω-chain of deflations on D. We have to change the proof of Theorem 4.1 only a little bit: let $\widetilde{D}$ be the set of all those ω-sequences $(x_n)_{n\in\mathbf{N}}$ which satisfy the additional condition $x_m \geq f_m(\bigvee^{\uparrow}_{n\in\mathbf{N}} x_n)$ for all $m \in \mathbf{N}$.

We leave it to the reader to check the details. ▌

It is a disturbing fact that we have not been able to prove this theorem for all continuous B-domains although there seems to be no reason why it should fail in the general case.

4.2 Lawson-compact domains

In this section we will study continuous dcpo's for which the Lawson-topology is compact. Our first task is to characterize Lawson-compactness by a property of Scott-quasicompact upper sets.

Definition. We call an upper set A in a poset D finitely generated if A is of the form $\uparrow M$ for a finite set $M \subseteq D$.

Lemma 4.10 *Let D be a continuous dcpo.*

(i) *Any upper set in D is the intersection of all its Scott-open neighborhoods.*

(ii) *Any Scott-quasicompact upper set $A \subseteq D$ is the (filtered) intersection of all its finitely generated Scott-neighborhoods.*

Proof. (i) If A is an upper set and if $x \in D$ is not contained in A then for each $a \in A$ we have $x \not\geq a$. By continuity there is $y_a \ll a$ such that x is not greater than y_a. Thus x is not contained in the open neighborhood $O = \bigcup_{a \in A} \twoheaduparrow y_a$ of A.

(ii) If O is an open neighborhood of the Scott-quasicompact upper set A then each element a of A is contained in some set of the form $\twoheaduparrow x_a$ with $x_a \in O$. By compactness there is some finite subcovering $\twoheaduparrow x_{a_1}, \ldots, \twoheaduparrow x_{a_n}$ of the covering $(\twoheaduparrow x_a)_{a \in A}$. The set $\uparrow x_{a_1} \cup \ldots \cup \uparrow x_{a_n}$ is a finitely generated Scott-neighborhood of A contained in O. The conclusion now follows directly from part (i). ▮

Part (ii) of the preceding lemma already indicates that we can represent Scott-quasicompact upper sets by directed collections of finite sets. The following theorem, which is due to M.E.Rudin (cf. [6]), will be an important tool in developing this idea further.

Theorem 4.11 *If $(\uparrow M_i)_{i \in I}$ is a filtered collection of upper sets generated by finite sets $M_i \neq \phi$ in a poset D then there is a directed subset J of $\bigcup_{i \in I} M_i$ which intersects each M_i nontrivially.*

Proof. Let $\mathcal{P}$ be the set of all subsets P of $\bigcup_{i\in I} M_i$ which intersect each M_i and for which $\uparrow(P\cap M_i)$ contains $\uparrow(P\cap M_j)$ whenever i is less or equal to j in I. $\mathcal{P}$ is a nonempty set since it contains $\bigcup_{i\in I} M_i$ itself. By Kuratowski's principle let $\mathcal{C}$ be a maximal chain in $\mathcal{P}$ and let J equal the intersection of all sets in $\mathcal{C}$. We prove that J has the required properties.

It is clear that J intersects all M_i since these are finite sets. In fact, J itself belongs to $\mathcal{P}$ as $J\cap M_i$ and $J\cap M_j$ equal $J'\cap M_i$ and $J'\cap M_j$, respectively, for some $J'\in\mathcal{P}$.

As for directedness, let a and b be two elements in J. By the maximality of $\mathcal{C}$ both $J\setminus\uparrow a$ and $J\setminus\uparrow b$ do not belong to $\mathcal{P}$. So for some $i_a\in I$ the set $M_{i_a}\cap(J\setminus\uparrow a)$ is empty and similarly for some $i_b\in I: M_{i_b}\cap(J\setminus\uparrow b)=\phi$. Then for i an upper bound of i_a and i_b in I we have $M_i\cap(J\setminus\uparrow a)=\phi=M_i\cap(J\setminus\uparrow b)$. Thus each element of $M_i\cap J$ is above both a and b. As this set is nonempty we get an upper bound for $\{a,b\}$ in J. ▌

Lemma 4.12 *If $(A_i)_{i\in I}$ is a filtered collection of Scott-quasicompact upper sets in a continuous dcpo D and if O is a Scott-open neighborhood of $A=\bigcap_{i\in I} A_i$ then there is some $i_0\in I$ such that A_{i_0} is contained in O.*

Proof. For $i\in I$ let $\mathcal{U}_i$ be the neighborhood filter of A_i ordered by superset inclusion. The set $J=\{(i,O)\mid i\in I, O\in\mathcal{U}_i\}$ is already directed but by applying Proposition 1.1 we may think of J as a lattice, in which every principal ideal is finite, together with a monotone mapping $j\mapsto(i_j,O_j)$. Since O_j is a neighborhood of the compact set A_{i_j} we find by Lemma 4.10 a finite set $M_j\subseteq O_j$ such that $A_{i_j}\subseteq\twoheaduparrow M_j\subseteq\uparrow M_j\subseteq O_j$. We want to choose the sets M_j in a monotone fashion and we can do so because every principal ideal in J is finite: if $M_{j'}$ is chosen for all $j'<j$ then A_{i_j} is contained in $O_j\cap\bigcap_{j'<j}\twoheaduparrow M_{j'}$ and again by Lemma 4.10 we find a finite set M_j such that $A_j\subseteq\twoheaduparrow M_j\subseteq\uparrow M_j\subseteq O_j\cap\bigcap_{j'<j}\twoheaduparrow M_{j'}$. This set M_j has the property that each of its members is above some element of $M_{j'}$ whenever $j'<j$.

Now assume the conclusion of our lemma is false. Then each set $A_i \setminus O$ is nonempty and hence for each $j \in J$ the set $M_j \setminus O$ is nonempty. The sets $\uparrow(M_j \setminus O)$ satisfy the hypothesis of Theorem 4.11 and thus there is a directed set K which intersects all $M_j \setminus O$.

Let x be the supremum of K. By Lemma 4.10 it belongs to each A_i and hence to $A \subseteq O$. By construction, no element of K itself belongs to O and this contradicts O being an open set. ▮

Proposition 4.13 *Let D be a dcpo with continuous function space and let A be a Scott-quasicompact set inside the the Scott-open set O. Then there is $f \ll id_D$ such that $f(A) \subseteq O$ holds.*

Proof. For all $a \in A \subseteq O$ there is $a' \in O$ with $a' \ll a$ and we have a function $f_a \ll id_D$ which maps a above a'. Thus, A is covered by the open sets $f_a^{-1}(O)$, $a \in A$, and by compactness we may choose a finite subcovering $f_{a_1}^{-1}(O), \ldots, f_{a_n}^{-1}(O)$. Let f be any upper bound for $\{f_{a_1}, \ldots, f_{a_n}\}$ in $\mathop{\downarrow\!\!\!\downarrow} id_D$. For any $a \in A$ we then have $a \in f_{a_i}^{-1}(O)$ for some $i \in \{1, \ldots, n\}$ and $f(a) \geq f_{a_i}(a) \in O$. ▮

Corollary 4.14 *If D is a dcpo with continuous function space and if A is a Scott-quasicompact upper set then $A = \bigcap_{f \ll id_D} \uparrow f(A)$.*

Theorem 4.15 (i) *If D is a continuous dcpo then every Scott-quasicompact upper set is Lawson-closed.*

(ii) *If D is a continuous and Lawson-compact dcpo then an upper set is Scott-quasicompact if and only if it is Lawson-closed.*

Proof. Part (i) is an immediate consequence of Lemma 4.10 (ii), part (ii) follows from General Topology. ▮

Theorem 4.16 *A continuous dcpo D is Lawson-compact if and only if the intersection of any collection of Scott-quasicompact upper sets is Scott-quasicompact.*

Proof. 'if' Let J be a covering of D by Lawson-open sets. By Alexander's Subbase Theorem we may assume that all open sets in J have the form $D \setminus \uparrow x$ or $\Uparrow x$, so we can write $D = \bigcup_{i \in I} D \setminus \uparrow x_i \cup \bigcup_{j \in J} \Uparrow y_j$. The first union equals $D \setminus \bigcap_{i \in I} \uparrow x_i$ and by assumption the set $\bigcap_{i \in I} \uparrow x_i$ is Scott-quasicompact and contained in $\bigcup_{j \in J} \Uparrow y_j$. Hence $\bigcap_{i \in I} \uparrow x_i$ is contained in some finite union $\Uparrow y_{j_0} \cup \ldots \cup \Uparrow y_{j_n}$. Lemma 4.12 tells us that a finite intersection $\uparrow x_{i_0} \cap \ldots \cap \uparrow x_{i_m}$ of principal filters is contained in that open set already. So D is covered by the sets $\Uparrow y_{j_0}, \ldots, \Uparrow y_{j_n}, D \setminus \uparrow x_{i_0}, \ldots, D \setminus \uparrow x_{i_m}$.

The converse is an immediate consequence of the preceding theorem. ∎

In the following we will try to find a condition on the element level which ensures Lawson-compactness.

Lemma 4.17 *A continuous dcpo D is Lawson-compact if and only if D is Scott-quasicompact and each pair of Scott-quasicompact upper sets has a quasicompact intersection.*

Proof. We will employ Theorem 4.16, so let $(A_s)_{s \in S}$ be any collection of Scott-quasicompact upper sets in D. If $S = \phi$ then $\bigcap_{s \in S} A_s = D$, else let $(O_t)_{t \in T}$ be an open covering of $A = \bigcap_{s \in S} A_s$. By Lemma 4.12 there is some finite subset S' of S such that $A' = \bigcap_{s \in S'} A_s$ is contained in $\bigcup_{t \in T} O_t$. A' is compact by assumption, hence some finite subcovering of $(O_t)_{t \in T}$ already covers A'. Since A' contains A, our proof is complete. ∎

Lemma 4.18 *A continuous dcpo D with property m is Lawson-compact if and only if D is Scott-quasicompact and for all pairs $a' \ll a$, $b' \ll b$ in D the set $mub(\{a, b\})$ is contained in a finite union of sets of the form $\Uparrow c$, $c \in mub(\{a', b'\})$.*

Proof. Of course, only the 'if'-part is interesting. Suppose we are given two Scott-quasicompact upper sets A and B and an open covering $(O_t)_{t\in T}$ of $A \cap B$. By Lemma 4.10 (ii) and Lemma 4.12 we find finitely generated upper sets $\uparrow M_A$ and $\uparrow M_B$ such that $A \subseteq \Uparrow M_A \subseteq \uparrow M_A$, $B \subseteq \Uparrow M_B \subseteq \uparrow M_B$ and $\uparrow M_A \cap \uparrow M_B \subseteq O = \bigcup_{t\in T} O_t$. We interpolate between A and $\Uparrow M_A$ once more by a finitely generated upper set M'_A and similarly for B, that is $A \subseteq \Uparrow M'_A \subseteq \uparrow M'_A \subseteq \Uparrow M_A$, $B \subseteq \Uparrow M'_B \subseteq \uparrow M'_B \subseteq \Uparrow M_B$.

The set $\uparrow M'_a \cap M'_B$ is generated by the set $\bigcup_{x\in M'_A, y\in M'_B} mub(\{x,y\})$ which by assumption is contained in a finite union of sets of the form $\Uparrow z$ with $z \in \bigcup_{x\in M_A, y\in M_B} mub(\{x,y\})$.

This proves that $A \cap B$ is contained in a finite union of sets O_t. ▮

The following corollary appears as the '2/3 SFP Theorem' in Plotkin's Pisa Lecture Notes ([20]).

Corollary 4.19 *An algebraic dcpo D is Lawson-compact if and only if $B(D)$ has property M.*

Proof. (if) Property M applied to the empty set tells us that D has finitely many minimal elements and is hence Scott-quasicompact. By Proposition 1.30, D has property m and we can apply Lemma 4.18 in order to get that D is Lawson-compact.

(only if) If D is Lawson-compact then $B(D)$ has property m: Let M be a finite set of compact elements and let x be an upper bound for M. By Kuratowski's principle, x is contained in some maximal chain $C \subseteq ub(M)$. If C has no minimal element then the open sets $D \setminus \downarrow c$, $c \in C$, cover $ub(M) = \bigcap_{m\in M} \uparrow m$ which we assume to be compact. This contradiction proves property m.

Having this, Lemma 4.18 immediately gives us property M. ▮

4.3 Continuous L-domains

In the following we collect some information about continuous L-domains with which we didn't want to overload Chapter 2.2.

Proposition 4.20 (i) *For D an L-domain the set $I(D)$ of ideals is an algebraic L-domain.*

(ii) *If D is a continuous L-domain then there is a projection from $I(D)$ onto D.*

(iii) *If E is a retract of a continuous L-domain D then E is a continuous L-domain.*

Proof. (i) By Proposition 1.17 we already know that $I(D)$ is an algebraic dcpo. It remains to show that it is also an L-domain. We use characterization (vii) of Theorem 2.10 for this. Let $\downarrow a_1, \downarrow a_2$ be principal ideals contained in an ideal J. By Proposition 2.8 we may form the supremum a of $\{a_1, a_2\}$ in J and clearly $\downarrow a$ is the unique minimal upper bound of $\{\downarrow a_1, \downarrow a_2\}$ in $\downarrow J$.

(ii) This follows directly from the corresponding part of Proposition 1.17.

(iii) Let $r\colon D \to D$ be a retraction on D. From the general result (Proposition 1.16) we already know that $E = im(r)$ is a continuous dcpo. Given elements $a_1, a_2 \leq x$ in the image of r we can form the supremum $a = a_1 \vee a_2$ in $\downarrow x$. We show that $r(a)$ is the supremum with respect to $\downarrow x \cap E$. Indeed, if $y \in \downarrow x \cap E$ is any upper bound of $\{a_1, a_2\}$ then we have $y \geq a$ and hence $y = r(y) \geq r(a)$. So we have suprema for finite sets and together with the fact that E is a dcpo this says that $\downarrow x$ is a complete lattice. ∎

So the objects of **cL** are exactly the continuous L-domains. In particular, continuous L-domains and Scott-continuous functions form a cartesian closed category.

For continuous dcpo's we have the following supplement to Theorem 2.10:

Theorem 4.21 *For a continuous dcpo with least element the following are equivalent:*

(i) *D is a continuous L-domain.*

(ii) *For each $x \in D$ the set $\twoheaddownarrow x$ is a $\vee$-semilattice with a smallest element.*

(iii) *D has infima for filtered sets and for any bounded pair $\{a_1, a_2\}$ of elements of D the set $\Downarrow(\downarrow a_1 \cap \downarrow a_2)$ is directed.*

Proof. The equivalence (i) $\iff$ (ii) corresponds to (iv) $\iff$ (v) in Theorem 2.10.

(i) $\implies$ (iii) By Theorem 2.9 the infimum of $a = a_1 \wedge a_2$ exists and we get from the definition of continuity that $\Downarrow a$ is a directed set. Filtered sets are connected, so to them Theorem 2.9 also applies.

(iii) $\implies$ (i) Given an element $x \in D$ we show that $\downarrow x$ is a complete lattice. The infimum of the empty set is equal to x. Let A be a nonempty subset of $\downarrow x$. For elements $a_1, a_2 \in A$ the infimum is equal to $\bigvee^{\uparrow} \Downarrow(\downarrow a_1 \cap \downarrow a_2)$ and so we have infima for finite sets. The set A is the directed union of finite subsets the infima of which form a filtered set. Thus $\bigwedge A$ exists in $\downarrow x$. ▌

Proposition 4.22 *Let D be a pointed continuous dcpo with property m. D is a continuous L-domain if and only if D does not contain the poset $\mathbf{X}_{\perp}^{\top}$ (Figure 2.2) as a retract.*

Proof. By Proposition 4.20, $\mathbf{X}_{\perp}^{\top}$ cannot occur as a retract in an L-domain. This proves the 'if'-part.

For the converse let D be a continuous dcpo with property m which is not an L-domain. By Theorem 4.21(iii) there exist a, a_1, a_2 in D such that $a_1, a_2 \leq a$ and $A = \Downarrow(\downarrow a_1 \cap \downarrow a_2)$ is not directed. So there are elements $y_1, y_2 \in A$ for which there is no upper bound in this set. By interpolation we find elements $y_1', y_2' \in A$ such that $y_1 \ll y_1'$ and $y_2 \ll y_2'$. We define a retraction $r \colon D \to D$ as follows:

$$r(x) = \begin{cases} a, & \text{if } x \not\leq a_1, x \not\leq a_2; \\ a_1, & \text{if } x \leq a_1, x \not\leq a_2; \\ a_2, & \text{if } x \not\leq a_1, x \leq a_2; \\ y_1', & \text{if } x \in \downarrow a_1 \cap \downarrow a_2 \cap \Uparrow y_1; \\ y_2', & \text{if } x \in \downarrow a_1 \cap \downarrow a_2 \cap \Uparrow y_2; \\ \perp, & \text{otherwise.} \end{cases}$$

It is clear that r is a retraction with image $\{a, a_1, a_2, y_1', y_2', \perp\}$ which is a copy of $\mathbf{X}_{\perp}^{\top}$ inside D. ▌

4.4 Maximal cartesian closed categories of continuous directed-complete partial orders

Lemma 2.13 is crucial for our results about the maximal subcategories in **ALG**. We begin this section with a generalization of this lemma to the continuous case.

Lemma 4.23 *Let D and E be pointed dcpo's with property m. If E is not an L-domain and if D is not Lawson-compact then $[D \longrightarrow E]$ is not continuous.*

Proof. Assume that $[D \longrightarrow E]$ is continuous although E is not an L-domain and D is not Lawson-compact. By Proposition 4.22 we know that E contains the poset $\mathbf{X}_\bot^\top$ as a retract, so we also have that $[D \longrightarrow \mathbf{X}_\bot^\top]$ is continuous.

From Lemma 4.18 we infer that there are elements $v_1'' \ll v_1$ and $v_2'' \ll v_2$ in D such that $mub(\{v_1, v_2\})$ is not covered by finitely many sets of the form $\Uparrow c$ with $c \in mub(\{v_1'', v_2''\})$. We have interpolating elements v_1', v_2' with $v_1'' \ll v_1' \ll v_1$ and $v_2'' \ll v_2' \ll v_2$.

The elements of $\mathbf{X}_\bot^\top$ we label as shown in Figure 2.2, that is $\bot < \{a_1', a_2'\} < \{a_1, a_2\} < \top$. Generalizing the notation of Proposition 1.39 we denote by $O \searrow e$ the function which maps the Scott-open set O onto e and everything else onto $\bot$.

Consider the functions $\Uparrow v_1' \searrow a_1'$, $\Uparrow v_1'' \searrow a_1'$ and $\Uparrow v_2' \searrow a_2'$, $\Uparrow v_2'' \searrow a_2'$. We claim that $\Uparrow v_i' \searrow a_i'$ is way-below $\Uparrow v_i'' \searrow a_i'$, $i = 1, 2$, in $[D \longrightarrow \mathbf{X}_\bot^\top]$. Indeed, a directed family of functions with supremum above $\Uparrow v_1'' \searrow a_1'$ must contain a member which maps $v_1' \in \Uparrow v_1''$ above the (compact) element a_1'. This function is then already above $\Uparrow v_1' \searrow a_1'$.

A minimal upper bound h for $\{\Uparrow v_1'' \searrow a_1', \Uparrow v_2'' \searrow a_2'\}$ in $[D \longrightarrow \mathbf{X}_\bot^\top]$ is given by the definition:

$$h(x) = \begin{cases} a_2, & \text{if } x \in \Uparrow v_1'' \cap \Uparrow v_2''; \\ a_1', & \text{if } x \in \Uparrow v_1'' \setminus \Uparrow v_2''; \\ a_2', & \text{if } x \in \Uparrow v_2'' \setminus \Uparrow v_1''; \\ \bot, & \text{otherwise.} \end{cases}$$

There must be a function g way-below h which is an upper bound for $\{\uparrow v_1' \searrow a_1', \uparrow v_2' \searrow a_2'\}$. This function must in any case map the elements of $mub(\{v_1, v_2\}) \subseteq \uparrow v_1' \cap \uparrow v_2'$ onto a_2.

On the other hand, we can give the definition of a directed family of functions f_A with supremum above h which contains no member above g. This contradiction will finish our proof.

Let A be a finite subset of $mub(\{v_1'', v_2''\})$. By assumption, $\uparrow A$ does not cover $mub(\{v_1, v_2\})$. We define $f_A\colon D \to \mathbf{X}_\bot^\top$ by

$$f_A(x) = \begin{cases} \top, & \text{if } x \in \uparrow v_1'' \cap \uparrow v_2'' \cap \uparrow A; \\ a_1, & \text{if } x \in \uparrow v_1'' \cap \uparrow v_2'' \setminus \uparrow A; \\ a_1', & \text{if } x \in \uparrow v_1'' \setminus \uparrow v_2''; \\ a_2', & \text{if } x \in \uparrow v_2'' \setminus \uparrow v_1''; \\ \bot, & \text{otherwise.} \end{cases}$$

The second set in this definition is always nonempty, so no f_A is above g. On the other hand, the supremum of all f_A, A a finite subset of $mub(\{v_1'', v_2''\})$, maps $\uparrow v_1'' \cap \uparrow v_2''$ onto $\top$ and is therefore above h. ▮

To proceed further we need the analogue of Lemma 2.2, that is:

Conjecture 4.24 *If D is a Lawson-compact dcpo with continuous function space then D is a continuous FB-domain.*

We have not formulated this conjecture with the U^∞-operator because even in a bifinite domain the U^∞-operator can produce infinite sets if it is applied to noncompact elements. This is to say that there is no canonical choice for a function of which we must prove that its image is finite. Instead, if some particular function is not a deflation then the reason can be that the domain does not belong to **cFB** but it can also be that we have simply picked the wrong function.

However, we do feel that with the machinery developed in Section 4.2 it should be possible to prove this conjecture. We continue and indicate what its consequences would be.

Theorem 4.25 (i) *The category* **cL** *is a maximal cartesian closed full subcategory of* $\mathbf{CONT}_\perp$.

(ii) *If* <u>*Conjecture 4.24*</u> *is true then there is only one more maximal cartesian closed full subcategory of* $\mathbf{CONT}_\perp$, *the category of continuous B-domains.*

Proof. (i) If **C** is a cartesian closed full subcategory of $\mathbf{CONT}_\perp$ properly containing **cL**, then there is an object E in **C** which is not an L-domain. The poset D depicted in Figure 1.4 does belong to **C** but is not Lawson-compact. Thus by Lemma 4.23 the function space $[D \longrightarrow E]$ cannot be continuous. Contradiction!

(ii) This follows immediately from Lemma 4.23. ▌

Lemma 4.26 *If D is a pointed dcpo with an ω-continuous function space $[D \longrightarrow D]$ then D is Lawson-compact.*

Proof. D has property m by Theorem 1.37 and is a continuous dcpo by Theorem 1.35. Assume that D is not Lawson-compact. From Lemma 4.23 we infer that D must be a continuous L-domain.

Let $v_1'' \ll v_1$, $v_2'' \ll v_2$ be elements such that $mub(\{v_1, v_2\})$ is not covered by a finite collection of sets of the form $\twoheaduparrow c$, $c \in mub(\{v_1'', v_2''\})$. From D being an L-domain we can infer that each element x of $mub(\{v_1, v_2\})$ is above a unique element of $mub(\{v_1'', v_2''\})$, which must therefore be way-below x.

Let v_1' interpolate between v_1'' and v_1, v_2' interpolate between v_2'' and v_2. Choose elements $a_1, a_2 \in mub(\{v_1, v_2\})$ and let a_1', a_2' be the unique minimal upper bounds of $\{v_1', v_2'\}$ which are below a_1 and a_2, respectively.

For each ~~finite~~ subset S of $mub(\{v_1'', v_2''\}) \cap {\downarrow} mub(\{v_1, v_2\})$ (this is an infinite set!) we define functions $f_S \colon D \to D$ and $g_S \colon D \to D$ as follows:

$$f_S(x) = \begin{cases} a_1, & \text{if } x \in \twoheaduparrow v_1'' \cap \twoheaduparrow v_2'' \cap \twoheaduparrow S; \\ a_2, & \text{if } x \in \twoheaduparrow v_1'' \cap \twoheaduparrow v_2'' \setminus \twoheaduparrow S; \\ v_1, & \text{if } x \in \twoheaduparrow v_1'' \setminus \twoheaduparrow v_2''; \\ v_2, & \text{if } x \in \twoheaduparrow v_2'' \setminus \twoheaduparrow v_1''; \\ \perp, & \text{otherwise.} \end{cases}$$

$$g_S(x) = \begin{cases} a_1', & \text{if } x \in \mathord{\Uparrow} v_1' \cap \mathord{\Uparrow} v_2' \cap \mathord{\Uparrow} S; \\ a_2', & \text{if } x \in \mathord{\Uparrow} v_1' \cap \mathord{\Uparrow} v_2' \setminus \mathord{\Uparrow} S; \\ v_1', & \text{if } x \in \mathord{\Uparrow} v_1' \setminus \mathord{\Uparrow} v_2'; \\ v_2', & \text{if } x \in \mathord{\Uparrow} v_2' \setminus \mathord{\Uparrow} v_1'; \\ \bot, & \text{otherwise.} \end{cases}$$

These functions are well defined since no element of D is above two different elements of $mub(\{v_1'', v_2''\})$.

For each S, the function g_S is way-below f_S: let $(h_i)_{i\in I}$ be a directed family of mappings with $\bigvee^{\uparrow}_{i\in I} h_i = f_S$ (cf. Proposition 1.5). Then there is $i_0 \in I$ such that h_{i_0} maps v_1' above v_1' and v_2' above v_2'. Thus it maps $\mathord{\Uparrow} v_1' \cap \mathord{\Uparrow} v_2' \cap \mathord{\Uparrow} S$ above v_1', v_2' and below a_1, that is, above a_1', and similarly, $\mathord{\Uparrow} v_1' \cap \mathord{\Uparrow} v_2' \setminus \mathord{\Uparrow} S$ above a_2'. So h_{i_0} is in fact greater than or equal to g_S.

If B is a countable base for $[D \longrightarrow D]$ then each of the intervals $[g_S, f_S]$ must contain at least one base element. We show that all these intervals are disjoint and since there are uncountably many of them this leads to a contradiction.

Let h be an element of $[g_{S_1}, f_{S_1}] \cap [g_{S_2}, f_{S_2}]$. Assume that s'' is an element of S_1 not contained in S_2. By construction there is an element $s \in mub(\{v_1, v_2\})$ above s'' and this element is mapped by h into $[a_1', a_1] \cap [a_2', a_2]$. But since D is an L-domain this latter set is empty. ▮

Theorem 4.27 *If Conjecture 4.24 is true then* ω-**CONT**$_\bot$ *contains a largest cartesian closed full subcategory:* ω-**cB**.

Turning to domains without least element we find that all the crucial lemmas of Chapter 3 are already formulated for dcpo's with a continuous function space. So there is not much proving needed in the remainder of this section.

Remember that we have introduced the classes **cUL**, **cFL**, **cUB**, and **cFB** as extensions of the corresponding classes of algebraic dcpo's via Theorem 1.23. We want to demonstrate that they have the expected internal structure.

Proposition 4.28 *Let D be a dcpo.*

(i) *D belongs to* **cUL** *if and only if D is the disjoint union of continuous L-domains.*

(ii) *D belongs to* **cUB** *if and only if D is the disjoint union of continuous B-domains.*

(iii) *D belongs to* **cFL** *if and only if D is well rooted and if every Scott-open principal filter in D is a continuous L-domain.*

(iv) *D belongs to* **cFB** *if and only if D is well rooted and if every Scott-open principal filter in D is a continuous B-domain.*

Proof. (i) and (ii) are trivial.

For the 'if'-part of (iii) simply note that the ideal completion of a well rooted dcpo D with continuous L-domains as open principal filters yields an algebraic FL-domain E and a projection from E onto D.

The 'if'-part of (iv) is a bit more involved (compare Theorem 3.11): we want to show that the set G of deflations on D is finitely separating (Theorem 4.5). So let $a_1' \ll a_1, \ldots, a_n' \ll a_n$ be elements of D for which we seek a separating deflation. We may assume that all root elements m appear as pairs $m \ll m$ among the $a_i' \ll a_i$ since a deflation on D must fix these anyway.

For m a minimal element of D let f_m be a deflation on $\uparrow m$ such that $f_m \circ f_m$ separates all pairs a_i', a_i where a_i belongs to $\uparrow m$. Such a deflation exists by assumption. For m a minimal upper bound of two minimal elements m_1, m_2 let f_m be a deflation on $\uparrow m$ for which $f_m \circ f_m$ separates not only those pairs $a_i' \ll a_i$ which are contained in $\uparrow m$, but also all pairs $f_{m_1} \circ f_{m_1}(x) \ll f_{m_1}(x)$ and $f_{m_2} \circ f_{m_2}(x) \ll f_{m_2}(x)$ for $x \in \uparrow m$. For any element x of $\uparrow m$ we then have $f_{m_1} \circ f_{m_1}(x) \leq f_m \circ f_m(f_{m_1}(x)) \leq f_m \circ f_m(x)$ and $f_{m_2} \circ f_{m_2}(x) \leq f_m \circ f_m(f_{m_2}(x)) \leq f_m \circ f_m(x)$, that is, $f_m \circ f_m$ is an upper bound for $\{f_{m_1} \circ f_{m_1}|_{\uparrow m}, f_{m_2} \circ f_{m_2}|_{\uparrow m}\}$.

It is clear that we can continue in this fashion for all elements in the root of D. The separating deflation f on D can be pasted together from the deflations on the open filters: $f(x) = f_{r_D(x)} \circ f_{r_D(x)}(x)$. ($r_D$ being the root function on D.) For the 'only if'-part of (iii) and (iv) first note that a retract of a well rooted dcpo is again well rooted: if $(r,e)\colon D \to E$ is a retraction-embedding pair and if r_D is the root function on D, then $f = r \circ r_D \circ e$ is a deflation on E. So Lemma 3.7 applies.

Let $\uparrow c$ be an open filter in E. The element $e(c)$ is not necessarily compact in D but equals the supremum of a directed collection $(d_i)_{i\in I}$ of compact elements. Thus we have $\bigvee^{\uparrow}_{i\in I} r(d_i) = r(\bigvee^{\uparrow}_{i\in I} d_i) = c$ and some d_i is mapped onto c by compactness. By restricting r and e we get that $\uparrow c$ is a retract of $\uparrow d_i$. Hence $\uparrow c$ is a continuous L-domain (or a continuous B-domain, respectively). ❚

Theorem 4.29 (i) *The categories* **cUL** *and* **cFL** *are maximal cartesian closed full subcategories of* **CONT**.

(ii) *If Conjecture 4.24 is true then there are exactly two more maximal cartesian closed full subcategories:* **cUB** *and* **cFB**.

Theorem 4.30 *If Conjecture 4.24 is true then* ω-**cFB** *is the largest cartesian closed full subcategory of* ω-**CONT**.

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Symbols:

Categories: