Models of Linear Polymorphism

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A thesis submitted
to The University of Birmingham
for the degree of Doctor of Philosophy

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February 2004

Abstract

This thesis is about the construction of order-theoretic models of second order linear logic. We examine the domain-theoretic model of *System* F given by Coquand, Gunter and Winskel. From the analysis we obtain a general framework to explain models of second order universal quantification using the notion of lax limit. The framework is then used to construct linear hyperdoctrines from suitable linear categories. Finally, we discuss a concrete instance of such a linear hyperdoctrine using the category of algebraic completely distributive lattices.

To Mathias and Noemi

Acknowledgements

I would like to thank the people that contributed to this thesis.

I am hugely grateful to my supervisor Achim Jung. Without his guidance and support this thesis would not have been possible. I greatly enjoyed working with him.

I also thank: Valeria de Paiva and Eike Ritter who suggested the topic of this thesis. Milly Maietti for many helpful discussions. Andrea Schalk for having shared her knowledge of linear categories with me. The *Fachbereich Mathematik* of the *TU Darmstadt*, in particular Klaus Keimel and Thomas Streicher, for supporting me since I left Birmingham. Edmund Robinson for encouraging me to undertake a PhD. My husband Mathias Kegelmann for uncountable help of any sort. The ω-chain in $[\![\forall X.X \multimap X]\!]$ is his. My parents for their constant support. All the friends that I met during these years.

Contents

Introduction

This thesis is about the construction of order-theoretic models of second order linear logic.

More concretely, we consider the $(\otimes, \mathrm{I}, \multimap, !, \forall)$-fragment, in the form of the term calculus for intuitionistic linear logic DILL ("dual intuitionistic linear logic" [4]) augmented with second order universal quantification *à la* Girard-Reynolds. We call the resulting term calculus $\mathrm{F}_{\mathrm{DILL}}$.

Second order lambda calculus or *System* F [41, 17] is a very powerful calculus. Its *impredicative* nature however poses difficulties for the construction of a model. A polymorphic function $f : \forall X.\, A(X)$ can be instantiated at the type $\forall X.\, A(X)$. The resulting function has type $\forall X.\, A\big(\forall X.\, A(X)\big)$. This means that the semantics of a polymorphic type, should correspond to an indexed product where the indexing family contains the product itself. Reynolds [43] showed that the impredicativity of *System* F prevents set-theoretic models other than the trivial one.

There are two techniques that have been successfully used to model the impredicativity of *System* F. The first one belongs to the area of domain theory and is the same that has been used to solve recursive domain equations. One possibility is to consider types as operators on a suitable universal domain [33]. Another one is Girard's construction using coherence spaces [18] and the similar construction on algebraic Scott-domains of Coquand, Gunter and Winskel [15]. Our analysis starts from the latter.

The second way to handle impredicativity of *System* F is that of realisability models. A short selection of the large amount of works related is [32, 23, 22, 24].

The main appeal of second order *linear* logic is the following. In conjunction with Reynolds' relational parametricity [42] it is possible to encode most data types in *System* F, including recursion and sums. But in a cartesian closed category with coproducts and initial object, fixed points cause the category to collapse [21] to the trivial category **1**. Plotkin suggested [38, 37] to circumvent this negative result using second order linear logic. He outlined a second order term calculus where primitive type constructions are linear ($\multimap$) and intuitionistic ($\to$) function types and second order universal quantification ($\forall$). In relationally parametric models of this calculus polymorphism and recursion can coexist. This again offers the possibility of having a relationally parametric model of polymorphism where most data types are encoded starting from a very small collection of primitive type constructions.

The details of this calculus and its relational semantics using partial functions on cpos never appeared. DILL (which has as primitive type constructions I, $\otimes$, $\multimap$ and $!$) is the closest account of the first order version of the calculus.

Linearity in conjunction with polymorphism has been considered in the following recent works. O'Hearn and Reynolds [35] use a polymorphic linear lambda-calculus to interpret Algol. However, they only need a predicative form of polymorphism. The absence of impredicativity means that the construction of a model does not present the technical difficulties that we address.

Abramski and Lenisa [1, 2] give an axiomatic characterisation of fully-complete models for ML-polymorphism using a category of PERs on linear combinatory algebra. They obtain what they call an *adjoint model* hyperdoctrine. However, in such a structure second order universal quantification is given only for intuitionistic types and not for linear ones.

Bierman, Pitts and Russo [12] consider an operational semantics for the polymorphic linear lambda calculus obtaining a programming language (`Lily`). The primitive type constructions of `Lily` are $\multimap$, $!$ and $\forall$. Any other type is represented using these constructions along Plotkin's indications.

As general notion of categorical semantics for $\mathrm{F}_{\mathrm{DILL}}$, we assume a linear version of the notion of *second order hyperdoctrine* [36] (or PL-category [45]). The fibred cartesian closed structure has to be replaced by a fibred structure suitable for linear logic.

This poses two main problems. This first one is which structure should the fibres have and which notion of morphisms between such fibres is suitable to interpret substitution of types for type variables. The second one is how to obtain such a *linear hyperdoctrine*.

The first question is answered by a careful analysis of two closely related presentations of categorical models of linear logic, namely that of *linear categories* and that of *symmetric monoidal adjunctions* [10, 11, 4].

To address the second question, we examine the domain-theoretic model of *System* F given by Coquand, Gunter and Winskel [15]. From this we obtain a general framework to explain models of second order universal quantification using the notion of *lax limit*. With this tool we construct a linear hyperdoctrine from a suitable category of domains, thus obtaining a sound model of $\mathrm{F}_{\mathrm{DILL}}$. We give an instance of this construction and compute some universal types in the concrete model.

The structure of the thesis is the following. Chapter 1 is devoted to categorical models of intuitionistic linear logic. We explain and analyse linear categories and symmetric monoidal adjunctions and define suitable functors between linear categories for modelling reindexing in a linear hyperdoctrine.

Chapter 2 introduces lax limits in Pos and in Pos-enriched categories as well as categories of functors and lax transformations. These two concepts will be fundamental for our construction of models of second order linear logic. We see how in categories enriched in the category Sup — of complete lattices and functions preserving arbitrary suprema — lax limits occur as a *lax bilimit* (in a sense to be made precise). We characterise lax bilimits in Sup-enriched categories and state some of their properties. Lax bilimits will be useful to obtain an instance of model for F_{DILL}. In addition, we study lax bilimits in categories enriched in a category of Z-complete posets and show that the usual domain-theoretic limit-colimit coincidence arises as a particular lax bilimit.

Part of the material in this chapter is similar to [46]. Nevertheless [46] is concerned with *lax* functors from the indiscrete category on a set of objects (*webs*) while here we are interested in ordinary functors. Theorem 2.28 extends a result given there.

In Chapter 3 we describe the construction of a linear hyperdoctrine from a suitable order-theoretic model of linear logic. This construction, with minimal modifications, also accounts for order-theoretic models of *System* F. It is therefore also an abstract presentation of the model in [15].

Finally, we give an instance of this construction using the category aCD of algebraic completely distributive lattices and functions preserving arbitrary suprema. For this model we calculate the interpretation of some closed universal types.

Appendix A contains the description of the term calculus F_{DILL} and its interpretation in a linear hyperdoctrine.

Appendix B is a collection of the most relevant background definitions.

We assume familiarity with basic order-theoretic concepts for which we refer to [3].

To typset the diagrams I used Paul Taylor's `diagrams` macro package.

Chapter 1

Categorical models of linear logic

In the literature there is a wealth of definitions and presentations of categorical models of linear logic. To obtain a categorical model of $\mathrm{F}_{\mathrm{DILL}}$ we need to choose one (sound and complete for DILL) and introduce a notion of functor between models that is suitable to model substitution of types for type variables. That is, we need to define a functor between categorical models of linear logic that preserves, strictly or up to isomorphisms, the structure that corresponds to the connectives of the (first order part of the) calculus (in our case, I, $\otimes$, $\multimap$, $!$).

The choice of DILL as the type theory for linear logic to work with restricts our attention to two categorical structures as models of linear logic. The first one is that of *linear category* [8, 10, 11], the second one is that presented with DILL in [4] and here called *symmetric monoidal adjunction for linear logic*.

These two structures are so closely related that they are sometimes regarded as equivalent. The second one is often seen as a simplification of the first which offers the intuition of an *intuitionistic world* suitably connected (via Girard's translation $!A \multimap B \cong A \to B$) to a *linear world*.

In this chapter we explain the advantages of the notion of linear category. To reach this aim we unravel the definition of linear category and obtain a better understanding of its nature and of its relation with symmetric monoidal adjunctions for linear logic.

Finally, we introduce a notion of functor between linear categories and define the category Lin of linear categories. This is what is used to obtain a categorical model of $\mathrm{F}_{\mathrm{DILL}}$ in the form of a *linear hyperdoctrine*. As an aside, we also make symmetric monoidal adjunctions for linear logic into a category and formulate the relationship between these two structures in terms of a reflection.

All the basic categorical structures that are not explicitly defined here are given in Appendix B.

1.1 Symmetric monoidal adjunction for linear logic

The categorical structure that is presented in [4] with the calculus DILL is that of a symmetric monoidal adjunction for linear logic.

Definition 1.1. A **symmetric monoidal adjunction for linear logic** $(F \dashv G \colon \mathbb{C} \to \mathbb{S})$ consists of the following:

1. A symmetric monoidal closed category $(\mathbb{S}, \otimes, I, \multimap, a, \lambda, \rho, s)$,
2. a cartesian category $(\mathbb{C}, \times, 1)$
3. a symmetric monoidal adjunction $(F, m^F, m^F_I) \dashv (G, m^G, m^G_I) \colon \mathbb{C} \to \mathbb{S}$ between the two.

A symmetric monoidal adjunction is an adjunction in the **2**-Cat of symmetric monoidal categories. This means that the unit and counit of the adjunction are assumed to be monoidal natural transformations (with respect to the obvious symmetric monoidal structure of the identity functors and of the two composition of the adjoint functors). In spite of this complexity, symmetric monoidal adjunctions are made appealing by the following characterisation due to Kelly [29, 26]:

Theorem 1.2. *An adjunction* $F \dashv G \colon \mathbb{X} \to \mathbb{A}$ *between monoidal categories* $(\mathbb{X}, \otimes_{\mathbb{X}}, I_{\mathbb{X}})$, $(\mathbb{A}, \otimes_{\mathbb{A}}, I_{\mathbb{A}})$ *is a* symmetric monoidal *adjunction if and only if* F *is a* strong *symmetric monoidal functor, that is if the natural transformation* $m^F_{X,Y} \colon FX \otimes_{\mathbb{A}} FY \to F(X \otimes_{\mathbb{X}} Y)$ *is a natural isomorphism and the map* $m^F_I \colon I_{\mathbb{A}} \to F(I_{\mathbb{X}})$ *is an isomorphism.*

Symmetric monoidal adjunctions were first considered as models for linear logic by Benton [6, 7]. There the cartesian category as well as the symmetric monoidal one are assumed to be closed and the resulting structure is called an LNL-category. The calculus LNL was introduced by Benton to capture the intuition of such a structure as a decomposition of the exponential !: In LNL the connection between intuitionistic logic (the cartesian closed category) and intuitionistic linear logic (the symmetric monoidal closed one) is no longer controlled by !, but by two new constructs that correspond to the left and right adjoint.

Working with DILL, I was puzzled by the fact that the calculus is soundly and completely modelled by a class of symmetric monoidal adjunctions, where two categories are given, while all of the interpretation happens in one of them.

I find the intuition behind monoidal adjunctions as an “intuitionistic world” together with a “linear world” slightly dangerous. It is of course precise for a calculus like LNL which has the two “worlds” already in its syntax. For a calculus like DILL it is somehow misleading: In the sequel we show that a symmetric monoidal adjunction occurs only as a convenient way of obtaining a linear category.

1.2 Linear category

We recall the original definition as given in [9, 10]:

Definition 1.3. A **linear category** consists of the following structure:

1. A symmetric monoidal closed category $(\mathbb{S}, \otimes, \mathrm{I}, \multimap, \alpha, \lambda, \rho, s)$, equipped with
2. a monoidal comonad $(\mathsf{T}, \mathsf{m}^{\mathsf{T}}, \mathsf{m}^{\mathsf{T}}_{\mathrm{I}}, \varepsilon, \delta)$ and
3. two monoidal natural transformations with components $e_A \colon \mathsf{T}A \to \mathrm{I}$ and $d_A \colon \mathsf{T}A \to \mathsf{T}A \otimes \mathsf{T}A$, $A \in |\mathbb{S}|$, such that
   a) $(\mathsf{T}A, e_A, d_A)$ is a commutative comonoid,
   b) e_A, d_A are coalgebra maps from the free coalgebra $(\mathsf{T}A, \delta_A)$,
   c) whenever $f \colon (\mathsf{T}A, \delta_A) \to (\mathsf{T}B, \delta_B)$ is a coalgebra map between free coalgebras, then it is also a comonoid morphism.

When we need to refer to a linear category in a concise way and there is no risk of ambiguity, we denote it with the pair $(\mathbb{S}, \mathsf{T})$.

This definition is quite complex and it implies a number of observations. For example the requirement that e_A, d_A are coalgebra maps from the free coalgebra $(\mathsf{T}A, \delta_A)$ embraces the observation that there are canonical coalgebra structures on their domains I and $\mathsf{T}A \otimes \mathsf{T}A$, defined in terms of the free coalgebra $(\mathsf{T}A, \delta_A)$.

In the sequel we unravel the assumptions of a linear category (mostly presenting results that already appear in a similar form in [10]) in the direction that is interesting for us, namely that of the interaction between the symmetric monoidal comonad and the commutative comonoid structure. This becomes clear in the category of T-coalgebras.

Notice that in this process the closed structure remains completely on a side: it can be assumed or not. Therefore whenever we mention a linear category in a statement about its commutative comonoids or its coalgebras, we do not implicitly assume that it is also closed.

To warm up, we give a lemma that we owe to Andrea Schalk (private communication). It shows that the definition of linear category as given in Definition 1.3 is slightly redundant.

Lemma 1.4. *Given a symmetric monoidal category* $(\mathbb{S}, \otimes, \mathrm{I})$*, a comonad* $(\mathsf{T}, \varepsilon, \delta)$ *on it, and a family of commutative comonoids* $\big\{(\mathsf{T}A, e_A, d_A)\big\}_{A \in |\mathbb{S}|}$ *the two following conditions are equivalent:*

1. *Every coalgebra map* $f \colon (\mathsf{T}A, \delta_A) \to (\mathsf{T}B, \delta_B)$ *between free coalgebras is also a comonoid morphism (that is, condition (3c) of Definition 1.3);*
2. *every component of* δ *is a comonoid morphism and the families* $\{e_A\}_{A \in |\mathbb{S}|}$ *and* $\{d_A\}_{A \in |\mathbb{S}|}$ *are natural;*

Proof. ($1 \Rightarrow 2$). For any $f \in \mathbb{S}(A, B)$ the map Tf is a coalgebra map between free coalgebras, therefore it is also a comonoid morphism; but Tf being a comonoid morphism for every $f \in \mathbb{S}(A, B)$ amounts exactly to naturality of $\{e_A\}_{A \in |\mathbb{S}|}$ and $\{d_A\}_{A \in |\mathbb{S}|}$. Moreover, δ is clearly a morphism between free coalgebras, hence a comonoid morphisms.

($2 \Rightarrow 1$). Let $f\colon (TA, \delta_A) \to (TB, \delta_B)$ be a coalgebra map. The first of the conditions for f to be a comonoid map is given by the outer hexagon in the following diagram:

$$\begin{tikzcd}
 & TA \arrow[dl, "e_A"'] \arrow[r, "f"] \arrow[d, "\delta_A"] & TB \arrow[d, "\delta_B"] \arrow[dr, "e_B"] & \\
I \arrow[dr, equal] & T^2A \arrow[r, "Tf"] \arrow[d, "e_{TA}"] & T^2B \arrow[d, "e_{TB}"] & I \arrow[dl, equal] \\
 & I \arrow[r, equal] & I &
\end{tikzcd}$$

The upper square commutes by assumption that f is a coalgebra map, the lower square by naturality of e, and the right as well as the left triangle commutes because every component of δ is a comonoid map.

As for preservation of d, consider the similar diagram:

$$\begin{tikzcd}
 & TA \arrow[dl, "d_A"'] \arrow[r, "f"] \arrow[d, "\delta_A"] & TB \arrow[d, "\delta_B"] \arrow[dr, "d_B"] & \\
TA \otimes TA \arrow[dr, "\delta_A \oplus \delta_A"'] & T^2A \arrow[r, "Tf"] \arrow[d, "d_{TA}"] & T^2B \arrow[d, "d_{TB}"] & TB \otimes TB \arrow[dl, "\delta_B \oplus \delta_B"] \\
 & T^2A \otimes T^2A \arrow[r, "Tf \otimes Tf"'] \arrow[d, "\varepsilon_{TA} \otimes \varepsilon_{TA}"'] & T^2B \otimes T^2B \arrow[d, "\varepsilon_{TB} \otimes \varepsilon_{TB}"] & \\
 & TA \otimes TA \arrow[r, "f \otimes f"'] & TB \otimes TB &
\end{tikzcd}$$

The upper part of the diagram is analogous to the previous one and commutes for the same reasons. The lower square commutes by naturality of ε. The outer square together with the comonad law $\varepsilon T \circ \delta = \mathrm{id}_T$ yields the wanted property:

$$\begin{tikzcd} TA \arrow[r, "f"] \arrow[d, "d_A"'] & TB \arrow[d, "d_B"] \\ TA \otimes TA \arrow[r, "f \otimes f"] & TB \otimes TB \end{tikzcd}$$

□

Since Definition 1.3 requires anyway naturality of e and d, condition (3c) can be replaced by the only requirement that the components of δ are comonoid morphisms. This is also the only way in which this condition is used in the sequel.

1.2.1 The category of coalgebras

We now recall that a monoidal comonad induces a monoidal category structure on the category of coalgebras (Eilenberg-Moore):

Proposition 1.5. *Let* $(\mathbb{S}, \otimes, I, a, \lambda, \rho, s)$ *be a symmetric monoidal category with a symmetric monoidal comonad* $(T, m^T, m^T_I, \varepsilon, \delta)$*. The category* $\mathbb{S}^T$ *of* T*-coalgebras inherits a symmetric monoidal structure from that of* $\mathbb{S}$*.*

Proof. The candidate symmetric monoidal structure of $\mathbb{S}^T$ is as follows: The unit T-coalgebra is $m^T_I\colon I \to TI$, if (A, x) and (B, y) are two coalgebras, their tensor is given by:

$$\begin{tikzcd} A \arrow[d, "x"'] \\ TA \end{tikzcd} \otimes^T \begin{tikzcd} B \arrow[d, "y"'] \\ TB \end{tikzcd} = \begin{tikzcd} A \otimes B \arrow[d, "x \otimes y"] \\ TA \otimes TB \arrow[d, "m^T_{A,B}"] \\ TA \otimes TB \end{tikzcd}$$

while the coherence isomorphisms of $\mathbb{S}^T$ are those of $\mathbb{S}$.

We need to check that (I, m^T_I) and $(A \otimes B, m^T_{A,B} \circ (x \otimes y))$ are coalgebras: For (I, m^T_I), $\varepsilon_I \circ m^T_I = id_I$ holds because ε is a monoidal natural transformation and $\delta_I \circ m^T_I = Tm^T_I \circ m^T_I$ because δ is a monoidal natural transformation. The analogous binary conditions that ε and δ have to satisfy as monoidal natural transformations provide the equation for $(A \otimes B, m^T_{A,B} \circ (x \otimes y))$ being a coalgebra.

Moreover, the natural isomorphisms a, λ, ρ and s are coalgebra maps: We give the proof only for ρ. In the diagram

$$\begin{tikzcd}
A \otimes I \arrow[r, "\rho_A"'] \arrow[d, "x \otimes id_I"'] & A \arrow[d, "x"] \\
TA \otimes I \arrow[r, "\rho_{TA}"] \arrow[d, "id_{TA} \otimes m^T_I"'] & TA \\
TA \otimes TA \arrow[r, "m^T_{A,I}"'] & T(A \otimes I) \arrow[u, "T\rho_A"']
\end{tikzcd}$$

the upper square commutes as ρ is natural and the lower square is one of the diagrams that T has to satisfy as monoidal functor. The outer square expresses the fact that ρ_A is a coalgebra map between $(A, x) \otimes^T (I, m^T_I)$ and (A, x). □

Remark. Condition (3b) of Definition 1.3 implicitly assumes this proposition, or at least the fact that there is a coalgebra structure on the unit I and a precise way of forming a coalgebra on $TA \otimes TA$ starting from the free coalgebra on A.

Remark. An analogous result does not hold for the Kleisli category. The direction of the natural transformation $m^T : T(-) \otimes T(-) \dot{\to} T(- \otimes -)$ works well for the Eilenberg-Moore category but it should be inverted to work for the Kleisli category. It is possible to get a feel for this if one attempts to define the tensor product of two morphisms in the Kleisli category. In the case of monads the situation is reversed: The monoidal structure carries over to the Kleisli category but not to the Eilenberg-Moore.

Now that we have a precise monoidal structure on the category of coalgebras, we may ask if the canonical adjunction $F^T \dashv G^T : \mathbb{S}^T \to \mathbb{S}$, where F^T is the forgetful functor and G^T is the free functor, is symmetric monoidal. This is easily established using Theorem 1.2: The identities $m^{(F^T)} = id$ and $m^{(F^T)}_I = id_I$ give to the left adjoint F^T the required strong (actually, strict) symmetric monoidal structure.

Notice that G is symmetric monoidal via $m^{(G^T)} = m^T$ and $m^{(G^T)}_I = m^T_I$. In order for this to make sense, it is worth to notice that m^T and m^T_I are in fact coalgebra maps;

this is expressed by the following diagrams whose commutativity is ensured by δ being a monoidal natural transformation:

$$\begin{tikzcd}
TA \otimes TB \arrow[r, "m^T_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & T(A \otimes B) \arrow[dd, "\delta_{A \otimes B}"] \\
T^2A \otimes T^2B \arrow[d, "m^T_{TA,TB}"'] & \\
T(TA \otimes TB) \arrow[r, "Tm^T_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "m^T_I"] \arrow[d, "m^T_I"'] & TI \arrow[d, "\delta_I"] \\
TI \arrow[r, "Tm^T_I"'] & T^2 I
\end{tikzcd}$$

So we have obtained the following proposition:

Proposition 1.6. *If* $(T, m^T, m^T_I, \varepsilon, \delta)$ *is a symmetric monoidal comonad on a symmetric monoidal category* $(\mathbb{S}, \otimes, I, \alpha, \lambda, \rho)$*, the canonical adjunction* $F^T \dashv G^T \colon \mathbb{S}^T \to \mathbb{S}$ *is symmetric monoidal with respect to the symmetric monoidal structure inherited by* $\mathbb{S}^T$.

1.2.2 The natural transformations e and d

At this point we want to focus on the natural transformations e and d in the definition of linear category and see how the requirements that they are asked to satisfy allow to extend them to natural transformations on the category of coalgebras.

Remark. When composing (symmetric) monoidal functors and natural transformations there is a precise way of seeing the resulting functor or natural transformation as monoidal. This fact is implicitly used in the definition of linear category. For example, the fact that $(T, m^T, m^T_I, \varepsilon, \delta)$ is a symmetric monoidal comonad means not only that (T, m^T, m^T_I) is a symmetric monoidal functor, but also that ε and δ are monoidal natural transformations. This carries the implicit assumption that there is a canonical way of seeing the identity functor $\mathrm{Id}_{\mathbb{S}}$ and the functor T^2 as symmetric monoidal. Moreover, the requirement that e and d are monoidal natural transformations presupposes a symmetric monoidal functor structure on the constant functor I and on $(T - \otimes T-)$ (given explicitly in Appendix B).

We start with a technical but very useful lemma which will help to obtain the commutativity of some diagrams in which not all of the subdiagrams commute.

Lemma 1.7. *Suppose* (T, ε, δ) *is a (not necessarily monoidal) comonad on a symmetric monoidal category* $\mathbb{S}$*. If* $\{(TA, e_A, d_A)\}_{A \in |\mathbb{S}|}$ *is a family of commutative comonoids, with* e *and*

d *natural transformations, and the components of* δ *are comonoid morphisms, then for any coalgebra* (A, x) *one has*

$$(x \otimes x) \circ (\varepsilon_A \otimes \varepsilon_A) \circ d_A \circ x = d_A \circ x.$$

Proof.

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "x"'] & TA \arrow[r, "d_A"] \arrow[d, "Tx"'] & TA \otimes TA \arrow[r, "\varepsilon_A \otimes \varepsilon_A"] \arrow[d, "Tx \otimes Tx"] & A \otimes A \arrow[d, "x \otimes x"] \\
TA \arrow[r, "\delta_A"] \arrow[dr, "d_A"'] & T^2A \arrow[r, "d_{TA}"] & T^2A \otimes T^2A \arrow[r, "\varepsilon_{TA} \otimes \varepsilon_{TA}"] & TA \otimes TA \\
& TA \otimes TA \arrow[ur, "\delta_A \otimes \delta_A"'] & &
\end{tikzcd}$$

In the diagram above, the left square commutes because (A, x) is a coalgebra and the middle one by naturality of d. The lower square commutes because δ_A is a comonoid morphism and the right square commutes by naturality of ε. Finally the lower outer edge of the diagram is $d_A \circ x$ since $\varepsilon_{TA} \circ \delta_A = \mathrm{id}_{TA}$ is one of the comonad laws. ∎

Natural families of commutative comonoids in the category of coalgebras. Let $\mathbb{S}$ be a symmetric monoidal category and $(T, m^T, m_I^T, \varepsilon, \delta)$ a symmetric monoidal comonad on it. Assume that there are natural transformations with components $e_A \colon TA \to I$ and $d_A \colon TA \to TA \otimes TA$ such that (TA, e_A, d_A) is a commutative comonoid, e_A and d_A are coalgebra morphisms, and each δ_A is a morphism of commutative comonoids.

Using e and d, we define two families of maps indexed by the T-coalgebras:

Definition 1.8. For any coalgebra (A, x) we define the following morphisms in $\mathbb{S}$:

$$\begin{aligned}
e^T_{(A,x)} &:= e_A \circ x \\
d^T_{(A,x)} &:= \varepsilon_A \otimes \varepsilon_A \circ d_A \circ x
\end{aligned}$$

The requirement that e and d are coalgebra maps corresponds to commutativity of the following two diagrams:

$$
\begin{tikzcd}
TA \arrow[r, "e_A"] \arrow[d, "{}^{1}\delta_A"'] & I \arrow[d, "m^T_I"] \\
T^2A \arrow[r, "Te_A"'] & TI
\end{tikzcd}
\qquad
\begin{tikzcd}
TA \arrow[r, "d_A"] \arrow[dd, "\delta_A"'] & TA \otimes TA \arrow[d, "\delta_A \otimes \delta_A"] \\
 & T^2A \otimes T^2A \arrow[d, "m^T_{TA,TA}"] \\
T^2A \arrow[r, "Td_A"'] & T(TA \otimes TA)
\end{tikzcd}
\tag{1.1}
$$

We see that also the newly defined e^T and d^T are coalgebra maps:

Lemma 1.9. *$e^T_{(A,x)}$ and $d^T_{(A,x)}$ are coalgebra maps.*

Proof. We need to check the commutativity of the following two diagrams:

$$
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "x"'] & TA \arrow[r, "e_A"] \arrow[d, "\delta_A"'] & I \arrow[d, "m^T_I"'] \\
TA \arrow[r, "Tx"'] & T^2A \arrow[r, "Te_A"'] & TI
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "x"] \arrow[dd, "x"'] & TA \arrow[r, "d_A"] \arrow[dd, "\delta_A"'] & TA \otimes TA \arrow[r, "\varepsilon_A \otimes \varepsilon_A"] \arrow[d, "\delta_A \otimes \delta_A"] & A \otimes A \arrow[d, "x \otimes x"] \\
 & & T^2A \otimes T^2A \arrow[r, "T\varepsilon_A \otimes T\varepsilon_A"] \arrow[d, "m^T_{TA,TA}"] & TA \otimes TA \arrow[d, "m^T_{A,A}"] \\
TA \arrow[r, "Tx"'] & T^2A \arrow[r, "Td_A"'] & T(TA \otimes TA) \arrow[r, "T(\varepsilon_A \otimes \varepsilon_A)"'] & T(A \otimes A)
\end{tikzcd}
$$

In the first diagram the left square is one of the coalgebra laws while the right square is the first of (1.1).

[1] Prop 13 and 15, p. 148 [10].

In the second diagram the left square is the same coalgebra law, the middle square is the second of (1.1), the lower left commutes by naturality of m^{T}, while the top left does not commute, but one can apply lemma 1.7 and the fact that $T\varepsilon_A \circ \delta_A = \mathrm{id}_{TA}$ (one of the comonad equations) and obtain that the outer diagram commutes. ☐

Now we can prove the naturality of e^T and d^T:

Proposition 1.10. *The families* e^T *and* d^T *define two natural transformations in the category of coalgebras:*

$$e^T \colon \mathrm{Id}_{\mathbb{S}^T} \xrightarrow{\bullet} (I, m_I^{\mathsf{T}})$$
$$d^T \colon \mathrm{Id}_{\mathbb{S}^T} \xrightarrow{\bullet} (- \otimes^T -)$$

Proof. Let $f \colon (A, x) \to (B, y)$ be any coalgebra map. To check naturality of e^T and d^T we need to prove that the following two (outer) diagrams commute:

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "f"'] & TA \arrow[r, "e_A"] \arrow[d, "Tf"'] & I \arrow[d, equal] \\
B \arrow[r, "y"'] & TB \arrow[r, "e_B"'] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "f"'] & TA \arrow[r, "d_A"] \arrow[d, "Tf"'] & TA \otimes TA \arrow[r, "\varepsilon_A \otimes \varepsilon_A"] \arrow[d, "Tf \otimes Tf"'] & A \otimes A \arrow[d, "f \otimes f"] \\
B \arrow[r, "y"'] & TB \arrow[r, "d_B"'] & TB \otimes TB \arrow[r, "\varepsilon_B \otimes \varepsilon_B"'] & B \otimes B
\end{tikzcd}$$

The leftmost squares of both diagrams commute by definition of coalgebra map. The rightmost squares of both diagrams commute by naturality of e and ε, respectively. The central square of the second diagram commutes by naturality of d. ☐

We now observe that the components of the natural transformations e^T and d^T at any free coalgebra (TA, δ_A) coincide with e_A and d_A:

Lemma 1.11. $e^T_{(TA,\delta_A)} = e_A$ *and* $d^T_{(TA,\delta_A)} = d_A$.

Proof. This is an almost immediate consequence of the fact that δ is a comonoid morphism. The equation that states that δ respects e is exactly the first equality: $e_{TA} \circ \delta_A = e_A$. The fact that δ respects d is expressed by the equation $d_{TA} \circ \delta_A = (\delta_A \otimes \delta_A) \circ d_A$. Therefore we have $d^T_{(TA,\delta_A)} := (\varepsilon_{TA} \otimes \varepsilon_{TA}) \circ d_{TA} \circ \delta_A = (\varepsilon_{TA} \otimes \varepsilon_{TA}) \circ (\delta_A \otimes \delta_A) \circ d_A = d_A$ where the last equality uses one of the comonad laws. ☐

So far we have extended the natural transformations e and d in $\mathbb{S}$ to related natural transformations e^T and d^T in the category of coalgebras.

We have also asked for each (TA, e_A, d_A) to be a commutative comonoid. We have seen that the category of coalgebras $\mathbb{S}^T$ has a symmetric monoidal structure induced by that of $\mathbb{S}$. Now we wonder whether $((A, x), e^T_{(A,x)}, d^T_{(A,x)})$ is also a commutative comonoid in the category of coalgebras. The answer is affirmative and is given by Proposition 1.13 below. We begin with a lemma.

Lemma 1.12. *Suppose* (T, ε, δ) *is a comonad on a symmetric monoidal category* $\mathbb{S}$. *If* $\{(TA, e_A, d_A)\}_{A \in |\mathbb{S}|}$ *is a family of commutative comonoids, with* e *and* d *natural transformations, and* δ *is a comonoid morphism, then for any coalgebra* (A, x) *one has the two equations:*

$$\mathrm{id}_A \otimes x \circ (\varepsilon_A \otimes \varepsilon_A) \circ d_A \circ x \;=\; (\varepsilon_A \otimes \mathrm{id}_{TA}) \circ d_A \circ x \tag{1.2}$$

$$x \otimes \mathrm{id}_A \circ (\varepsilon_A \otimes \varepsilon_A) \circ d_A \circ x \;=\; (\mathrm{id}_{TA} \otimes \varepsilon_A) \circ d_A \circ x \tag{1.3}$$

Proof. From Lemma 1.7 we have $(x \otimes x) \circ (\varepsilon_A \otimes \varepsilon_A) \circ d_A \circ x = d_A \circ x$ and hence

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "x"'] & TA \arrow[r, "d_A"] & TA \otimes TA \arrow[r, "\varepsilon_A \otimes \varepsilon_A"] & A \otimes A \arrow[d, "x \otimes x"'] \arrow[dr, "\mathrm{id}_A \oplus x"] & \\
TA \arrow[rrr, "d_A"'] & & & TA \otimes TA \arrow[r, "\varepsilon_A \otimes \mathrm{id}_{TA}"'] & A \otimes TA
\end{tikzcd}$$

where the leftmost triangle commutes by one of the comonad laws. A similar argument proves the second equation. □

Proposition 1.13. *In the symmetric monoidal category* $(\mathbb{S}^T, \otimes^T, I^T)$ *of* T*-coalgebras the triple* $\left((A, x), e^T_{(A,x)}, d^T_{(A,x)}\right)$ *is a commutative comonoid.*

Proof. We need to check the commutativity of four diagrams: two for left and right unit laws, one for associativity, and the last one for commutativity. For the unit laws consider the diagram:

$$\begin{tikzcd}
A \arrow[rr, "\rho_A^{-1}"] \arrow[d, "x"'] & & A \otimes I \arrow[d, "x \otimes \mathrm{id}_I"] \\
TA \arrow[rr, "\rho_{TA}^{-1}"] \arrow[d, "d_A"'] & & TA \otimes I \arrow[d, equal] \\
TA \otimes TA \arrow[rr, "\mathrm{id}_{TA} \otimes e_A"] \arrow[d, "\varepsilon_A \otimes \varepsilon_A"'] \arrow[dr, "\varepsilon_A \oplus \mathrm{id}_{TA}"] & & TA \otimes I \arrow[d, "\varepsilon_A \otimes \mathrm{id}_I"] \\
A \otimes A \arrow[r, "\mathrm{id}_A \otimes x"'] & A \otimes TA \arrow[r, "\mathrm{id}_A \otimes e_A"'] & A \otimes I
\end{tikzcd}$$

The upper square commutes by naturality of ρ. The middle square commutes because of the assumption that (TA, e_A, d_A) is a commutative comonoid. The lower left triangle does not commute, but its arrows precomposed with $d_A \circ x$ are equal by Equation 1.2. Finally, notice that the right edge is the identity by one of the comonad laws.

An analogous argument works for the isomorphism λ using Equation 1.3.

Next, we want to check associativity given by the diagram:

$$\begin{tikzcd}
A \arrow[r, "d^T_{(A,x)}"] \arrow[d, "d^T_{(A,x)}"'] & A \otimes A \arrow[r, "\mathrm{id}_A \otimes d^T_{(A,x)}"] & A \otimes (A \otimes A) \arrow[d, "a_{A,A,A}"] \\
A \otimes A \arrow[rr, "d^T_{(A,x)} \otimes \mathrm{id}_A"'] & & (A \otimes A) \otimes A
\end{tikzcd}$$

We show its commutativity by the following calculation:

$$\begin{aligned}
& a \circ (\mathrm{id}_A \otimes d^T_{(A,x)}) \circ d^T_{(A,x)} = \\
&= a \circ \big(\mathrm{id}_A \otimes (\varepsilon_A \otimes \varepsilon_A)\big) \circ (\mathrm{id}_A \otimes d_A) \circ (\mathrm{id}_A \otimes x) \circ (\varepsilon_A \otimes \varepsilon_A) \circ d_A \circ x && \text{by def.} \\
&= a \circ \big(\mathrm{id}_A \otimes (\varepsilon_A \otimes \varepsilon_A)\big) \circ (\varepsilon_A \otimes d_A) \circ d_A \circ x && \text{by eq. 1.2} \\
&= a \circ \big(\varepsilon_A \otimes (\varepsilon_A \otimes \varepsilon_A)\big) \circ (\mathrm{id}_{TA} \otimes d_A) \circ d_A \circ x \\
&= \big((\varepsilon_A \otimes \varepsilon_A) \otimes \varepsilon_A\big) \circ a \circ (\mathrm{id}_{TA} \otimes d_A) \circ d_A \circ x && \text{by nat. of } a \\
&= \big((\varepsilon_A \otimes \varepsilon_A) \otimes \varepsilon_A\big) \circ (d_A \otimes \mathrm{id}_{TA}) \circ d_A \circ x && (TA, e_A, d_A) \text{ comonoid} \\
&= \big((\varepsilon_A \otimes \varepsilon_A) \otimes \mathrm{id}_A\big) \circ (d_A \otimes \varepsilon_A) \circ d_A \circ x \\
&= \big((\varepsilon_A \otimes \varepsilon_A) \otimes \mathrm{id}_A\big) \circ (d_A \otimes \mathrm{id}_A) \circ (x \otimes \mathrm{id}_A) \circ (\varepsilon_A \otimes \varepsilon_A) \circ d_A \circ x && \text{by eq. 1.3} \\
&= d^T_{(A,x)} \otimes \mathrm{id}_A \circ d^T_{(A,x)} && \text{by def.}
\end{aligned}$$

Finally, to shows that $\big((A, x), e^T_{(A,x)}, d^T_{(A,x)}\big)$ is a *commutative* comonoid, consider the diagram:

$$\begin{tikzcd}
& A \arrow[dl, "x"'] \arrow[dr, "x"] & \\
TA \arrow[rr, equal] \arrow[d, "d_A"'] & & TA \arrow[d, "d_A"] \\
TA \otimes TA \arrow[rr, "s_{TA,TA}"] \arrow[d, "\varepsilon_A \otimes \varepsilon_A"'] & & TA \otimes TA \arrow[d, "\varepsilon_A \otimes \varepsilon_A"] \\
A \otimes A \arrow[rr, "s_{A,A}"'] & & A \otimes A
\end{tikzcd}$$

The middle square says that (TA, e_A, d_A) is a commutative comonoid while the lower square commutes by naturality of s. $\square$

Remark. Since $\big((A,x), e^T_{(A,x)}, d^T_{(A,x)}\big)$ is a commutative comonoid for any coalgebra (A,x) the fact that $e^T \colon \mathrm{Id}_{\mathbb{S}^T} \stackrel{\bullet}{\to} (I, m^T_I)$ and $d^T \colon \mathrm{Id}_{\mathbb{S}^T} \stackrel{\bullet}{\to} (- \otimes^T -)$ are natural transformations in the category of coalgebras coincides with the fact that any coalgebra morphism is a comonoid morphism.

So far we obtained that the category of coalgebras $\mathbb{S}^T$ of a linear category $\mathbb{S}$ is a symmetric monoidal category with monoidal structure induced by that of $\mathbb{S}$, that on every coalgebra (A,x) there is a commutative comonoid $((A,x), e^T_{(A,x)}, d^T_{(A,x)})$, and that the families e^T and d^T are natural in $\mathbb{S}^T$ and therefore any coalgebra map is a comonoid map.

Diagonal structure. Some authors call a family of commutative comonoids of a symmetric monoidal category $(\mathbb{X}, \otimes, I)$ indexed by the objects of $\mathbb{X}$ a *diagonal structure* if this family "respects" the monoidal structure of $\mathbb{X}$. This means that the comonoid on I has the identity on I as counit and that the comonoid on a tensor $X \otimes Y$ is the one obtained by "forming the tensor" of the comonoids on X and Y (a formal definition can be found in the appendix).

The following proposition shows that the family of commutative comonoids on the category of T-coalgebras does respect the monoidal structure.

Proposition 1.14. *Let* $(\mathbb{S}, \otimes, I)$ *be a symmetric monoidal category with symmetric monoidal comonad* $(T, m^T, m^T_I, \varepsilon, \delta)$ *and natural transformations* e *and* d*, such that for each* A *in* $\mathbb{S}$ (TA, e_A, d_A) *is a commutative comonoid* e_A *and* d_A *are coalgebra morphisms, and* δ_A *is a comonoid morphism. Then in the category of coalgebras* $\mathbb{S}^T$ *the family of commutative comonoids* $((A, x), e^T_{(A,x)}, d^T_{(A,x)})$ *indexed by the* T*-coalgebras form a diagonal structure.*

Proof. We need to prove that, given any two coalgebras (A, x) and (B, y), the following three diagrams commute:

$$\begin{tikzcd} I \arrow[r, equal, "e^T_{(I,m^T_I)}"] & I \end{tikzcd} \qquad \begin{tikzcd} & A \otimes B \arrow[dl, "e^T_{(A,x)} \oplus e^T_{(B,y)}"'] \arrow[dr, "e^T_{(A,x)\oplus T(B,y)}"] & \\ I \otimes I \arrow[rr, "\cong"'] & & I \end{tikzcd}$$

$$\begin{tikzcd} & A \otimes B \arrow[dl, "d^T_{(A,x)} \oplus d^T_{(B,y)}"'] \arrow[dr, "d^T_{(A,x)\oplus T(B,y)}"] & \\ (A \otimes A) \otimes (B \otimes B) \arrow[rr, "\cong"'] & & (A \otimes B) \otimes (A \otimes B) \end{tikzcd}$$

For the first diagram we have $e^T_{(I,m^T_I)} = e_I \circ m^T_I = id_I$ since e is monoidal. For the second diagram, the right edge corresponds to the upper path and the left to the lower path in the following

$$\begin{tikzcd} A \otimes B \arrow[d, "x \otimes y"'] & \\ TA \otimes TB \arrow[r, "m^{A,B}"] \arrow[d, "e_A \otimes e_B"'] & T(A \otimes B) \arrow[d, "e_{A \otimes B}"] \\ I \otimes I \arrow[r, "\cong"'] & I \end{tikzcd}$$

where the square commutes because e is monoidal. The third diagram is similar to the second one although somewhat more complex. Again, the right edge corresponds to the upper path and the left path to lower path in the following:

$$\begin{tikzcd}
A \otimes B \arrow[d, "x \otimes y"'] & & \\
TA \otimes TB \arrow[rr, "m^T_{A,B}"] \arrow[d, "d_A \otimes d_B"'] & & T(A \otimes B) \arrow[d, "d_{A \otimes B}"] \\
(TA \otimes TA) \otimes (TB \otimes TB) \arrow[r, "sw"'] \arrow[d, "(\varepsilon_A \otimes \varepsilon_A) \otimes (\varepsilon_B \otimes \varepsilon_B)"'] & (TA \otimes TB) \otimes (TA \otimes TB) \arrow[r, "m^T_{A,B} \otimes m^T_{A,B}"'] & T(A \otimes B) \otimes T(A \otimes B) \arrow[d, "\varepsilon_{A \otimes B} \otimes \varepsilon_{A \otimes B}"] \\
(A \otimes A) \otimes (B \otimes B) \arrow[rr, "sw"'] & & (A \otimes B) \otimes (A \otimes B)
\end{tikzcd}$$

The upper square commutes because d is monoidal. The lower square can be seen as expressing the fact that $(- \otimes -)\varepsilon$ is a monoidal natural transformation from $(- \otimes -)T$ to $(- \otimes -)$. □

Let us summarise: The category of coalgebras of a linear category has a commutative comonoid structure on each object, any coalgebra map is a comonoid map and the family of comonoids respects the monoidal structure, that is to say, the Eilenberg-Moore category is a *category of comonoids* (cf. the definition of linear category recently given in [25]).

There is an interesting proposition, which is regarded as category theory folklore, that connects the naturality of the families of counits and comultiplications of a diagonal structure with the quite strong property that the considered monoidal structure must be a finite product structure (the result in this form appears in [28]):

Proposition 1.15. *Let $\mathbb{S}$ be a symmetric monoidal category and $\{A, t_A, \Delta_A\}_{A \in |\mathbb{A}|}$ a diagonal structure on $\mathbb{S}$. Then:*

1. *If $\{t_A\}_A$ is a natural transformation from the identity on $\mathbb{S}$ to the constant I functor, then I is a terminal.*

2. *If in addition $\{\Delta_A\}_A$ is natural from the identity functor $\mathbb{S}$ to $(- \otimes -)$, then $\otimes$ is the categorical product.*

Proof. The object I is weakly terminal, and if $f\colon A \to I$ is any arrow to I then naturality means

$$\begin{tikzcd}
A \arrow[r, "t_A"] \arrow[d, "f"'] & I \arrow[d, equal] \\
I \arrow[r, equal, "t_I"'] & I
\end{tikzcd}$$

which implies $f = t_A$.

Suppose $\Delta\colon \mathbb{S} \to (- \otimes -)$ is natural. To show $\otimes = \times$ we first have to define the projections:

$$\begin{tikzcd}
A \otimes I \arrow[d, "\cong"'] & A \otimes B \arrow[l, "A \otimes t_B"'] \arrow[r, "t_A \otimes B"] \arrow[dl, "\pi_0"] \arrow[dr, "\pi_1"] & I \otimes B \arrow[d, "\cong"] \\
A & & B
\end{tikzcd}$$

If we are given two morphisms $f\colon C \to A$ and $g\colon C \to B$ we define $\langle f, g \rangle$ to be the composition

$$\langle f, g \rangle\colon C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{f \otimes g} A \otimes B.$$

Now, we have to consider the composition with the projections:

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\Delta"'] & A \arrow[d, "\Delta"'] \arrow[ddr, equal] & \\
C \otimes C \arrow[r, "f \otimes f"] \arrow[d, "f \otimes g"'] \arrow[dr, "f \otimes t"] & A \otimes A \arrow[d, "A \otimes t"] & \\
A \otimes B \arrow[r, "A \otimes t"'] & A \otimes I \arrow[r, "\cong"] & A
\end{tikzcd}$$

The top square commutes by naturality, the bottom one since we already know that I is a terminal, and the triangle on the right is one of the axioms defining comonoids. Hence, we have shown $\pi_0 \circ \langle f, g \rangle = f$, and the proof for $\pi_1 \circ \langle f, g \rangle = g$ is analogous.

To show the uniqueness of $\langle f, g\rangle$ consider another mediating arrow $h\colon C \to A \otimes B$:

$$\begin{tikzcd}
C \arrow[rr, "h"] \arrow[d, "\Delta"'] & & A \otimes B \arrow[r, equal] \arrow[dl, "\Delta"'] \arrow[d, "\Delta\otimes\Delta"] & A \otimes B \arrow[dd, equal] \\
C \otimes C \arrow[r, "h\otimes h"'] & (A \otimes B) \otimes (A \otimes B) \arrow[r, "\cong", "i"'] \arrow[dr, "(A\oplus t)\oplus(t\oplus B)"'] & (A \otimes A) \otimes (B \otimes B) \arrow[d, "(A\otimes t)\otimes(t\otimes B)"] & \\
& & (A \otimes I) \otimes (I \otimes B) \arrow[r, "\cong"] & A \otimes B
\end{tikzcd}$$

For most sub-diagrams commutativity is obvious. The rightmost rectangle is the tensor of the equations $(A \otimes t) \circ \Delta = A$ and $(t \otimes B) \circ \Delta = B$, both of which are instances of one of the axioms for comonoids. To understand

$$(A \otimes t_A) \otimes (t_B \otimes B) \circ i = (A \otimes t_B) \otimes (t_A \otimes B)$$

we have to take a closer look at the isomorphism i. It is made up from the isomorphisms saying that $\otimes$ is associative and commutative. These isomorphisms are natural transformations by definition. Note that $I \otimes I \cong I$ and so $I \otimes I$ is also terminal which implies that the commutativity morphism $\kappa_{I,I}\colon I \otimes I \to I \otimes I$ is the identity. This concludes the proof that the bottom middle triangle commutes. If we trace the arrows at the lower edge of the diagram we see that we have shown $h = \langle f, g\rangle$. □

As an immediate consequence we have that the category of coalgebras of a linear category is a cartesian category:

Proposition 1.16. *If* $(\mathbb{S}, \otimes, I, \multimap)$ *with symmetric monoidal comonad* $T = (T, m^T, m^T_I, \varepsilon, \delta)$ *and natural transformations* e *and* d *is a linear category, the monoidal structure* $\otimes^T$, (I, m^T_I) *inherited by the category* $\mathbb{S}^T$ *of* T*-coalgebras is a finite product structure.*

Proof. This is an immediate corollary of Propositions 1.14 and 1.15. □

The results obtained in this section are summed up in the following table, which represents the parts of the definition of linear category (on the left) with the structure that they force on the category of coalgebras (on the right).

$\mathbb{S}$ smc (T, ε, δ) comonad	$F^T \dashv G^T \colon \mathbb{S}^T \to \mathbb{S}$
$\mathbb{S}$ smc (T, ε, δ) symm. monoidal comonad	$\mathbb{S}^T$ smc, $F^T \dashv G^T$ symm. monoidal
$\mathbb{S}$ smc (T, ε, δ) symm. monoidal comonad e_A, d_A coalg. maps and nat. (TA, e_A, d_A) comm. comonoid δ comonoid map	$\mathbb{S}^T$ smc, $F^T \dashv G^T$ symm. monoidal $e^T{}_{(A,x)}$, $d^T{}_{(A,x)}$ coalg. maps and natural in $\mathbb{S}^T$ $((A, x), e^T_{(A,x)}, d^T_{(A,x)})$ comm. comonoid $e^T_{(TA,\delta_A)} = e_A$ and $d^T_{(TA,\delta_A)} = d_A$
$\mathbb{S}$ smc (T, ε, δ) symm. monoidal comonad e_A, d_A coalg. maps and nat. (TA, e_A, d_A) comm. comonoid δ comonoid map e, d monoidal	$\mathbb{S}^T$ smc, $F^T \dashv G^T$ symm. monoidal $e^T{}_{(A,x)}$, $d^T{}_{(A,x)}$ coalg. maps and natural in $\mathbb{S}^T$ $((A, x), e^T_{(A,x)}, d^T_{(A,x)})$ diag. structure $e^T_{(TA,\delta_A)} = e_A$ and $d^T_{(TA,\delta_A)} = d_A$

The last row of the table becomes the following if we add the closure property for $\mathbb{S}$ (only to complete the definition of linear category):

$\mathbb{S}$ linear category with symm. monoidal structure $(\otimes, I)$ and symm. monoidal comonad (T, ε, δ)	$F^T \dashv G^T$ symm. monoidal $(\mathbb{S}^T, \otimes^T, (I, m^T_I))$ cartesian

1.3 Comparison

As already mentioned, there is a strong connection between linear categories and symmetric monoidal adjunctions for linear logic.

From linear categories to symmetric monoidal adjunctions for linear logic
Proposition 1.16 shows that a linear category can be seen as a monoidal adjunction for linear logic (Definition 1.1) by considering the canonical adjunction with the category of coalgebras.

This way of obtaining a symmetric monoidal adjunction for linear logic from a linear category (so that the comonad induced by the adjunction correspond to the original comonad of the linear category) is not unique. To mention one possibility, the full subcategory of finite products of free coalgebras in the category of coalgebras also gives such a symmetric monoidal adjunction. This construction appears in [10, 7]. It is used [4] to prove completeness of the equational theory of DILL with respect to symmetric monoidal adjunctions for linear logic. Some other constructions are also considered in [10, 7].

From symmetric monoidal adjunctions for linear logic to linear categories
Given a symmetric monoidal adjunction, it is possible to construct a linear category. So far, the possibility that there might be different linear categories connected with a given symmetric monoidal adjunction has not been excluded nor considered. Nevertheless only one construction is ever presented. This is the following.

Assume that $(F \dashv G \colon \mathbb{C} \to \mathbb{S})$ is a symmetric monoidal adjunction for linear logic. Let $(\mathbb{S}, \otimes, I)$ be the symmetric monoidal structure on $\mathbb{S}$ and $(\mathbb{C}, \times, 1)$ the finite product structure on $\mathbb{C}$. The category $\mathbb{S}$ is then a linear category: The symmetric monoidal comonad $(T = FG, \varepsilon, \delta = F\eta G)$ is the one induced by the symmetric monoidal adjunction $F \dashv G$. The natural transformations $e\colon T \overset{\bullet}{\to} I$ and $d\colon T \overset{\bullet}{\to} (T - \otimes T-)$ are given by

$$e_A := (m^F_I)^{-1} \circ F(\star_{GA})$$
$$d_A := (m^F_{GA,GA})^{-1} \circ F(\Delta_{GA})$$

using the map $\star$ into the terminal object and the diagonal map Δ in $\mathbb{C}$. It is tedious but straightforward to check that they satisfy all the requirements in the definition of linear category.

One can also check that starting from a linear category and considering the symmetric monoidal adjunction with the category of coalgebras, the natural transformations e and d as defined above do coincide with the original ones: See Lemma 1.11.

Despite this fact, it is not clear in general if it could be possible to pick two other natural transformations e' and d' and obtain a different linear category structure on $(\mathbb{S}, \otimes, I)$ for the same symmetric monoidal comonad $(T, m^T_I, m^T, \varepsilon, \delta)$.

1.3.1 How many linear categories?

Proposition 1.16 relates a linear category $(\mathbb{S}, \otimes, I)$ with a symmetric monoidal comonad T to a precise symmetric monoidal adjunction for linear logic. Namely, the cartesian category is the category of coalgebras. Its finite product structure is the symmetric monoidal structure $(\otimes^T, I^T)$ induced by the symmetric monoidal structure on $\mathbb{S}$ via the symmetric monoidal comonad T.

In this category, the coalgebra maps $e^{T}_{(A,x)}$ and $d^{T}_{(A,x)}$ of Definition 1.8 are uniquely determined as the map into the terminal object and the diagonal map. Moreover, recalling Lemma 1.11, $e_A = e^{T}_{(TA,\delta_A)}$ and $d_A = d^{T}_{(TA,\delta_A)}$.

For any other pair of natural transformations $e'\colon T \xrightarrow{\bullet} I$ and $d'\colon T \xrightarrow{\bullet} (T - \otimes T-)$ satisfying the properties required in the definition of linear category, one obtains $e'^{T}_{(A,x)} = e^{T}_{(A,x)}$ and $d'^{T}_{(A,x)} = d^{T}_{(A,x)}$ by uniqueness, therefore $e'_A = e_A$ and $d'_A = d_A$.

That is, *there is at most one linear category structure* on a symmetric monoidal closed category $\mathbb{S}$ with a given symmetric monoidal comonad T.

I adopt the name "linear exponential comonad" from [25] (where a similar conclusion on the nature of linear category is implicit):

Definition 1.17. Let $(\mathbb{S}, \otimes, I)$ be a symmetric monoidal category. A *linear exponential comonad* on $\mathbb{X}$ is a symmetric monoidal comonad $(T, m^{T}_{I}, m^{T}, \varepsilon, \delta)$ such that the symmetric monoidal structure $(\otimes^{T}, I^{T})$ inherited by the category $\mathbb{S}^{T}$ of T-coalgebras is a finite product structure.

We have obtained the following:

Theorem 1.18. *Let $(\mathbb{S}, \otimes, I, \multimap)$ be a symmetric monoidal closed category with a symmetric monoidal comonad $(T, m^{T}, m^{T}_{I}, \varepsilon, \delta)$. The following conditions are equivalent:*

1. $(\mathbb{S}, T)$ *is a linear category.*
2. T *is a linear exponential comonad.*
3. *There exists a cartesian category $\mathbb{C}$ and a monoidal adjunction $(F \dashv G\colon \mathbb{C} \to \mathbb{S})$ inducing the given comonad.*

I would prefer to adopt the second property as concise definition of linear category.

The original definition remains important: Whenever one needs to check if a given symmetric monoidal comonad is a linear exponential comonad, to exhibit natural transformations e and d with the required properties can be easier than showing that the symmetric monoidal structure inherited by the category of coalgebras is a finite product structure; or it could be convenient to use property (3) above and exhibit a monoidal adjunction for the comonad.

The advantage of property (2) in Theorem 1.18 is that it makes clear what the nature of a linear category is (the fact that there is no choice for the natural transformations e and d) using a canonical construction for the comonad.

1.3.2 Why linear categories?

As already mentioned, the calculus DILL [4] is presented together with an interpretation in a symmetric monoidal adjunction for linear logic. This is proved to give a sound and complete model for the equational theory of the calculus. Nevertheless, all of the interpretation of DILL takes place in the symmetric monoidal category. The only property that is important in the interpretation is that it is a linear category. As a consequence, there is no need for a notion of morphism between models in the form of a morphism of monoidal adjunction (as briefly given there). We therefore choose linear categories as the appropriate notion of categorical model for linear logic and proceed by defining suitable functors between them.

Notice, however, that the natural way to present the completeness result for the interpretation of DILL is via the construction of a symmetric monoidal adjunction for linear logic. It is therefore convenient to keep the tools of Theorem 1.18 at hand.

1.4 Morphisms

We introduce a notion of functor between linear categories that preserves the structure of linear category strictly: Such functors are needed to model substitution of types for type variables, hence we want at least to preserve the linear category structure up to isomorphisms. Nevertheless, in the construction of a model for F_{DILL} that we present, reindexing does preserve the structure on the nose.

Definition 1.19. Consider two linear categories: $(\mathbb{S}, \otimes, I, \multimap)$ with linear exponential comonad $(\mathsf{T}, \mathsf{m}^{\mathsf{T}}, \mathsf{m}^{\mathsf{T}}_{I}, \varepsilon, \delta)$ and $\mathbb{S}' = (\mathbb{S}', \otimes', I', \multimap')$ with linear exponential comonad $(\mathsf{T}', \mathsf{m}^{\mathsf{T}'}, \mathsf{m}^{\mathsf{T}'}_{I}, \varepsilon', \delta')$. A *linear functor* is a functor $\mathsf{K}\colon \mathbb{S} \to \mathbb{S}'$ that satisfies:

1. K is strict symmetric monoidal closed,
2. $\mathsf{KT} = \mathsf{T}'\mathsf{K}$ as monoidal functors,
3. $\mathsf{K}\varepsilon = \varepsilon'_{\mathsf{K}}$, and
4. $\mathsf{K}\delta = \delta'_{\mathsf{K}}$.

Linear categories and linear functors form a category Lin with obvious identity and composition.

With $\mathsf{KT} = \mathsf{T}'\mathsf{K}$ as monoidal functors, we mean that $\mathsf{K}\mathsf{m}^{\mathsf{T}} \circ \mathsf{m}^{\mathsf{K}}{}_{\mathsf{T}} = \mathsf{T}'\mathsf{m}^{\mathsf{K}} \circ \mathsf{m}^{\mathsf{T}'}{}_{\mathsf{K}}$ and $\mathsf{K}\mathsf{m}^{\mathsf{T}}_{I} \circ \mathsf{m}^{\mathsf{K}}_{I} = \mathsf{T}'\mathsf{m}^{\mathsf{K}}_{I} \circ \mathsf{m}^{\mathsf{T}'}_{I}$.

Remark. Given the definition of linear category, one would feel inclined to ask for a morphism of linear categories also to preserves the natural transformations e and d, that is $\mathsf{Ke} = \mathsf{e}'_{\mathsf{K}}$ and $\mathsf{Kd} = \mathsf{d}'_{\mathsf{K}}$. This is not necessary, as it follows from the rest of the requirements as we soon see. This is consistent with the observation that the natural transformations e and d are strictly connected to the nature of the comonad (that is, with its being a linear exponential comonad) and not independent additional structure.

1.4.1 Products in the category of coalgebras

Let $(\mathbb{S}, \otimes, I)$ be a symmetric monoidal category with a linear exponential comonad $(T, m^T, m^T_I, \varepsilon, \delta)$.

Let us consider the symmetric monoidal structure inherited by the category $\mathbb{S}^T$ of T-coalgebras, which is, by definition, a finite product structure.

The terminal object is the coalgebra (I, m^T_I). Given T-coalgebras (A, x) and (B, y) their product is the coalgebra $(A \otimes B, m^T_{A,B} \circ x \otimes y)$.

For any T-coalgebra (A, x), let $\star_A \colon (A, x) \to (I, m^T_I)$ be the unique morphism to the terminal object and $\Delta_A \colon (A, x) \to (A, x) \otimes (A, x)$ the diagonal map.

Consider the families of coalgebra morphisms

$$e_A := \star_{(TA,\delta_A)} \tag{1.4}$$
$$d_A := \Delta_{(TA,\delta_A)} \tag{1.5}$$

indexed by $A \in |\mathbb{S}|$.

Lemma 1.20. *The coalgebra morphisms* $\star_{(A,x)}$ *and* $\Delta_{(A,x)}$ *decompose as follows:*

$$\star_{(A,x)} = \star_{(TA,\delta_A)} \circ x = e_A \circ x \tag{1.6}$$
$$\Delta_{(A,x)} = \varepsilon_A \otimes \varepsilon_A \circ \Delta_{(TA,\delta_A)} \circ x = \varepsilon_A \otimes \varepsilon_A \circ d_A \circ x. \tag{1.7}$$

Proof. The first equation is obvious. The second is obtained by composing $\varepsilon_A \otimes \varepsilon_A$ after the equality $x \otimes x \circ \Delta_{(A,x)} = \Delta_{(TA,\delta_A)} \circ x$ that expresses the naturality condition for Δ at the coalgebra map x. □

Given two coalgebras (A, x) and (B, y) the support map for the first projection is given by:

$$A \otimes B \xrightarrow{id_A \otimes y} A \otimes TB \xrightarrow{id_A \otimes e_B} A \otimes I \xrightarrow{\rho_A} A$$

Similarly, the second projection is given by:

$$A \otimes B \xrightarrow{x \otimes id_B} TA \otimes B \xrightarrow{e_A \otimes id_B} I \otimes B \xrightarrow{\lambda_B} B$$

If $f \colon (C, z) \to (A, x)$ and $g \colon (C, z) \to (B, y)$ are coalgebra morphisms, the support of the pairing $\langle f, g\rangle \colon (C, z) \to (A, x) \otimes (B, y)$ is given by:

$$C \xrightarrow{z} TC \xrightarrow{d_C} TC \otimes TC \xrightarrow{\varepsilon_C \otimes \varepsilon_C} C \otimes C \xrightarrow{f \otimes g} A \otimes B.$$

1.4.2 Morphisms of comonads

The following is the dual of the definition of morphism of monads as given in [47]:

Definition 1.21. Let $(\mathsf{T}, \varepsilon, \delta)$ be a comonad on a category $\mathbb{S}$ and $(\mathsf{T}', \varepsilon', \delta')$ a comonad on a second category $\mathbb{S}'$. A *morphism of comonads* (K, α) is a functor $\mathsf{K} \colon \mathbb{S} \to \mathbb{S}'$ together with a natural transformation $\alpha \colon \mathsf{KT} \stackrel{\bullet}{\to} \mathsf{T}'\mathsf{K}$ satisfying the equations

$$\mathsf{K}\varepsilon = \varepsilon'_{\mathsf{K}} \circ \alpha \tag{1.8}$$

$$\mathsf{T}'\alpha \circ \alpha_{\mathsf{T}} \circ \mathsf{K}\delta = \delta'_{\mathsf{K}} \circ \alpha. \tag{1.9}$$

Such a functor K *lifts* to a functor K^{em} between the relative categories of coalgebras. The functor $\mathsf{K}^{\mathrm{em}} \colon \mathbb{S}^{\mathsf{T}} \to {\mathbb{S}'}^{\mathsf{T}'}$ is defined as follows. For any T-coalgebra (A, x)

$$\mathsf{K}^{\mathrm{em}}(A, x) = (\mathsf{K}A, \alpha_A \circ \mathsf{K}x) \tag{1.10}$$

and for any coalgebra morphism f,

$$\mathsf{K}^{\mathrm{em}} f = \mathsf{K}f. \tag{1.11}$$

Notice that $(\mathsf{K}A, \alpha_A \circ \mathsf{K}x)$ is a T'-coalgebra:

$$\begin{tikzcd}
\mathsf{K}A \arrow[r, "\mathsf{K}x"] \arrow[d, "\mathsf{K}x"'] & \mathsf{KT}A \arrow[r, "\alpha_A"] \arrow[d, "\mathsf{K}\delta_A"] & \mathsf{T}'\mathsf{K}A \arrow[dd, "\delta'_{\mathsf{K}A}"] \\
\mathsf{KT}A \arrow[r, "\mathsf{KT}x"] \arrow[d, "\alpha_A"'] & \mathsf{KT}^2A \arrow[d, "\alpha_{\mathsf{T}A}"] & \\
\mathsf{T}'\mathsf{K}A \arrow[r, "\mathsf{T}'\mathsf{K}x"'] & \mathsf{T}'\mathsf{KT}A \arrow[r, "\mathsf{T}'\alpha_A"'] & {\mathsf{T}'}^2\mathsf{K}A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{K}A \arrow[r, "\mathsf{K}x"] \arrow[d, equal] & \mathsf{KT}A \arrow[d, "\alpha_A"] \arrow[dl, "\mathsf{K}\varepsilon_A"'] \\
\mathsf{K}A & \mathsf{T}'\mathsf{K}A \arrow[l, "\varepsilon'_{\mathsf{K}A}"]
\end{tikzcd}$$

On the left the commutativity of the upper square follows from the analogous comonad law for x, the lower square commutes by naturality of α, while the right square corresponds to Equation 1.9 (notice that this last condition also says that α_A is a coalgebra morphism $\mathsf{K}^{\mathrm{em}}(\mathsf{T}A, \delta_A) \to (\mathsf{T}'\mathsf{K}A, \delta'_{\mathsf{K}A})$). Similarly, in the diagram on the right the commutativity of the upper triangle follows from the analogous comonad law for x, while the lower triangle corresponds to Equation 1.8.

Definition 1.22. For symmetric monoidal categories $(\mathbb{S}, \otimes, I)$ and $(\mathbb{S}', \otimes', I')$ and corresponding symmetric monoidal comonads $(\mathsf{T}, \mathsf{m}^{\mathsf{T}}, \mathsf{m}^{\mathsf{T}}_{I}, \varepsilon, \delta)$ and $(\mathsf{T}', {\mathsf{m}^{\mathsf{T}}}', {\mathsf{m}^{\mathsf{T}}_{I}}', \varepsilon', \delta')$, a *morphism of symmetric monoidal comonads* is a pair (K, α) where K is a symmetric monoidal functor $\mathbb{S} \to \mathbb{S}'$ and $\alpha \colon \mathsf{KT} \stackrel{\bullet}{\to} \mathsf{T}'\mathsf{K}$ is a monoidal natural transformation satisfying Equations 1.8 and 1.9.

That α is a monoidal natural transformation means commutativity of the two following diagrams:

$$
\begin{tikzcd}
KTA \otimes' KTB \arrow[r, "\alpha_A \otimes' \alpha_B"] \arrow[d, "m^K_{TA,TB}"'] & T'KA \otimes' T'KB \arrow[d, "m^{T'}_{KA,KB}"] \\
K(TA \otimes TB) \arrow[d, "Km^T_{A,B}"'] & T'(KA \otimes' KB) \arrow[d, "T'm^K_{A,B}"] \\
KT(A \otimes B) \arrow[r, "\alpha_{A \otimes B}"'] & T'K(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
I' \arrow[r, equal] \arrow[d, "m^K_I"'] & I' \arrow[d, "m^{T'}_I"] \\
KI \arrow[d, "Km^T_I"'] & T'I' \arrow[d, "T'm^K_I"] \\
KTI \arrow[r, "\alpha_I"'] & T'KI
\end{tikzcd}
$$

Because of these conditions on α, the functor K^{em} is a symmetric monoidal closed functor with respect to the symmetric monoidal structure inherited by the categories of coalgebras, and this via the monoidal structure of K: The diagram on the left implies that each $m^K_{A,B}$ is a coalgebra morphism

$$K^{em}(A, x) \otimes^{T'} K^{em}(B, y) \to K^{em}\big((A, x) \otimes^{T} (B, y)\big)$$

while the second one is exactly the condition that the morphism m^K_I has to satisfy to be a coalgebra morphism $(I'm^{T'}_I) \to K^{em}(I, m^T_I)$.

Strong symmetric monoidal functors and products. Let $\mathbb{C}$ and $\mathbb{D}$ be two categories with chosen products ($\times$ and $\times'$) and terminal objects (1 and $1'$). These are obviously symmetric monoidal structures on $\mathbb{C}$ and $\mathbb{D}$. Let $F \colon \mathbb{C} \to \mathbb{D}$ be a functor. The condition that F is a *strong* symmetric monoidal functor amounts to the fact that F preserves the finite product structure up to (*coherent*) isomorphisms. For any $C \in |\mathbb{C}|$ and $D \in |\mathbb{D}|$, let $\star_C \colon C \to 1$ and $\star'_D \colon D \to 1'$ be, respectively, the unique morphism into the terminal object in $\mathbb{C}$ and $\mathbb{D}$. Similarly, let $\Delta_C \colon C \to C \times C$ and $\Delta'_D \colon D \to D \times' D$ be the diagonal maps. It is easy to check that the two following equalities hold:

$$F(\star_C) = m^F_I \circ \star'_{FC} \tag{1.12}$$
$$F(\Delta_C) = m^F_{C,C} \circ \Delta'_{FC}. \tag{1.13}$$

From these observations we obtain the following. Consider two symmetric monoidal categories $(\mathbb{S}, \otimes, I)$ and $(\mathbb{S}', \otimes', I')$ with respective linear exponential comonads $(T, m^T, m^T_I, \varepsilon, \delta)$ and $(T', m^{T'}, m^{T'}_I, \varepsilon', \delta')$. In the following let e and d denote the coalgebra morphism from Equations 1.4 and 1.5 and, moreover, let e' and d' be the analogous ones for $\mathbb{S}'$.

Proposition 1.23. *Let* $(\mathbb{S}, \otimes, \mathrm{I})$ *and* $(\mathbb{S}', \otimes', \mathrm{I}')$ *be symmetric monoidal categories with respective linear exponential comonads* $(\mathsf{T}, \mathsf{m}^{\mathsf{T}}, \mathsf{m}^{\mathsf{T}}_{\mathrm{I}}, \varepsilon, \delta)$ *and* $(\mathsf{T}', \mathsf{m}^{\mathsf{T}'}, \mathsf{m}^{\mathsf{T}'}_{\mathrm{I}}, \varepsilon', \delta')$*. If* (K, α) *is a morphism of symmetric monoidal comonads and the functor* K *is* strong *monoidal, we get the following equation relating the coalgebra morphisms* e *and* e'

$$\mathsf{K}\mathsf{e}_{\mathsf{A}} = \mathsf{m}^{\mathsf{K}}_{\mathrm{I}} \circ \mathsf{e}'_{\mathsf{KTA}} \circ \alpha_{\mathsf{TA}} \circ \mathsf{K}\delta_{\mathsf{A}}. \tag{1.14}$$

Similarly, for d *and* d' *we get*

$$\mathsf{K}\mathsf{d}_{\mathsf{A}} = \mathsf{m}^{\mathsf{K}}_{\mathsf{TA},\mathsf{TA}} \circ (\varepsilon_{\mathsf{KTA}} \otimes \varepsilon_{\mathsf{KTA}}) \circ \mathsf{d}'_{\mathsf{KTA}} \circ \alpha_{\mathsf{TA}} \circ x. \tag{1.15}$$

Proof. By the definition of linear exponential comonad, the symmetric monoidal structure inherited by the categories of coalgebras $\mathbb{S}^{\mathsf{T}}$ and $\mathbb{S}'^{\mathsf{T}'}$ are finite product structures. Since we assume (K, α) to be a morphism of symmetric monoidal comonads with K *strong* symmetric monoidal, the functor K^{em} is a strong symmetric monoidal functor with respect to these finite products structures. Equation 1.12 for the functor to K^{em} and the object $(\mathsf{TA}, \delta_{\mathsf{A}})$ amounts to

$$\mathsf{K}^{\mathrm{em}}\big(\star_{(\mathsf{TA},\delta)}\big) = \mathsf{m}^{\mathsf{K}}_{\mathrm{I}} \circ \star'_{\mathsf{K}^{\mathrm{em}}(\mathsf{TA},\delta_{\mathsf{A}})}.$$

By recalling the definitions of e and e' as well as that of K^{em}, and the decomposition (1.6), this becomes:

$$\begin{aligned}
\mathsf{K}\mathsf{e}_{\mathsf{A}} &= \mathsf{m}^{\mathsf{K}}_{\mathrm{I}} \circ \star'_{(\mathsf{KTA},\alpha_{\mathsf{TA}} \circ \mathsf{K}\delta_{\mathsf{A}})} \\
&= \mathsf{m}^{\mathsf{K}}_{\mathrm{I}} \circ \star'_{(\mathsf{T}'\mathsf{KTA},\delta'_{\mathsf{KTA}})} \circ \alpha_{\mathsf{TA}} \circ \mathsf{K}\delta_{\mathsf{A}}) \\
&= \mathsf{m}^{\mathsf{K}}_{\mathrm{I}} \circ \mathsf{e}'_{\mathsf{KTA}} \circ \alpha_{\mathsf{TA}} \circ \mathsf{K}\delta_{\mathsf{A}})
\end{aligned}$$

Similarly, equality (1.13) becomes

$$\mathsf{K}^{\mathrm{em}}\big(\Delta_{(\mathsf{TA},\delta)}\big) = \mathsf{m}^{\mathsf{K}}_{\mathsf{TA},\mathsf{TA}} \circ \Delta'_{\mathsf{K}^{\mathrm{em}}(\mathsf{TA},\delta_{\mathsf{A}})}$$

which, unravelled using the definitions of d and d' and the decomposition (1.7), yields the wanted result. □

Corollary 1.24. *Let* $(\mathbb{S}, \otimes, \mathrm{I}, \multimap)$ *and* $(\mathbb{S}', \otimes', \mathrm{I}', \multimap)$ *be linear categories with respective linear exponential comonads* $(\mathsf{T}, \mathsf{m}^{\mathsf{T}}, \mathsf{m}^{\mathsf{T}}_{\mathrm{I}}, \varepsilon, \delta)$ *and* $(\mathsf{T}', \mathsf{m}^{\mathsf{T}'}, \mathsf{m}^{\mathsf{T}'}_{\mathrm{I}}, \varepsilon', \delta')$*. If* $\mathsf{K}\colon \mathbb{S} \to \mathbb{S}'$ *is a linear functor, then*

$$\mathsf{K}\mathsf{e} = \mathsf{e}'_{\mathsf{K}} \quad \textit{and} \quad \mathsf{K}\mathsf{d} = \mathsf{d}'_{\mathsf{K}}.$$

Proof. For a linear functor K, we have that $\mathsf{K}^{\mathrm{em}}(\mathsf{A}, x) = (\mathsf{KA}, \mathsf{K}x)$. The equality 1.12 for the functor to K^{em} and the object $(\mathsf{TA}, \delta_{\mathsf{A}})$ is:

$$\mathsf{K}^{\mathrm{em}}\big(\star_{(\mathsf{TA},\delta)}\big) = \star'_{\mathsf{K}^{\mathrm{em}}(\mathsf{TA},\delta_{\mathsf{A}})}.$$

By recalling the definitions of e and e' and the decomposition (1.6) this becomes

$$Ke_A = \star'_{(KTA, K\delta_A)} = \star'_{(T'KA, \delta'_{KA})} = e'_{KA}.$$

Similarly, equality (1.13) is

$$K^{em}\big(\Delta_{(TA,\delta)}\big) = \Delta'_{K^{em}(TA,\delta_A)}$$

which, by using the definitions of d and d' and the decomposition (1.7), becomes

$$Kd_A = \Delta'_{(KTA, K\delta_A)} = \Delta'_{(T'KA, \delta'_{KA})} = d'_{KA}.$$

□

1.4.3 A category of comonads and a category of adjunctions

Now we consider a functor of symmetric monoidal adjunction for linear logic in the form of a particular *transformation of adjoints* as defined in [31]. If we relax part of the definition of a linear functor to admit strong monoidal closed functors, we obtain that the results of this subsection are still valid when we specialise categories and functors to the respective models of linear logic.

What we are going to present now are rather obvious general facts for which I found no explicit reference in the literature.

Definition 1.25. Let Cmad be the following category: Objects are comonads (T, ε, δ) on a category $\mathbb{S}$. A morphism from (T, ε, δ) on $\mathbb{S}$ to $(T', \varepsilon', \delta')$ on $\mathbb{S}'$ is a *strict* morphisms of comonads, that is a functor $K\colon \mathbb{S} \to \mathbb{S}'$ satisfying

1. $KT = T'K$,
2. $K\varepsilon = \varepsilon'_K$, and
3. $K\delta = \delta'_K$.

Identity and composition are as expected.

Definition 1.26. Let Adj be the following category: Objects are adjunctions $F \dashv G\colon \mathbb{A} \to \mathbb{S}$ between any two categories. A morphism from an adjunction $F \dashv G\colon \mathbb{A} \to \mathbb{S}$ with unit η and counit ε, to an adjunction $F' \dashv G'\colon \mathbb{A}' \to \mathbb{S}'$ with unit η' and counit ε', is a pair of functors (K, H) where $K\colon \mathbb{S} \to \mathbb{S}'$, $H\colon \mathbb{A} \to \mathbb{A}'$ such that

1. $KF = F'K$,
2. $HG = G'K$, and
3. $H\eta = \eta'_H$ or, equivalently, $K\varepsilon = \varepsilon'_K$.

Identity and composition are as expected.

We identify the category of Cmad with a precise subcategory of the category of adjunctions that we are now going to define. An adjunction $F \dashv G \colon \mathbb{A} \to \mathbb{S}$ with unit η and counit ε induces a comonad $(T = FG, \varepsilon, \delta = F\eta G)$ on $\mathbb{S}$. If one considers the class of adjunctions that arise as the Eilenberg-Moore construction of a comonad, one obtains a full subcategory $\mathsf{Adj}_{\mathsf{em}}$ of Adj:

Definition 1.27. Let $\mathsf{Adj}_{\mathsf{em}}$ be the full subcategory of Adj whose objects are those adjunctions given by the Eilenberg-Moore construction for a comonad.

Morphisms in the subcategory $\mathsf{Adj}_{\mathsf{em}}$ have a particular form:

Lemma 1.28. *Any Adj-morphism (K, H) between objects of $\mathsf{Adj}_{\mathsf{em}}$ is of the form (K, K^{em}).*

Proof. Let (K, H) be a morphism in Adj from $F^T \dashv G^T$ to $F^S \dashv G^S$. By definition of Adj-morphism, one has that $F^S H = KF^T$ while, by construction, F^T and F^S are the forgetful functors that assign to any coalgebra its support. Any structural map of T-coalgebra $x \colon A \to TA$ is both an object and a morphism in the category $\mathbb{S}^T$. The condition $H\eta = \eta'_H$ states exactly that H acts on $x \colon A \to TA$ as an object precisely as it acts on $x \colon A \to TA$ as a morphism. Therefore from the equality $F^S H = KF^T$ we deduce that $H(A, x) = K^{\mathsf{em}}(A, x)$. $\square$

We therefore obtain:

Proposition 1.29. *The category Cmad is isomorphic to $\mathsf{Adj}_{\mathsf{em}}$.*

The fact that the Eilenberg-Moore category $\mathbb{S}^T$ is a terminal object in the category of adjunctions inducing the comonad T on $\mathbb{S}$ translates into the following proposition.

Proposition 1.30. *Given an adjunction $F \dashv G \colon \mathbb{A} \to \mathbb{S}$ with unit η and counit ε and the corresponding comonad $(T = FG, \varepsilon, \delta = F\eta G)$ on $\mathbb{S}$, the unique comparison functor $R \colon \mathbb{A} \to \mathbb{S}^T$ is the unit of a reflection $\mathcal{R}_{\mathsf{em}}$ of Adj in $\mathsf{Adj}_{\mathsf{em}}$.*

Proof. The candidate arrow that reflects an object of Adj into $\mathsf{Adj}_{\mathsf{em}}$ is given by the comparison functor R paired up with the identity functor:

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "R"] \arrow[d, "F"', shift right=2] \arrow[d, phantom, "\dashv"] & \mathbb{B}^T \arrow[d, "F^T"', shift right=2] \arrow[d, phantom, "\dashv"] \\
\mathbb{B} \arrow[u, "G"', shift right=2] \arrow[r, equal] & \mathbb{B} \arrow[u, "G^T"', shift right=2]
\end{tikzcd}$$

We now show that $(R, \mathrm{Id}_{\mathbb{B}})$ is a universal arrow from $F \dashv G$ to $\mathsf{Adj}_{\mathsf{em}}$. Let S be a comonad on a category $\mathbb{C}$ and (K, H) an Adj-map from $F \dashv G \colon \mathbb{S} \to \mathbb{B}$ to $F^S \dashv G^S \colon \mathbb{C}^S \to \mathbb{C}$, as pictured in the diagram:

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "H"] \arrow[d, shift right=2, "F"'] \arrow[d, phantom, "\dashv"] & \mathbb{C}^S \arrow[d, shift right=2, "F^S"'] \arrow[d, phantom, "\dashv"] \\
\mathbb{B} \arrow[r, "K"'] \arrow[u, shift right=2, "G"'] & \mathbb{C} \arrow[u, shift right=2, "G^S"']
\end{tikzcd}$$

We want to prove that (K, K^{em}) is the unique map from $F^T \dashv G^T$ to $F^S \dashv G^S$ such that $(K, K^{em}) \circ (\mathrm{Id}_{\mathbb{B}}, R) = (K, H)$.

To see that $K^{em} \circ R = H$ on objects, recall that the component of the unit of the canonical adjunction $F^S \dashv G^S \colon \mathbb{C}^S \to \mathbb{C}$ at an S-coalgebra (C, c) is c itself seen as a coalgebra morphism. Therefore, if η' is the unit of $F^S \dashv G^S$, the component of η' at (C, c) coincides with c, that is $\eta'_{(C,c)} = c$. Hence, for any object A in $\mathbb{S}$

$$KF\eta_A = F^S H\eta_A = F^S \eta'_{HA}$$

since (K, H) is a morphism in Adj and the last one is the structural map of the S-coalgebra HA. But for any object A of $\mathbb{S}$, the coalgebra RA is by construction the T-coalgebra $(FA, F\eta_A)$, hence $K^{em}RA = (KFA, KF\eta_A) = HA$.

Once this is established on objects, the property on morphisms is very easy, as an S-coalgebra morphism coincides with its support, that is its image under F^S. So for $f \in \mathbb{S}(A, B)$ one has $F^S Hf = KFf = KF^T Rf = KRf$, that is $Hf = K^{em}RF$.

Uniqueness follows immediately from Lemma 1.28. □

1.4.4 The categories sLin and SMAL

We now specialise the definitions of Cmad and Adj to obtain respectively a category sLin of linear categories and *strong* linear functors and a category SMAL of symmetric monoidal adjunctions for linear logic.

Definition 1.31. Given two linear categories $(\mathbb{S}, \otimes, I, \multimap)$ with linear exponential comonad $(T, m^T, m^T_I, \varepsilon, \delta)$ and $\mathbb{S}' = (\mathbb{S}', \otimes', I', \multimap')$ with linear exponential comonad $(T', m^{T'}, m^{T'}_I, \varepsilon', \delta')$, a functor $K \colon \mathbb{S} \to \mathbb{S}'$ is a *strong* linear functor if

1. K is strong symmetric monoidal closed,

2. $KT = T'K$ as monoidal functors,

3. $K\varepsilon = \varepsilon'_K$, and

4. $K\delta = \delta'_K$.

Linear categories and strong linear functors form a category sLin with obvious identity and composition.

Definition 1.32. A *morphism* between two symmetric monoidal adjunctions for linear logic $(F \dashv G \colon \mathbb{C} \to \mathbb{S})$, with unit and counit (η, ε), and $(F' \dashv G' \colon \mathbb{C}' \to \mathbb{S}')$, with unit and counit (η', ε'), is a pair of functors (K, L) where $K \colon \mathbb{S} \to \mathbb{S}'$ and $L \colon \mathbb{C} \to \mathbb{C}'$ such that

1. K is strong symmetric monoidal closed,
2. L preserves products and terminal objects,
3. $KF = F'L$ and $LG = G'K$ as monoidal functors, and
4. $K\varepsilon = \varepsilon'_K$ or, equivalently, $L\eta = \eta'_L$.

Symmetric monoidal adjunctions for linear logic with the given morphisms form a category SMAL with obvious identity and composition.

With $KF = F'L$ and $LG = G'K$ as monoidal functors, we mean that $Km^F \circ m^K{}_F = F'm^L \circ m^{F'}{}_L$ and $Km^F_I \circ m^K_I = F'm^L_I \circ m^{F'}_I$, and similarly for G.

Let $K \colon (\mathbb{S}, T) \to (\mathbb{S}', T')$ be a strong linear functor. We know that K^{em}, as a strong symmetric monoidal functor between categories with chosen finite products, preserves those finite products up to coherent isomorphisms. Therefore, it preserves products and terminal objects. This means that (K, K^{em}) is a SMAL-functor between $(F^T \dashv G^T \colon \mathbb{S}^T \to \mathbb{S})$ and $(F^{T'} \dashv G^{T'} \colon \mathbb{S}'^{T'} \to \mathbb{S}')$.

1.4.5 How the categories sLin and SMAL relate

The category sLin is a subcategory of Cmad, while SMAL a subcategory of Adj. Let $\mathsf{SMAL}_{\mathsf{em}}$ be the intersection of SMAL with $\mathsf{Adj}_{\mathsf{em}}$. We obtain:

Proposition 1.33. *The category* sLin *is isomorphic to the full subcategory* $\mathsf{SMAL}_{\mathsf{em}}$ *of* SMAL.

Proof. The isomorphism of Cmad and $\mathsf{Adj}_{\mathsf{em}}$ restricts to an isomorphism of sLin and $\mathsf{SMAL}_{\mathsf{em}}$: If $(\mathbb{S}, \otimes, I)$ is a linear category with a linear exponential comonad $(T, m^T, m^T_I, \varepsilon, \delta)$, then the associated adjunction $F^T \dashv G^T$ with the category $\mathbb{S}^T$ of coalgebras is a symmetric monoidal adjunction (Proposition 1.6). Moreover the symmetric monoidal structure inherited by $\mathbb{S}^T$ is a finite product structure (Proposition 1.16). The adjunction $F^T \dashv G^T$ is therefore a symmetric monoidal adjunction for linear logic. If K is a strong linear functor then (K, K^{em}) is a map in the category $\mathsf{SMAL}_{\mathsf{em}}$.

Conversely, as we observed in Section 1.3.1, an adjunction $F^T \dashv G^T \colon \mathbb{S}^T \to \mathbb{S}$ corresponds to a unique linear category structure on $\mathbb{S}$. ☐

Proposition 1.34. *The category* $\mathsf{SMAL_{em}}$ *is a full reflective subcategory of* SMAL.

Proof. We prove that R is a strong monoidal functor, via m^F and m^F_I. For any $A \in |\mathbb{A}|$ we have $RA = (F\eta_A \colon FA \to FGFA)$. Moreover, for $A, B \in |\mathbb{C}|$ the morphisms $m^F_{A,B}$ and m^F_I are coalgebra morphisms as this corresponds to the fact that $F\eta$ is a monoidal natural transformation:

$$
\begin{tikzcd}
FA \otimes FB \arrow[r, "m^F_{A,B}"] \arrow[d, "F\eta_A \otimes F\eta_B"'] & F(A \times B) \arrow[dd, "F\eta_{A\times B}"] \\
TFA \otimes TFB \arrow[d, "m^T_{FA,FB}"'] & \\
T(FA \otimes FB) \arrow[r, "Tm^F_{A,B}"'] & TF(A \times B)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "m^F_I"] \arrow[d, "m^T_I"'] & F1 \arrow[d, "F\eta_1"] \\
TI \arrow[r, "Tm^F_I"] & TF1
\end{tikzcd}
$$

This means that for any SMAL object $F \dashv G \colon \mathbb{C} \to \mathbb{S}$ the comparison functor $R \colon \mathbb{C} \to \mathbb{S}^T$ (where $T = FG$) is such that $(\mathrm{Id}_{\mathbb{S}}, R)$ is a SMAL morphism. That is, the result of Proposition 1.30 restricts to SMAL and $\mathsf{SMAL_{em}}$. ☐

Notice that in any symmetric monoidal adjunction $F \dashv G \colon \mathbb{C} \to \mathbb{S}$ the left adjoint F is *strong* symmetric monoidal, but in general not *strict*. The symmetric monoidal structure of the comparison functor coincides with that of F. This means that this result does not hold if we require strict symmetric monoidal functors as morphisms between symmetric monoidal adjunctions.

Chapter 2

Lax limits and lax bilimits

We look at a simple construction that yields a lax limit for functors from a small category into certain subcategories of the category Pos of partial orders and order preserving maps. Just as limits provide a functor that is right adjoint to the diagonal [31, Section X.1], this lax limit construction gives a right adjoint to the diagonal into the category of functors and *lax transformations* (Definition 2.7 and Proposition 2.9).

It is nice to see that this adjunction is what allows the modelling universal quantification, both for *System* F in [15] and for $\mathrm{F}_{\mathrm{DILL}}$ here, as explained in Chapter 3.

For (small) diagrams into the category Sup of complete lattices with maps preserving arbitrary suprema the universal lax cone consists of upper adjoint maps whose lower adjoint combine to form a lax colimit.

This leads to the topic of lax bilimits. We show how the lax bilimit construction also carries over to diagrams in an arbitrary Sup-enriched category, provided a lax limit exists. This is done with an approach similar to the categorical analysis of the limit-colimit coincidence of [39].

Then, we give a characterisation of those lax cones-cocones of adjoint pairs that provide a lax bilimit for a diagram in a Sup-enriched category and discuss some properties of lax bilimits (that will be useful in Chapter 3): Preservation of lax bilimits by Sup-enriched functors and interaction with continuity-like conditions.

Finally, we extend the construction and characterisation of lax bilimits to a larger class of diagrams into a $\mathbb{Z}$-enriched category, where $\mathbb{Z}$ is a category of posets which have suprema for certain subsets (e.g. Dcpo, the category of directed complete partial orders and maps preserving directed suprema).

This last result permits us to show how the usual limit-colimit coincidence [44, 49, 50, 39] (in the general case of a directed diagram in the category of adjoint pairs of a Dcpo-category [3]) can be seen as a particular case of lax bilimit.

2.1 Lax limits

Let $\mathbb{I}$ be a small category and consider a diagram of shape $\mathbb{I}$ in Pos, that is a functor:

$$\mathbb{A} \colon \mathbb{I} \to \mathsf{Pos}$$

The lax limit for such a diagram in Pos can be described in terms of *lax threads*, in the same way that a limit in the category of sets and functions can be described in terms of threads for the diagram (the set of threads for a diagram corresponds to the construction of a limit via products and equalisers).

Definition 2.1. We define the set of **lax threads** for $\mathbb{A}$:

$$A = \left\{ \langle x_i \rangle \in \prod_{i \in |\mathbb{I}|} \mathbb{A}i \;\middle|\; \mathbb{A}h(x_i) \leq x_j \text{ for all } i, j \in |\mathbb{I}| \text{ and } h \in \mathbb{I}(i,j) \right\}$$

Point-wise comparison makes A into a poset and the obvious projections $p_i \colon A \to \mathbb{A}i$ extracting the i-th components are order preserving maps.

When the diagram takes values in certain subcategories $\mathbb{X}$ of Pos— such as the category Dcpo of directed complete orders and Scott-continuous functions, or the category Sup of complete lattices and maps that preserve arbitrary suprema — A inherits the structure of an $\mathbb{X}$-object point-wise and the projections p_i are $\mathbb{X}$-morphisms.

The definitions that follow make sense in the general context of a Pos-enriched category (for enriched categories refer to Appendix B) and are used later in this context.

Definition 2.2. A **lax cone** for a diagram $\mathbb{A} \colon \mathbb{I} \to \mathbb{X}$ into a Pos-enriched category $\mathbb{X}$ is a family of arrows $\{a_i \colon A \to \mathbb{A}i\}_{i \in |\mathbb{I}|}$ such that for any $h \colon i \to j$ in $\mathbb{I}$:

$$\begin{tikzcd}
 & A \arrow[dl, "a_i"'] \arrow[dr, "a_j"] \arrow[d, phantom, "\leq"] & \\
\mathbb{A}i \arrow[rr, "\mathbb{A}h"'] & {} & \mathbb{A}j
\end{tikzcd}$$

Definition 2.3. Let $\{u_k \colon A \to B_k\}_{k \in K}$ be a family of maps indexed by some class K in a Pos-enriched category $\mathbb{X}$. We say that the family $\{u_k\}_{k \in K}$ is **jointly order reflecting** if for any pair of parallel maps $f, g \colon C \to A$ in $\mathbb{X}$ one has:

$$(u_k \circ f \leq u_k \circ g \quad \text{for all} \quad k \in K) \quad \Rightarrow \quad f \leq g$$

Similarly, a family of maps $\{v_k \colon B_k \to A\}_{k \in K}$ is jointly order reflecting if for any pair of parallel maps $m, n \colon A \to C$ in $\mathbb{X}$ one has:

$$(m \circ v_k \leq n \circ v_k \quad \text{for all} \quad k \in K) \quad \Rightarrow \quad m \leq n$$

Definition 2.4. A **lax limit** for $\mathbb{A}\colon \mathbb{I} \to \mathbb{X}$ is a universal lax cone $\{a_i\colon A \to \mathbb{A}i\}_{i\in|\mathbb{I}|}$, in the sense that for any other lax cone $\{b_i\colon B \to \mathbb{A}i\}_{i\in|\mathbb{I}|}$ there exists a unique morphism $m\colon B \to A$ such that $a_i \circ m = b_i$ for any $i \in |\mathbb{I}|$. In addition we require that the universal lax cone is jointly order reflecting[1].

Remark. Let $\mathbb{A}\colon \mathbb{I} \to \mathbb{X}$ be a small diagram in a subcategory $\mathbb{X}$ of Pos. The family of projection from the set of lax threads

$$\{p_i\colon A \to \mathbb{A}i\}_{i\in|\mathbb{I}|}$$

which, as observed above, constitute a lax cone over $\mathbb{A}$, obviously enjoys the property of being jointly order reflecting.

Proposition 2.5. *The set of lax threads for a small diagram $\mathbb{A}\colon \mathbb{I} \to \mathsf{Pos}$ (respectively Dcpo, Sup) together with the projections $p_i\colon A \to \mathbb{A}i$ is a lax limit for $\mathbb{A}$ in Pos (Dcpo, Sup).*

Proof. By construction, the projections p_i form a lax cone for $\mathbb{I}$. Assume $\{b_i\colon B \to \mathbb{A}i\}_{i\in|\mathbb{I}|}$ is another lax cone for $\mathbb{A}$ (that is, $\mathbb{A}h \circ b_i \leq b_j$ for any $h\colon i \to j$ in $\mathbb{I}$). Define a map $m\colon B \to A$ by $mx = \langle b_i x\rangle$ for any $x \in B$. The map m takes values in A — since $\{b_i\}_{i\in|\mathbb{I}|}$ is a lax cone for $\mathbb{A}$ — and it is obviously order preserving.

By construction of m, we have that $p_i \circ m = b_i$ for any $i \in |\mathbb{I}|$. If $n\colon B \to A$ is another map such that $p_i \circ n = b_i$ for any $i \in |\mathbb{I}|$, then it coincides necessarily with f.

When $\{b_i\colon B \to \mathbb{A}i\}_{i\in|\mathbb{I}|}$ is a lax cone in Dcpo (Sup), the mediating function m defined above is obviously a map that preserves directed (arbitrary) suprema, since every b_i does and suprema in A are taken point-wise. That is, $\{p_i\colon B \to \mathbb{A}i\}_{i\in|\mathbb{I}|}$ is also a universal cone for a diagram $\mathbb{A}$ taking values in Dcpo (Sup). ☐

Corollary 2.6. *The categories Pos, Dcpo and Sup have small lax limits. The lax threads construction of Definition 2.1 provides (chosen) lax limits.*

2.1.1 Lax limit as right adjoint to the diagonal

If a category $\mathbb{X}$ has $\mathbb{I}$-limits, the diagonal functor into the category of functors and natural transformations

$$\mathrm{diag}_{\mathbb{I}}\colon \mathbb{X} \to \mathrm{Funct}(\mathbb{I}, \mathbb{X})$$

which assigns to any object the constant functor at that object, has a right adjoint

$$\lim_{\mathbb{I}}\colon \mathrm{Funct}(\mathbb{I}, \mathbb{X}) \to \mathbb{X}$$

The same happens for a Pos-enriched category $\mathbb{X}$ that has lax limits of shape $\mathbb{I}$, if we change the category of functors accordingly.

[1] The meaning of this assumption will become clear after having introduced the notion of *lax transformations* (Definition 2.7).

Definition 2.7. Let $F, G\colon \mathbb{I} \to \mathbb{X}$ be functors from a category $\mathbb{I}$ to a Pos-enriched category $\mathbb{X}$. A **lax transformation**[2] $s\colon F \overset{\odot}{\to} G$, is a family $\{s_i\colon Fi \to Gi\}_{i \in |\mathbb{I}|}$ of maps in $\mathbb{X}$ such that for any $h\colon i \to j$ in $\mathbb{I}$:

$$\begin{tikzcd} Fi \arrow[r, "s_i"] \arrow[d, "Fh"'] \arrow[dr, phantom, "\geq"] & Gi \arrow[d, "Gh"] \\ Fj \arrow[r, "s_j"'] & Gj \end{tikzcd}$$

Obviously a natural transformation $s\colon F \overset{\bullet}{\to} G$ is in particular a lax transformation.

Definition 2.8. Let $\mathbb{I}$ be a category and $\mathbb{X}$ a Pos-enriched category. Define the category $\mathbb{X}^{\mathrm{lax}\,\mathbb{I}}$ to be the category whose objects are functors $\mathbb{I} \to \mathbb{X}$ and morphisms are lax transformations between such functors. Composition is component-wise: If $s\colon F \overset{\odot}{\to} G$ and $t\colon G \overset{\odot}{\to} H$ are lax transformations, the composite $t \bullet s\colon F \overset{\odot}{\to} H$ is given by the family $(t \bullet s)_i := t_i \circ s_i$ for any $i \in |I|$.

Remark. If $\mathbb{I}$ is not small, there is a *class* of lax transformations $F \overset{\odot}{\to} G$ for any pair of functors $F, G\colon \mathbb{I} \to \mathbb{X}$, that is, $\mathbb{X}^{\mathrm{lax}\,\mathbb{I}}(F, G)$ is not a set. However, component-wise comparison provides an obvious order between lax transformations in $\mathbb{X}^{\mathrm{lax}\,\mathbb{I}}(F, G)$ and composition in $\mathbb{X}^{\mathrm{lax}\,\mathbb{I}}$ preserves this order. So — apart from set-theoretic problems of size — $\mathbb{X}^{\mathrm{lax}\,\mathbb{I}}$ is again order enriched. Therefore, in the following we are also going to call such categories Pos-enriched; from the standpoint of foundations this can be justified by assuming a sufficient number of Grothendieck universes.

Proposition 2.9. *Let $\mathbb{I}$ be a category and $\mathbb{X}$ a Pos-enriched category. If $\mathbb{X}$ has (given) lax limits of shape $\mathbb{I}$, the assignment of the vertex of the given universal lax cone to a functor $\mathbb{I} \to \mathbb{X}$ is the object part of a functor*

$$\mathrm{laxlim}_{\mathbb{I}}\colon \mathbb{X}^{\mathrm{lax}\,\mathbb{I}} \to \mathbb{X}$$

which is right adjoint to the diagonal:

$$\mathrm{diag}_{\mathbb{I}}\colon \mathbb{X} \to \mathbb{X}^{\mathrm{lax}\,\mathbb{I}}$$

Proof. Functoriality of $\mathrm{laxlim}_{\mathbb{I}}$ follows from the fact that the image of an arrow is defined by the universal property of lax limiting cones. The fact that it is a right adjoint to the diagonal functor is straightforward given the definition of lax limit. $\square$

[2] This is a particular case of what is called *left exact transformation* in [48].

This adjunction is essentially what is used to model second order universal quantification both for the model of *System* F based on Scott-domains of [15] (even if in the exposition there this is not apparent) and in the model of F_{DILL} described in Chapter 3.

On the definition of lax limit. A lax cone

$$\{a_i \colon A \to \mathbb{A}i\}_{i \in |\mathbb{I}|}$$

over a diagram $\mathbb{A} \colon \mathbb{I} \to \mathbb{X}$ corresponds exactly to a lax transformation $\mathrm{diag}_{\mathbb{I}}\, A \overset{\odot}{\to} \mathbb{A}$, and for any $B \in |\mathbb{X}|$ it determines a morphism

$$\mathbb{X}(B, A) \to \mathbb{X}^{\mathrm{lax}\,\mathbb{I}}\left(\mathrm{diag}_{\mathbb{I}}\, B, \mathbb{A}\right)$$

by composition; this is obviously natural in B and order preserving. The lax cone $\{a_i\}_{i \in |\mathbb{I}|}$ is universal for $\mathbb{A}$ exactly when this is a natural isomorphism. The condition that each

$$\mathbb{X}(B, A) \to \mathbb{X}^{\mathrm{lax}\,\mathbb{I}}(\mathrm{diag}_{\mathbb{I}}\, B, \mathbb{A})$$

is an *order* isomorphism amounts to the further requirement that $\{a_i\}_{i \in |\mathbb{I}|}$ is jointly order reflecting. Therefore:

Lemma 2.10. *Let $\mathbb{X}$ be a* Pos*-enriched category and $\mathbb{I}$ a category. Then* $\mathrm{laxlim}_{\mathbb{I}} \colon \mathbb{X}^{\mathrm{lax}\,\mathbb{I}} \to \mathbb{X}$*, provided it exists, is a* Pos*-enriched functor and therefore* $\mathrm{diag}_{\mathbb{I}} \dashv \mathrm{laxlim}_{\mathbb{I}}$ *is a* Pos*-adjunction.*

Proof. A limiting lax cone is required to be jointly order reflecting. This readily implies that, when it exists, the functor $\mathrm{laxlim}_{\mathbb{I}}$ preserves the component-wise order between lax transformations. The diagonal functor is obviously a Pos-functor. For any functor $F \colon \mathbb{I} \to \mathbb{X}$ and any $B \in |\mathbb{B}|$ we therefore have Pos-isomorphisms

$$\mathbb{X}^{\mathrm{lax}\,\mathbb{I}}(\mathrm{diag}_{\mathbb{I}}\, B, F) \cong \mathbb{X}(B, \mathrm{laxlim}_{\mathbb{I}}\, F)$$

natural in F and B. □

Also for the category Dcpo we obtain a similar result:

Lemma 2.11. *Let $\mathbb{X}$ be a* Dcpo*-enriched category and $\mathbb{I}$ a category. Then the category $\mathbb{X}^{\mathrm{lax}\,\mathbb{I}}$ is also* Dcpo*-enriched. Moreover, when it exists, the functor* $\mathrm{laxlim}_{\mathbb{I}}$ *is* Dcpo*-enriched.*

Proof. Composition in $\mathbb{X}$ preserves directed suprema, therefore the component-wise supremum of a directed family of lax transformations is also a lax transformation. Moreover, composition of lax transformations preserves these suprema. That is to say, $\mathbb{X}^{\mathrm{lax}\,\mathbb{I}}$ is Dcpo-enriched.

Next we show that, whenever $\mathrm{laxlim}_{\mathbb{I}}$ exists it is a Dcpo-enriched functor. Let $F, G\colon \mathbb{I} \to \mathbb{X}$ be two functors and consider a directed family of lax transformations $\{s_k\colon F \overset{\odot}{\to} G\}_{k\in K}$ and let $s := \bigvee^{\uparrow}_{k\in K} s_k$. Since $\mathrm{laxlim}_{\mathbb{I}}$ is Pos-enriched (Lemma 2.10), the family of $\mathbb{X}$-morphisms $\{\mathrm{laxlim}_{\mathbb{I}}\, s_k\}_{k\in K}$ is also directed and $\bigvee^{\uparrow}_{k\in K}(\mathrm{laxlim}_{\mathbb{I}}\, s_k) \leq \mathrm{laxlim}_{\mathbb{I}}\, s$. If $\{p^F_i\colon \mathrm{laxlim}_{\mathbb{I}}\, F \to Fi\}_{i\in|\mathbb{I}|}$ and $\{p^G_i\colon \mathrm{laxlim}_{\mathbb{I}}\, G \to Gi\}_{i\in|\mathbb{I}|}$ are a lax limit over F and G, respectively, we have

$$p^G_i \circ \Big(\bigvee^{\uparrow}_{k\in K} \mathrm{laxlim}_{\mathbb{I}}\, s_k\Big) = \bigvee^{\uparrow}_{k\in K} \Big(p^G_i \circ \mathrm{laxlim}_{\mathbb{I}}\, s_k\Big) =$$

$$= \bigvee^{\uparrow}_{k\in K} \Big((s_k)_i \circ p^F_i\Big) = s_i \circ p^F_i = p^G_i \circ \mathrm{laxlim}_{\mathbb{I}}\, s$$

for any $i \in |\mathbb{I}|$, where the second and fourth equality use the property of $\mathrm{laxlim}_{\mathbb{I}}\, s_i$ and, respectively, $\mathrm{laxlim}_{\mathbb{I}}\, s$ of being the mediating morphism into a lax limit. Since the family $\{p^G_i\}_{i\in|\mathbb{I}|}$ is jointly order reflecting, we conclude that $\bigvee^{\uparrow}_{k\in K}(\mathrm{laxlim}_{\mathbb{I}}\, s_k) = \mathrm{laxlim}_{\mathbb{I}}\, s$. □

2.2 Lax bilimits in Sup

For a diagram $\mathbb{A}$ of shape $\mathbb{I}$ in Sup, the above construction of a lax limit via lax threads is still valid, as already pointed out. What happens when the diagram $\mathbb{A}$ takes values in Sup is that this construction gives also a (jointly order reflecting) universal lax cocone for $\mathbb{A}$. The universal lax cone and cocone combine in a precise way, which we discuss in Section 2.3 after giving the main results.

Definition 2.12. A **lax cocone**[3] for a diagram $\mathbb{A}\colon \mathbb{I} \to \mathbb{X}$ into a Pos-enriched category $\mathbb{X}$ is a family of arrows $\{f_i\colon \mathbb{A}i \to A\}_{i\in|\mathbb{I}|}$ such that for any $h\colon i \to j$ in $\mathbb{I}$:

$$\begin{tikzcd} & A \arrow[d, phantom, "\geq"] & \\ \mathbb{A}i \arrow[ur, "f_i"] \arrow[rr, "\mathbb{A}h"'] & {} & \mathbb{A}j \arrow[ul, "f_j"'] \end{tikzcd}$$

Definition 2.13. A **lax colimit** for $\mathbb{A}\colon \mathbb{I} \to \mathbb{X}$ is a universal lax cocone $\{f_i\colon \mathbb{A}i \to A\}_{i\in|\mathbb{I}|}$, in the sense that for any other lax cocone $\{g_i\colon \mathbb{A}i \to B\}_{i\in|\mathbb{I}|}$ there exists a unique morphism $m\colon A \to B$ such that $m \circ f_i = g_i$ for any $i \in |\mathbb{I}|$. In addition we require the universal cocone to be jointly order reflecting.

[3] If a lax cone corresponds to a *left* lax transformation, in the words of [48], a lax cocone corresponds to a *right* lax transformation.

Proposition 2.14 (Lax bilimits in Sup). *Let* $\mathbb{A}\colon \mathbb{I} \to \mathsf{Sup}$ *be a small diagram,* A *the set of lax threads for* $\mathbb{A}$ *and* $\{p_i\colon A \to \mathbb{A}i\}_{i\in|\mathbb{I}|}$ *the family of projections onto the* i*th-components. Then*

1. *the maps* $\{p_i\}_{i\in|\mathbb{I}|}$ *— that as seen above constitute a universal lax cone for* $\mathbb{A}$ *— are upper adjoint maps,*
2. *the corresponding lower adjoint maps* $\{e_i\colon \mathbb{A}i \to A\}_{i\in|\mathbb{I}|}$ *form a lax colimit for* $\mathbb{A}$*,*
3. $\bigvee_{i\in|\mathbb{I}|} e_i \circ p_i = \mathrm{id}_A$*, and*
4. $p_j \circ e_i = \bigvee_{h\colon i\to j} \mathbb{A}h$*, for any* i *and* j *in* $|\mathbb{I}|$*.*

Proof. To begin with, we obtain a candidate lower adjoint e_k to p_k for any $k \in |\mathbb{I}|$ and establish (4). Only after proving (1) and (3) we are able to show (2).

For any index $k \in |\mathbb{I}|$ consider the maps $a_i^k\colon \mathbb{A}k \to \mathbb{A}i$ defined by

$$a_i^k = \bigvee_{h\colon k\to i} \mathbb{A}h.$$

Since composition in Sup preserves arbitrary suprema, these maps form a lax cone for $\mathbb{A}$: We have

$$\begin{tikzcd} & \mathbb{A}k \arrow[dl, "a_i^k"'] \arrow[dr, "a_j^k"] \arrow[d, phantom, "\leq"] & \\ \mathbb{A}i \arrow[rr, "\mathbb{A}h"'] & {} & \mathbb{A}j \end{tikzcd}$$

for any $h\colon i \to j$ in $\mathbb{I}$. Hence for any $k \in |\mathbb{I}|$ there exists a unique map $e_k\colon \mathbb{A}k \to A$ in Sup such that

$$p_i \circ e_k = a_i^k = \bigvee_{h\colon k\to i} \mathbb{A}h.$$

This provides the equality in (4).

(1) To see that the map e_i is lower adjoint to p_i for any $i \in |\mathbb{I}|$ notice that $p_i \circ e_i = \bigvee_{h\colon i\to i} \mathbb{A}h \geq \mathrm{id}_{\mathbb{A}i}$. Furthermore, using the fact that composition in Sup preserves arbitrary suprema and that $\{p_i\}_{i\in|\mathbb{I}|}$ is a lax cone for $\mathbb{A}$, we see that for any index j

$$p_j \circ e_i \circ p_i = \bigvee_{h\colon i\to j} \mathbb{A}h \circ p_i \leq p_j.$$

The family $\{p_i\}_{i\in|\mathbb{I}|}$ is jointly order reflecting, hence the above inequality implies

$$e_i \circ p_i \leq \mathrm{id}_A\,.$$

(3) From the adjunction $e_i \dashv p_i$ we have that $\bigvee_{i\in|\mathbb{I}|} e_i \circ p_i \leq \mathrm{id}_A$. Moreover, for any index j we use the other adjunction inequality to get $p_j \circ \bigvee_{i\in|\mathbb{I}|} e_i \circ p_i \geq p_j \circ e_j \circ p_j \geq p_j$ and therefore also $\bigvee_{i\in|\mathbb{I}|} e_i \circ p_i \geq \mathrm{id}_A$, that is

$$\bigvee_{i\in|\mathbb{I}|} e_i \circ p_i = \mathrm{id}_A\,.$$

(2) We now show that $\{e_i \colon \mathbb{A}i \to A\}_{i\in|\mathbb{I}|}$ is a lax cocone for $\mathbb{A}$. For any $h \colon i \to j$ in $\mathbb{I}$ and any $k \in |\mathbb{I}|$ we have

$$p_k \circ e_j \circ \mathbb{A}h = \bigvee_{g\colon j\to k} \mathbb{A}g \circ \mathbb{A}h \leq \bigvee_{h\colon i\to k} \mathbb{A}h = p_k \circ e_i$$

and therefore (using again that family of projections is jointly order reflecting) $e_j \circ \mathbb{A}h \leq e_i$ as required.

To see that $\{e_i \colon \mathbb{A}i \to A\}_{i\in|\mathbb{I}|}$ is a lax colimit for $\mathbb{A}$ consider another lax cocone $\{c_i \colon \mathbb{A}i \to C\}_{i\in|\mathbb{I}|}$. We can define a morphism $g \colon A \to C$ by taking

$$g = \bigvee_{i\in|\mathbb{I}|} c_i \circ p_i.$$

Then we have that for any $i \in |\mathbb{I}|$

$$g \circ e_i = (\bigvee_{j\in|\mathbb{I}|} c_j \circ p_j) \circ e_i = \bigvee_{j\in|\mathbb{I}|} c_j \circ (p_j \circ e_i) = \bigvee_{j\in|\mathbb{I}|} c_j \circ (\bigvee_{h\colon i\to j} \mathbb{A}h) \leq c_i$$

using the explicit description of $p_j \circ e_i$ and the fact that $\{c_i\}_{i\in|\mathbb{I}|}$ is a lax cocone for $\mathbb{A}$. But it is also true that $g \circ e_i \geq c_i \circ p_i \circ e_i \geq c_i$, therefore $g \circ e_i = c_i$ and this for any $i \in |\mathbb{I}|$.

Let $m \colon A \to C$ be another map such that $m \circ e_i = c_i$ for any $i \in |\mathbb{I}|$. Using (3) we obtain $m = m \circ \bigvee_{i\in|\mathbb{I}|} e_i \circ p_i = \bigvee_{i\in|\mathbb{I}|} m \circ e_i \circ p_i = \bigvee_{i\in|\mathbb{I}|} c_i \circ p_i = g$ as required. The fact that the cocone $\{e_i \colon \mathbb{A}i \to A\}_{i\in|\mathbb{I}|}$ is jointly order reflecting follows from (3). $\square$

Remark. In the above proof, we only use the properties of the family $\{p_i\}_{i\in|\mathbb{I}|}$ of projections that it forms a universal lax cone over $\mathbb{A}$ and that it is a family of jointly order reflecting morphisms. Therefore, this kind of combination of lax limit and lax colimit can be studied in the more general context of Sup-enriched categories. For this reason we carry on the analysis in this case. However, the lax thread construction is only possible for a small diagram into Sup. For an arbitrary diagram in a Sup-enriched category we need to assume the existence of a lax limit.

2.3 Lax bilimits in a Sup-enriched category

Here we see how, under certain requirements, the construction of lax bilimits given in the section above for a diagram in Sup can be repeated for a diagram $\mathbb{A}\colon \mathbb{I} \to \mathbb{X}$ where $\mathbb{X}$ is a Sup-enriched category.

To be able to do so, since this time we do not have the possibility to consider the set of lax threads for $\mathbb{A}$, we need to assume that the the lax limit of the diagram exists in $\mathbb{X}$.

Theorem 2.15 (Lax bilimits in a Sup-enriched category). *Let $\mathbb{A}\colon \mathbb{I} \to \mathbb{X}$ be an arbitrary diagram, where $\mathbb{X}$ is a Sup-enriched category. Assume that $\{p_i\colon A \to \mathbb{A}i\}_{i\in|\mathbb{I}|}$ is a lax limit for $\mathbb{A}$. Then*

1. *the arrows $\{p_i\}_{i\in|\mathbb{I}|}$ that constitute a universal lax cone for $\mathbb{A}$ are upper adjoint maps,*
2. *the corresponding lower adjoint maps $\{e_i\colon \mathbb{A}i \to A\}_{i\in|\mathbb{I}|}$ form a lax colimit for $\mathbb{A}$,*
3. *$\bigvee_{i\in|\mathbb{I}|} e_i \circ p_i = \mathrm{id}_A$, and*
4. *$p_j \circ e_i = \bigvee_{h\colon i\to j} \mathbb{A}h$ for any i and j in $|\mathbb{I}|$.*

Proof. The proof of Proposition 2.14 has been structured in such a way that it is now easy to repeat the various steps adapting them to the assumptions given here. What cannot be given is the set of lax threads and for this reason we assume the existence of the lax limit. All the rest of the proof carries over. ☐

Summing up, if a diagram in a Sup-enriched category has a lax limit it has also a lax colimit, given by the lower adjoint maps to the universal lax cone. For this reason we used — so far informally — the word lax bilimit.

Definition 2.16. Let $\mathbb{A}\colon \mathbb{I} \to \mathbb{X}$ be a diagram into a Pos-enriched category $\mathbb{X}$. We say that $\mathbb{X}$ has a **lax bilimit** in $\mathbb{X}$ when there are a lax limit $\{f_i\colon A \to \mathbb{A}_i\}_{i\in|\mathbb{I}|}$ and a lax colimit $\{g_i\colon \mathbb{A}i \to A\}_{i\in|\mathbb{I}|}$ with $g_i \dashv f_i$ for any $i \in |\mathbb{I}|$.

Remark. There is a relevant difference to the meaning of the word *bilimit* as it is normally used in domain theory: Here beside laxness, a lax limit and a lax colimit coexist over the *same diagram* $\mathbb{A}$, while in the usual limit-colimit coincidence one has a lax colimit on a diagram of embeddings together with a lax limit on the corresponding diagram of projections. In Section 2.5 we show how the limit-colimit coincidence of domain theory is nevertheless a particular case of lax bilimit.

Remark. Consider a diagram $\mathbb{A}\colon \mathbb{I} \to \mathbb{X}$ in a Pos-enriched category $\mathbb{X}$. Suppose that there is a family of adjoint pairs $\{l_k \dashv u_k\colon B \to A\}_{k\in K}$ indexed by a class K such that the family $\{u_k\}_{k\in K}$ is jointly order reflecting. Then the supremum $\bigvee_{k\in K}(l_k \circ u_k)$ exists in $\mathbb{X}(A, A)$ and equals id_A: The identity is clearly an upper bound for all $l_k \circ u_k$. If f is another upper bound, then we have

$$u_k \circ f \geq u_k \circ l_k \circ u_k = u_k$$

for any $k \in K$ which implies $f \geq \mathrm{id}_A$. Hence, id_A is the least upper bound of the family $\{l_k \circ u_k\}_{k \in K}$. On the other hand, if $\bigvee_{k \in K}(l_k \circ u_k)$ exists and equals id_A, we have that the families $\{u_k\}_{k \in K}$ and $\{l_k\}_{k \in K}$ are both jointly order reflecting.

Next, we show that in a Sup-enriched category properties (3) and (4) of Theorem 2.15 characterise the lax bilimit among lax cones and cocones of adjoint pairs.

Proposition 2.17 (Characterisation of lax bilimits). *Let $\mathbb{A}\colon \mathbb{I} \to \mathbb{X}$ be an arbitrary diagram, where $\mathbb{X}$ is a Sup-enriched category. Assume that*

- *$\{p_i\colon A \to \mathbb{A}i\}_{i \in |\mathbb{I}|}$ is a lax cone for $\mathbb{A}$ and $e_i \dashv p_i$ (and therefore $\{e_i\colon \mathbb{A}i \to A\}_{i \in |\mathbb{I}|}$) is a lax cocone for $\mathbb{A}$), or, equivalently, that*
- *$\{e_i\colon \mathbb{A}i \to A\}_{i \in |\mathbb{I}|}$) is a lax cocone for $\mathbb{A}$ and $e_i \dashv p_i$ (and therefore $\{p_i\colon \mathbb{A} \to \mathbb{A}i\}_{i \in |\mathbb{I}|}$) is a lax cone for $\mathbb{A}$).*

Then $\{p_i\}_{i \in |\mathbb{I}|}$ is a lax limit for $\mathbb{A}$ and $\{e_i\}_{i \in |\mathbb{I}|}$ is a lax colimit if and only if the two conditions below are satisfied:

1. *$\bigvee_{i \in |\mathbb{I}|} e_i \circ p_i = \mathrm{id}_A$ and*
2. *$p_j \circ e_i = \bigvee_{h\colon i \to j} \mathbb{A}h$ for any i and j in $|\mathbb{I}|$.*

Proof. Theorem 2.15 provides the *only if* part. We show that the given conditions are also sufficient. First we prove that $\{p_i\colon A \to \mathbb{A}i\}_{i \in |\mathbb{I}|}$ is a lax limit for $\mathbb{A}$. Let $\{b_i\colon B \to \mathbb{A}i\}_{i \in |\mathbb{I}|}$ be another lax cone for $\mathbb{A}$. Define $f\colon B \to A$ to be

$$f := \bigvee_{i \in |\mathbb{I}|} e_i \circ b_i.$$

Then

$$p_i \circ f = p_i \circ \bigvee_{j \in |\mathbb{I}|} e_j \circ b_j \geq p_i \circ e_i \circ b_i \geq b_i$$

using $e_i \dashv p_i$. But we also get

$$p_i \circ f = p_i \circ \bigvee_{j \in |\mathbb{I}|} e_j \circ b_j = \bigvee_{j \in |\mathbb{I}|} p_i \circ e_j \circ b_j = \bigvee_{j \in |\mathbb{I}|} \Big(\bigvee_{h\colon j \to i} \mathbb{A}h\Big) \circ b_j \leq b_i$$

using, respectively, that composition preserve arbitrary suprema, assumption (2) and the fact that $\{b_i\}_{i \in |\mathbb{I}|}$ is lax cone. Therefore $p_i \circ f = b_i$ for any $i \in |\mathbb{I}|$. If $m\colon B \to A$

is another arrow such that $p_i \circ m = b_i$ for any $i \in |\mathbb{I}|$, using assumption (1) and preservation of suprema by composition, we see that

$$m = (\bigvee_{i \in |\mathbb{I}|} e_i \circ p_i) \circ m = \bigvee_{i \in |\mathbb{I}|} e_i \circ p_i \circ m = \bigvee_{i \in |\mathbb{I}|} e_i \circ b_i = f.$$

The lax cone $\{p_i\}_{i \in |\mathbb{I}|}$ is therefore universal. Assumption (1) insures that it is also jointly order reflecting.

To show that $\{e_i \colon \mathbb{A}i \to A\}_{i \in |\mathbb{I}|}$ is a lax colimit for $\mathbb{A}$, assume $\{c_i \colon \mathbb{A}i \to C\}_{i \in |\mathbb{I}|}$ is another lax cocone for A and take as mediating map $g \colon A \to C$ defined by

$$g = \bigvee_{i \in |\mathbb{I}|} c_i \circ p_i$$

and mirror the steps above. □

Observe that if B is the vertex of a lax cone and cocone of adjoint pairs for $\mathbb{A}$, then the comparison maps into and from the lax bilimit are adjoint:

Proposition 2.18. *Let $\mathbb{A} \colon \mathbb{I} \to \mathbb{X}$ be an arbitrary diagram — where $\mathbb{X}$ is a* Sup*-enriched category — and assume that $\mathbb{A}$ has a lax bilimit*

$$\{p_i \colon A \to \mathbb{A}i, e_i \colon \mathbb{A}i \to A\}_{i \in |\mathbb{I}|}.$$

If $\{c_i \dashv b_i\}_{i \in |\mathbb{I}|}$ are pairs of adjoint maps such that $\{b_i \colon B \to \mathbb{A}i\}_{i \in |\mathbb{I}|}$ is a lax cone for $\mathbb{A}$ and $\{c_i \colon \mathbb{A}i \to B\}_{i \in |\mathbb{I}|}$ is a lax cocone, then the comparison maps $f := \bigvee_{i \in |\mathbb{I}|} e_i \circ b_i$ and $g := \bigvee_{i \in |\mathbb{I}|} c_i \circ p_i$ given respectively by universality of $\{p_i\}_{i \in |\mathbb{I}|}$ and $\{e_i\}_{i \in |\mathbb{I}|}$ are such that $f \dashv g$.

Proof. The function f is the comparison map into the lax limit and g is the comparison map into the lax colimit as seen in the proof of Proposition 2.17.

We have

$$f \circ g = (\bigvee_{i \in |\mathbb{I}|} e_i \circ b_i) \circ (\bigvee_{i \in |\mathbb{I}|} c_i \circ p_i) = \bigvee_{i \in |\mathbb{I}|} e_i \circ b_i \circ c_i \circ p_i \geq \bigvee_{i \in |\mathbb{I}|} e_i \circ p_i = \mathrm{id}_A$$

where the equalities and inequalities above are established using, respectively, the definition, the fact that composition preserves arbitrary suprema, the property $c_i \dashv b_i$ and Proposition 2.17.(1).

Moreover,

$$g \circ f = g \circ (\bigvee_{i \in |\mathbb{I}|} e_i \circ p_i) \circ f = \bigvee_{i \in |\mathbb{I}|} g \circ e_i \circ p_i \circ f = \bigvee_{i \in |\mathbb{I}|} c_i \circ b_i \leq \mathrm{id}_B$$

using, respectively, Proposition 2.17.(1), the fact that composition preserves arbitrary suprema, the fact that g and f are mediating morphisms and the property $c_i \dashv b_i$. □

As an immediate consequence of the characterisation of lax bilimits we obtain that they are preserved by Sup-enriched functors.

Corollary 2.19. *Any* Sup*-enriched functor* $F\colon \mathbb{X} \to \mathbb{Y}$ *preserves lax bilimits.*

Proof. Let $\mathbb{A}\colon \mathbb{I} \to \mathbb{X}$ be a diagram in $\mathbb{X}$. By Proposition 2.17, a family of adjoint pairs

$$\begin{tikzcd} \mathbb{A}i \arrow[r, shift left, "e_i"] \arrow[r, phantom, "\perp"] & A \arrow[l, shift left, "p_i"] \end{tikzcd}$$

forming a lax cone and a lax cocone for $\mathbb{A}$ is a lax bilimit for $\mathbb{A}$ if and only if

$$\bigvee_{i \in |\mathbb{I}|} e_i \circ p_i = \mathrm{id}_A \qquad \text{and} \qquad p_j \circ e_i = \bigvee_{h\colon i \to j} \mathbb{A}h \quad \text{for any } i \text{ and } j \text{ in } |\mathbb{I}|\,.$$

Any Sup-enriched functor $F\colon \mathbb{X} \to \mathbb{Y}$ preserves these conditions and therefore

$$\begin{tikzcd} F\mathbb{A}i \arrow[r, shift left, "Fe_i"] \arrow[r, phantom, "\perp"] & FA \arrow[l, shift left, "Fp_i"] \end{tikzcd}$$

is a lax bilimit for $F\mathbb{A}$. $\square$

2.3.1 Lax bilimits and continuity-like conditions

Definition 2.20. Let A be a complete lattice and $x, y \in A$. We say that x is **completely below** y and write $x \lll y$, if for any subset $B \subseteq A$ such that $\bigvee B \geq y$ there exists a $b \in B$ such that $b \geq x$. The elements of A that are completely below themselves are called **completely prime**:

$$\mathsf{CP}(A) = \{a \in A \mid a \lll a\}$$

In analogy with the definitions of continuous and algebraic dcpo (which involve the order of approximation $\ll$, see [3]) there is a notion of *prime-continuous* (or *super-continuous*) and *prime-algebraic* (or *super-algebraic*) complete lattice [3, 20]:

Definition 2.21. Let A be a complete lattice.

- A is **prime-continuous** if for any $x \in A$

$$x = \bigvee \{a \in A \mid a \lll x\}.$$

- A is **prime-algebraic** if for any $x \in A$

$$x = \bigvee \{a \in \mathsf{CP}(A) \mid a \leq x\}.$$

Lemma 2.22. *Consider a pair of lower-upper adjoints $l \dashv u$ in Pos. If u preserves arbitrary suprema then l preserves the $\lll$ relation.*

Proof. Assume $x \lll x'$ in X where $l \colon X \to Y$. For any $A \subseteq Y$ such that $\bigvee A$ exists and $\bigvee A \geq lx'$ we obtain, using that u preserves arbitrary suprema and is the upper adjoint to l,

$$\bigvee(uA) = u(\bigvee A) \geq x'.$$

As $x \lll x'$, there exists an $a \in A$ such that $ua \geq x$ and hence using the adjunction $a \geq lx$. We conclude that $lx \lll lx'$. □

The property of lower adjoint maps in Sup to preserve the $\lll$ relation ensures that lax bilimits preserve prime-continuity and prime-algebraicity, as we show in the following

Proposition 2.23. *Let $\mathbb{A} \colon \mathbb{I} \to \mathsf{Sup}$ be a diagram which has a lax bilimit:*

$$\begin{tikzcd} \mathbb{A}i \arrow[r, shift left, "e_i"] \arrow[r, phantom, "\perp"] & A \arrow[l, shift left, "p_i"] \end{tikzcd}$$

If each $\mathbb{A}i$ is prime-continuous, then A is prime-continuous. If each $\mathbb{A}i$ is prime-algebraic, then A is prime-algebraic.

Proof. Since we assume that $\{e_i \dashv p_i\}_{i \in |\mathbb{I}|}$ is a lax bilimit on $\mathbb{A}$, we have $\bigvee_i e_i \circ p_i = \mathrm{id}_A$. If each $\mathbb{A}i$ is prime-continuous, we have

$$\begin{aligned} x &= \bigvee_{i \in |\mathbb{I}|} (e_i \circ p_i)x && (\mathbb{A}i \text{ prime-cont.}) \\ &= \bigvee_{i \in |\mathbb{I}|} e_i\big(\bigvee \{a \in A \mid a \lll p_i(x)\}\big) && (e_i \text{ linear}) \\ &= \bigvee_{\substack{i \in |\mathbb{I}| \\ a \lll p_i(x)}} e_i(a) \end{aligned}$$

for every $x \in A$ since $a \lll p_i(x)$ implies $e_i(a) \lll e_i(p_i(x)) \leq x$, as e_i preserves the $\lll$ relation. The complete lattice A is therefore prime-continuous. A similar argument proves that if each $\mathbb{A}i$ is prime-algebraic, also A is prime-algebraic. □

This result is used in Chapter 3 where it is applied to show that the candidate functor to model universal quantification takes values in the appropriate category.

2.3.2 Just a little example

Let aCD be the full subcategory of Sup of algebraic completely distributive lattices. This category coincides with that of prime-algebraic complete lattices:

Proposition 2.24. *A complete lattice* A *is an algebraic completely distributive lattice if and only if for any* $x \in A$ *one has*

$$x = \bigvee\{a \in \mathsf{CP}(A) \mid a \leq x\},$$

that is, A *is prime-algebraic.*

Proof. The "if" part is an immediate consequence of Raney's result on continuity and distributivity [40] (also occurring as Theorem 7.1.3 in [3]): A complete lattice is completely distributive if and only if it is prime-continuous.

The "only if" part appears in [34] as Proposition 4.2 and in [19] as Proposition 3.4 □

This property of an algebraic completely distributive lattice of being generated by its completely prime elements can be expressed in terms of lax bilimits as we now show.

Consider the category $\mathsf{CP}^{\mathsf{op}}(A)$ obtained by reversing the order of the completely prime elements of A and the diagram

$$L\colon \mathsf{CP}^{\mathsf{op}}(A) \to \mathsf{Sup}$$

defined by:

$$La \;=\; \begin{tikzcd} a \arrow[d, dash] \\ \bot \end{tikzcd} \qquad (L_{ba}\colon La \to Lb) \;=\; \begin{tikzcd} a \arrow[r, mapsto] \arrow[d, dash] & b \arrow[d, dash] \\ \bot \arrow[r, mapsto] & \bot \end{tikzcd}$$

There are adjoint pairs (which are in fact embedding projection pairs)

$$\begin{tikzcd} La \arrow[r, shift left, "e_a"] \arrow[r, phantom, "\perp"] & A \arrow[l, shift left, "p_a"] \end{tikzcd}$$

where e_a is the inclusion of La in A while p_a is obtained by mapping every element in $\{x \in A \mid x \geq a\}$ to a and everything else to $\bot$. Such a p_a preserves arbitrary suprema (and is therefore a map in Sup) exactly when a is completely prime in A.

As a consequence of the fact that completely prime elements generate A by suprema, we have that

$$\bigvee_{a \in \mathsf{CP}(A)} e_a \circ p_a = \mathrm{id}_A \,.$$

Moreover, for any $a, b \in \mathsf{CP}(A)$

$$p_b \circ e_a \;=\; \left\{ \begin{array}{ll} L_{ba} & \text{if } b \le a \\ \bot & \text{otherwise} \end{array} \right\} \;=\; \bigvee\nolimits_{b \le a} L_{ba}.$$

Obviously $\{p_a\}_{a \in \mathsf{CP}(A)}$ is a lax cone and $\{e_a\}_{a \in \mathsf{CP}(A)}$ is a lax cocone for L. Using the characterisation of lax bilimit given in Proposition 2.17 we see that A is isomorphic to the lax (bi)limit of L:

$$A \cong \Big\{ \langle x_a \rangle \in \prod_{a \in \mathsf{CP}(A)} La \;\Big|\; (x_a = a) \Rightarrow (x_b = b) \text{ for any } b \le a \text{ in } A \Big\}$$

The reader who is only interested in details relevant to the model of F_{DILL} given in Chapter 3 does not need to read the rest of this chapter.

2.4 One more generalisation

Theorem 2.15 and Proposition 2.17 — stated for Sup-enriched categories — can be proved for some diagrams in a larger class of Pos-enriched categories.

Definition 2.25. A *subset system* [51] on Pos, is a function Z which assigns to each poset P a set $Z[P]$ of subsets of P such that: (a) there exists a poset P such that $Z[P]$ contains some non-empty set; and (b) if $f : P \to Q$ is a monotone function and $S \in Z[P]$ then its direct image $f[S] := \{f(s) \mid s \in S\}$ is in $Z[Q]$.

The elements of $Z[P]$ are called the *Z-sets* of P. A *Z-poset* is a poset P such that $P \in Z[P]$.

Definition 2.26. Let Z be a subset system on Pos. A poset is *Z-complete* if each of its Z-sets has a supremum. A monotone function is *Z-continuous* if it preserves all existing suprema of Z-sets.

A category $\mathbb{Z}$ is called a *Z-category* if it is the category of all Z-complete posets and all Z-continuous functions for a subset system Z on Pos.

A possible *shape* of Z-sets is given by their cardinality. In this case, a Z-category is, for cardinality ≤ 1, the category of posets with bottom element and strict monotone functions, for finite cardinality, the category of $\vee$-semilattices and functions preserving finite suprema. Other possibilities are to consider directed sub-posets (the Z-category becomes Dcpo) or bounded subsets (the Z-category becomes BC).

Assume that $\mathbb{Z}$ is a symmetric monoidal closed Z-category and consider a $\mathbb{Z}$-enriched category $\mathbb{X}$ and a diagram $\mathbb{A}\colon \mathbb{I} \to \mathbb{X}$. If the shape of Z-sets is a matter of cardinality and the set of objects of $\mathbb{I}$ has that cardinality, the statement and the proof of Theorem 2.15 and Proposition 2.17 still make sense.

The crucial point is the possibility of taking suprema in the hom-sets over a set of functions indexed by the objects of $\mathbb{I}$ (as in point (3) of Theorem 2.15).

One notable case in which Theorem 2.15 does not apply is that of $\mathbb{Z} = \mathsf{Dcpo}$. The usual domain-theoretic limit-colimit coincidence for a directed diagram of continuous embedding-projection pairs in a Dcpo-category $\mathbb{X}$ can, however, also be obtained as a particular case of lax bilimit. To do this we need to change the original diagram. Unfortunately, in this process directedness is lost. To be able to take a supremum of the form $\bigvee_{i\in|\mathbb{I}|} e_i \circ p_i$, we need to ensure that the assignment $i \mapsto e_i \circ p_i$ still provides us with a directed set even though $\mathbb{I}$ is merely a category.

We propose a sufficient condition to impose on the shape of a diagram into a $\mathbb{Z}$-enriched category and again obtain the results on lax bilimits which works also for $\mathbb{Z} = \mathsf{Dcpo}$.

Lemma 2.27. *Let $\mathbb{A}\colon \mathbb{I} \to \mathbb{X}$ be a (small) diagram in a Pos-enriched category $\mathbb{X}$ such that $\mathbb{I}$ has a subcategory $\mathbb{J}$ full on objects which is a poset.*

1. *If $\{p_i\colon A \to \mathbb{A}i\}_{i\in|\mathbb{I}|}$ is a lax cone over $\mathbb{A}$ such that for any map $(i \leq j)$ in $\mathbb{J}$, one has $\mathbb{A}(i \leq j) \circ p_i = p_i$, then for any lax cocone $\{c_i\colon \mathbb{A}i \to C\}_{i\in|\mathbb{I}|}$ over $\mathbb{A}$, the assignment*
$$i \mapsto c_i \circ p_i$$
reverses the order in $\mathbb{J}$.

2. *Similarly, if $\{e_i\colon \mathbb{A}i \to A\}_{i\in|\mathbb{I}|}$ is a lax cocone over $\mathbb{A}$ such that for any map $(i \leq j)$ in $\mathbb{J}$, one has $e_j \circ \mathbb{A}(i \leq j) = e_i$, then for any lax cone $\{b_i\colon B \to \mathbb{A}i\}_{i\in|\mathbb{I}|}$ over $\mathbb{A}$, the assignment*
$$i \mapsto e_i \circ b_i$$
preserves the order in $\mathbb{J}$.

Proof. For any $i, j \in |\mathbb{I}|$ for which there is a map $(i \leq j)$ in $\mathbb{J}$, from $\mathbb{A}(i \leq j) \circ p_i = p_j$ one obtains immediately:
$$c_j \circ p_j = c_j \circ \mathbb{A}(i \leq j) \circ p_i \leq c_i \circ p_i$$
Similarly, from $e_i = e_j \circ \mathbb{A}(i \leq j)$ we infer
$$e_i \circ b_i = e_j \circ \mathbb{A}(i \leq j) \circ b_i \leq e_j \circ b_j.$$
$\square$

This lemma is a simple tool , though somewhat *ad hoc*, to reach our aim of giving a generalisation of the result on lax bilimits which covers the case $\mathbb{Z} = \mathsf{Dcpo}$ (and hence of justifying the usual limit-colimit coincidence in terms of lax bilimits).

Theorem 2.28. *Let $\mathbb{Z}$ be a symmetric monoidal closed Z-category and $\mathbb{A}\colon \mathbb{I} \to \mathbb{X}$ a diagram in a $\mathbb{Z}$-category $\mathbb{X}$. Assume that $\mathbb{I}$ has a subcategory $\mathbb{J}$ full on objects such that $\mathbb{J}^{op}$ is a Z-poset. Let $\{p_i\colon A \to \mathbb{A}i\}_{i\in|\mathbb{I}|}$ be a lax limit for $\mathbb{A}$. Moreover assume that*

1. *the set $\{\mathbb{A}h \mid h \in \mathbb{I}(i,j)\}$ is a Z-poset for any $i, j \in |\mathbb{I}|$,*
2. *for any map $(i \le j)$ in $\mathbb{J}$, one has $\mathbb{A}(i \le j) \circ p_i = p_j$.*

Then

1. *the arrows $\{p_i\}_{i\in|\mathbb{I}|}$ that constitute a lax limit for $\mathbb{A}$ are upper adjoint maps,*
2. *the corresponding lower adjoint maps $\{e_i\colon \mathbb{A}i \to A\}_{i\in|\mathbb{I}|}$ form a lax colimit for $\mathbb{A}$,*
3. *$\bigvee_{i\in|\mathbb{I}|} e_i \circ p_i = \mathrm{id}_A$, and*
4. *$p_j \circ e_i = \bigvee_{h\colon i\to j} \mathbb{A}h$ for any i and j in $|\mathbb{I}|$.*

Proof. The proof of Theorem 2.15 can be repeated unchanged since now:

- assumption 1 guarantees that $\left\{\mathbb{A}h \;\middle|\; h \in \mathbb{I}(i,j)\right\}$ is a Z-poset and it is therefore possible to define $a_i^k = \bigvee_{h\in\mathbb{I}(k,i)} \mathbb{A}h$,
- assumption 2 via Lemma 2.27 ensures that for any lax cocone $\{c_i\colon \mathbb{A}i \to B\}_{i\in|\mathbb{I}|}$ over $\mathbb{A}$ the set $\left\{c_i \circ p_i \;\middle|\; i \in |\mathbb{I}|\right\}$ is a Z-poset and it is therefore possible to take $\bigvee_{i\in|\mathbb{I}|} c_i \circ p_i$,
- composition preserves suprema of shape Z.

$\square$

Proposition 2.29. *Let $\mathbb{Z}$ be a symmetric monoidal closed Z-category and $\mathbb{A}\colon \mathbb{I} \to \mathbb{X}$ a diagram in a $\mathbb{Z}$-category $\mathbb{X}$. Assume that $\mathbb{I}$ has a subcategory $\mathbb{J}$ full on objects such that $\mathbb{J}^{op}$ is also a subcategory of $\mathbb{I}$ and $\mathbb{J}^{op}$ is a Z-poset. Assume moreover that*

1. *there are a lax cone $\{p_i\colon A \to \mathbb{A}i\}_{i\in|\mathbb{I}|}$ and a lax cocone $\{e_i\colon \mathbb{A}i \to A\}_{i\in|\mathbb{I}|}$ over $\mathbb{A}$ with $e_i \dashv p_i$ for any $i \in |\mathbb{I}|$,*
2. *for any map $(i \le j)$ in $\mathbb{J}$, both*
   a) $\mathbb{A}(i \le j) \circ p_i = p_j$ and
   b) $e_j \circ \mathbb{A}(i \le j) = e_i$,

3. *the set* $\{\mathbb{A}h \mid h \in \mathbb{I}(i,j)\}$ *is a Z-poset and* $p_j \circ e_i = \bigvee_{h\in\mathbb{I}(i,j)} \mathbb{A}h$ *for any* $i, j \in |\mathbb{I}|$*, and*

4. $\bigvee_{i\in|\mathbb{I}|} e_i \circ p_i = \mathrm{id}_A$.

Then $\{p_i\}_{i\in|\mathbb{I}|}$ *is a lax limit and* $\{e_i\}_{i\in|\mathbb{I}|}$ *is a lax colimit over* $\mathbb{A}$.

Proof. First of all notice that the existence of $\mathbb{J}$ together with either of conditions (2a) or (2b) ensures that it makes sense to consider $\bigvee_{i\in|\mathbb{I}|} e_i \circ p_i$ as a consequence of Lemma 2.27.

To see that $\{e_i \colon \mathbb{A}i \to A\}_{i\in|\mathbb{I}|}$ is a lax colimit for $\mathbb{A}$ we use assumption (2a). This allows us to construct a mediating morphism from a given lax cocone and repeat the steps in the proof of Proposition 2.17. Analogously, to show that $\{p_i \colon A \to \mathbb{A}i\}_{i\in|\mathbb{I}|}$ is a lax limit we use assumption (2b). □

2.5 The limit-colimit coincidence

We now show that the limit-colimit coincidence for a directed diagram of adjoint pairs in a **Dcpo**-enriched category is a special case of lax bilimit.

The following is standard (it is a slight generalisation of [39]).

Let $\mathbb{I}$ be a directed set and $\mathbb{X}$ a **Dcpo**-enriched category. The kind of diagram that is of interest here is

$$\mathbb{A} \colon \mathbb{I} \to \mathbb{X}^{lu}$$

where $\mathbb{X}^{lu}$ is the category of adjoint pairs on $\mathbb{X}$. Such a diagram can also be seen as a pair of diagrams, one $\mathbb{A}^l \colon \mathbb{I} \to \mathbf{Dcpo}^l$ into the category of lower adjoint maps, the other $\mathbb{A}^u \colon \mathbb{I}^{op} \to \mathbf{Dcpo}^u$ into the category of upper adjoint maps.

To stick to a standard and pleasant notation for such diagrams adopted by [3], we call

- A_i the object $\mathbb{A}i$, for any $i \in \mathbb{I}$, and
- (e_{ji}, p_{ij}) the lower-upper adjoint pair $\mathbb{A}(i \leq j)$ for any $i \leq j$ in $\mathbb{I}$.

Assuming that the limit of the diagram $\mathbb{A}^u$ of upper adjoint maps exists in the ambient category $\mathbb{X}$, we have a universal cone:

$$\{p_i \colon A \to A_i\}_{i\in\mathbb{I}}$$

One notices then, that for any i,j in $\mathbb{I}$, the set of maps:

$$\left\{p_{jk} \circ e_{ki} \in \mathbb{X}(A_i, A_j) \;\middle|\; k \geq i, j\right\}$$

is directed, and therefore we have maps:

$$\bigvee_{k\geq i,j} p_{jk} \circ e_{ki} =: a^i_j \colon A_i \to A_j$$

These form a cone for $\mathbb{A}^u$. For any $i \in \mathbb{I}$ there exists therefore a unique mediating map

$$e_i \colon A_i \to A$$

such that for any i and j in $\mathbb{I}$ we have

$$p_j \circ e_i = a_i^j = \bigvee_{k \geq i,j} p_{jk} \circ e_{ki}.$$

It turns out that each e_i is a lower adjoint for the respective p_i and that $\{e_i\}_{i \in \mathbb{I}}$ is a universal cone for $\mathbb{A}^l$ in $\mathbb{X}$. For this reason one says that $\{e_i, p_i \colon A_i \to A\}_{i \in \mathbb{I}}$ is a *bilimit* for the diagram.

Moreover one has that $\bigvee\!\!\!\!\uparrow_{i \in \mathbb{I}} e_i \circ p_i = \mathrm{id}_A$.

2.5.1 What does this have to do with lax bilimits?

Let's modify the diagram we started with a bit.

Instead of taking the directed set $\mathbb{I}$ as shape of our diagram, we take the category $\tilde{\mathbb{I}}$ whose objects are the elements of $\mathbb{I}$ and whose morphisms are paths of finite length in $\mathbb{I}$ seen as an undirected graph. We write $(j, v_{n-1}) \cdot (v_{n-1}, v_{n-2}) \cdot \ldots \cdot (v_1, i)$ for a path $p \colon i \rightsquigarrow j$ in $\tilde{\mathbb{I}}$. The unit on i in $\tilde{\mathbb{I}}$ is (i, i), the 0-length path at i and composition of paths is simply given by concatenation. Clearly, $\tilde{\mathbb{I}}$ is a category.

We force our diagram to have shape $\tilde{\mathbb{I}}$ by "stretching" $\mathbb{A}$ over the zigzag of a path in $\tilde{\mathbb{I}}$. Let

$$\tilde{\mathbb{A}} \colon \tilde{\mathbb{I}} \to \mathsf{Dcpo}$$

be the functor defined by:

on objects $\tilde{\mathbb{A}}i = \mathbb{A}i = A_i$ for any $i \in \mathbb{I}$, and

on morphisms for a path $p \colon i \rightsquigarrow j$:

$$\begin{array}{rcll} \tilde{\mathbb{A}}p & = & \mathrm{id}_{A_i} & \text{if } p \text{ has lenght } 0 \\ \tilde{\mathbb{A}}p & = & h_{jm} \circ \tilde{\mathbb{A}}q & \text{if } p = (j, m) \cdot q \\ & & \text{where} & h_{jm} = \begin{cases} e_{jm} & \text{if } m \leq j \\ p_{jm} & \text{if } j \leq m \end{cases} \end{array}$$

We show via Proposition 2.29 with $\mathbb{Z} = \mathsf{Dcpo}$ that the family $\{e_i \dashv p_j\}_{i \in \mathbb{I}}$ is a lax bilimit for $\tilde{\mathbb{A}}$.

$\tilde{\mathbb{I}}$ has $\mathbb{I}$ and $\mathbb{I}^{\mathrm{op}}$ as subcategories.

For any $i \leq j$ in $\mathbb{I}^{op}$, we obviously have

$$\tilde{\mathbb{A}}(j, i) \circ p_i = p_{ji} \circ p_i = p_j.$$

Similarly, for any $i \leq j$ in $\mathbb{I}$:

$$e_j \circ \tilde{\mathbb{A}}(j, i) = e_j \circ e_{ji} = e_i$$

It is easy to prove by induction on the length of a path $p\colon i \rightsquigarrow j$ that $\{p_i\}_{i \in \mathbb{I}}$ is a lax cone over $\tilde{\mathbb{A}}$ and that $\{e_i\}_{i \in \mathbb{I}}$ is a lax cocone for $\tilde{\mathbb{A}}$.

Next, we need to check that the set $\{\tilde{\mathbb{A}}p \mid p\colon k \rightsquigarrow i \text{ in } \tilde{\mathbb{I}}\}$ is directed.

Lemma 2.30. *For any* $k \in \mathbb{I}$*, the set* $P = \{\tilde{\mathbb{A}}p \mid p\colon k \rightsquigarrow i \text{ in } \tilde{\mathbb{I}}\}$ *is a directed set in* $\mathbb{X}(A_k, A_i)$.

Proof. First we consider the subset P' of paths $p\colon k \rightsquigarrow i$ given by only two legs, the first going up, the second going down.

$$P' = \{\tilde{\mathbb{A}}p \mid p = (i, m) \cdot (m, k) \text{ for any } m \geq i, k\} = \{p_{im} \circ e_{mk} \mid \text{ for any } m \geq i, k\}$$

P' is directed since for any $m' \geq m$,

$$p_{im'} \circ e_{m'k} = p_{im} \circ p_{mm'} \circ e_{m'm} \circ e_{mk} \geq p_{im} \circ e_{mk}$$

A general path $p\colon k \rightsquigarrow i$ differs essentially from such a two-legged one only for a finite number of points in which one upward leg follows a downward one (we say *essentially* because — being $\mathbb{A}$ a functor into Dcpo^{ep}— two adjacent legs with the same orientation in a path p can be replaced by one single one, obtaining a path p' such that $\tilde{\mathbb{A}}p = \tilde{\mathbb{A}}p'$). But in a diamond shape determined by one $m \geq k, i$ and one $n \leq k, i$ in $\mathbb{I}$

$$\begin{tikzcd}
& A_m \arrow[dr, "p_{im}"] & \\
A_k \arrow[ur, "e_{mk}"] \arrow[dr, "p_{nk}"'] & & A_i \\
& A_n \arrow[ur, "e_{in}"'] &
\end{tikzcd}$$

one has always that

$$\begin{aligned}
p_{im} \circ e_{mk} &\geq p_{im} \circ e_{mn} \circ p_{nm} \circ e_{mk} \\
&= p_{im} \circ e_{im} \circ e_{mn} \circ p_{nk} \circ p_{km} \circ e_{mk} \\
&\geq e_{mn} \circ p_{nk}
\end{aligned}$$

Therefore, paths different form the two-legged ones considered in P' have image under $\tilde{\mathbb{A}}$ below at least one of the elements of P' and do not affect directedness of the set P guaranteed by directedness of P'. $\square$

Moreover, we know that $\bigvee^{\uparrow}_{i \in \mathbb{I}} e_i \circ p_i = \mathrm{id}_D$. Therefore, by Proposition 2.29 we obtain that $\{e_i \dashv p_j : A_i \to A\}_{i \in \mathbb{I}}$ is a lax bilimit for $\tilde{\mathbb{A}}$.

Chapter 3

Models in categories of domains

We now show how to obtain a *linear hyperdoctrine* starting from a suitable subcategory $\mathbb{X}$ of Dcpo that is a model of linear logic.

The material of this chapter follows the constructions of [15] to give a model of *System* F using Scott domains, while the presentation is radically different. From this analysis we get a model of second order linear logic and, in addition, an abstract account of the model of *System* F given in [15].

The first step is to see that the linear category structure of $\mathbb{X}$ lifts to any category of functors and lax transformations $(\mathbb{X}^{\mathfrak{l}})^{\mathrm{lax}\,\mathbb{I}}$ and that the functor $(\mathbb{X}^{\mathfrak{l}})^{\mathrm{lax}\,(-)}$ (on any category of categories) defines a (split) indexed linear category.

The second step is to show how continuity and bifiniteness allow to control the size of such an indexed linear category so that it is suitable to model second order universal quantification.

In Section 3.6 we briefly comment on the relation to the model of [15].

At the end of the chapter we give an instance of the model using the category aCD of algebraic completely distributive lattices and morphisms preserving arbitrary suprema. We give a short review of the properties of aCD that make it suitable for the construction of a linear hyperdoctrine and then compute the interpretation of some closed universal types.

3.1 Categories of lax transformations

Let $\mathbb{A}$ be a category and $\mathbb{X}$ a Pos-enriched category. In Section 2.1.1 we have defined the category $\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}$ of functors $\mathbb{A} \to \mathbb{X}$ and lax transformations.

We show how from a Pos-enriched linear category $\mathbb{X}$ we can obtain a linear category of functors and lax transformations.

The central result of this section is Proposition 3.7 which gives an account of the closed structure of our candidate linear category. The rest is a useful but rather straightforward collection of observations which takes care of all the rest of the structure of a linear category.

3.1.1 Some simple facts about $\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}$

Remark. As already pointed out in Section 2.1.1, if $\mathbb{A}$ is not a small category, $\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}$ is not Pos-enriched, but it behaves like a Pos-enriched category: For any $F, G\colon \mathbb{A} \to \mathbb{X}$, the class $\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}(F, G)$ is not a set; nevertheless, point-wise comparison induces on it an obvious order relation and composition preserve the order in each component. We will work with $\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}$ as if it was Pos-enriched.

Lemma 3.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be Pos-enriched categories. Any Pos-enriched functor*

$$M\colon \mathbb{X} \to \mathbb{Y}$$

induces a Pos-enriched functor

$$M^{\mathrm{lax}\,\mathbb{A}}\colon \mathbb{X}^{\mathrm{lax}\,\mathbb{A}} \to \mathbb{Y}^{\mathrm{lax}\,\mathbb{A}}$$

defined by postcomposing functors and lax transformations with M (by saying that $M^{\mathrm{lax}\,\mathbb{A}}$ is Pos-enriched we merely mean that it preserves the component-wise order between lax transformations).

Proof. Consider a lax transformation $s\colon F \overset{\odot}{\to} G$ in $\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}$. Since M is Pos-enriched, we have

$$\begin{tikzcd} MFA \arrow[r, "Ms_A"] \arrow[d, "MFa"'] \arrow[dr, phantom, "\geq"] & MGA \arrow[d, "MGa"] \\ MFA' \arrow[r, "Ms_{A'}"'] & MGA' \end{tikzcd}$$

for any $A \in |\mathbb{A}|$. So, $M^{\mathrm{lax}\,\mathbb{A}}(s)$ is the lax transformation $MF \to MG$ whose component at $A \in |\mathbb{A}|$ is Ms_A.

$M^{\mathrm{lax}\,\mathbb{A}}$ is clearly a functor.

If $s, t\colon F \overset{\odot}{\to} G$ are lax transformation in $\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}$ such that $s_A \leq t_A$ for any $A \in |\mathbb{A}|$, then obviously also $Ms_A \leq Mt_A$, that is $M^{\mathrm{lax}\,\mathbb{A}}$ preserves the component-wise order between lax transformations. $\square$

Lemma 3.2. *A lax transformation $\beta\colon M \overset{\odot}{\to} N$ between Pos-enriched functors $M, N\colon \mathbb{X} \to \mathbb{Y}$ induces a lax transformation*

$$\hat{\beta}\colon M^{\mathrm{lax}\,\mathbb{A}} \overset{\odot}{\to} N^{\mathrm{lax}\,\mathbb{A}}$$

If $\beta\colon M \overset{\bullet}{\to} N$ is a natural transformation, then $\hat{\beta}$ is also natural.

Proof. For any $F \in |\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}|$, the family of $\mathbb{Y}$-morphisms

$$\beta_F := \{\beta_{FA}\}_{A \in |\mathbb{A}|}$$

obviously defines a lax transformation $\beta_F : MF \overset{\odot}{\to} NF$. For any lax transformation $s : F \overset{\odot}{\to} G$ we have the following inequality in a diagram of lax transformations:

$$\begin{tikzcd}
MF \arrow[r, "\beta_F", "\odot"'] \arrow[d, "Ms"', "\odot"] \arrow[dr, phantom, "\geq"] & NF \arrow[d, "Ns", "\odot"'] \\
MF' \arrow[r, "\odot", "\beta_{F'}"'] & NF'
\end{tikzcd}$$

since the inequality at any $A \in |\mathbb{A}|$ follows from the fact that β is a lax transformation.
In $\beta : F \overset{\bullet}{\to} G$ is natural, we obtain natural transformations

$$\beta_F \colon MF \overset{\bullet}{\to} NF$$

which are particular lax transformations. Naturality of $\hat{\beta}$ in F is straightforward: For any lax transformation $s \colon F \overset{\odot}{\to} F'$ the following diagram of natural and lax transformations

$$\begin{tikzcd}
MF \arrow[r, "\beta_F", "\bullet"'] \arrow[d, "Ms"', "\odot"] & NF \arrow[d, "Ns", "\odot"'] \\
MF' \arrow[r, "\bullet", "\beta_{F'}"'] & NF'
\end{tikzcd}$$

commutes by naturality of β. □

3.1.2 $(\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}}$ as a linear category

Let $\mathbb{A}$ be a category, $\mathbb{X}$ a Pos-enriched category and $\mathbb{X}^l$ ($\mathbb{X}^e$) its subcategory of lower adjoint maps (embeddings). Obviously, for any functor

$$F \colon \mathbb{A} \to \mathbb{X}^l \quad (F \colon \mathbb{A} \to \mathbb{X}^e)$$

there is a functor

$$F^u \colon \mathbb{A}^{op} \to \mathbb{X} \quad (F^p \colon \mathbb{A}^{op} \to \mathbb{X}).$$

The category we are interested in is the following.

Definition 3.3. We denote by $(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}$ (respectively $(\mathbb{X}^{e})^{\mathrm{lax}\,\mathbb{A}}$) the **full** subcategory of $\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}$ whose objects are functors $F\colon \mathbb{A} \to \mathbb{X}^{l}$ ($F\colon \mathbb{A} \to \mathbb{X}^{e}$).

Remark. This definition feels somewhat awkward: Objects of $(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}$ are functors into the category $\mathbb{X}^{l}$ of lower adjoint maps while the components of a lax transformation between two such functors are not required to be lower adjoints. The reason to consider functors into $\mathbb{X}^{l}$ is to be able to deal with contravariance in the definition of the exponential between two functors.

In the sequel, any time we use adjoint pairs we could have used embedding-projection pairs instead. We stop therefore to use the double notation: $\mathbb{X}^{l}$ might also be $\mathbb{X}^{e}$.

An observation that is useful in some of the proofs below:

Lemma 3.4. *Let $\mathbb{X}$ be a* Pos*-enriched category. A family of $\mathbb{X}$-morphisms $\{s_A : FA \to GA\}_{A\in|\mathbb{A}|}$ is a lax transformation $s : F \overset{\odot}{\to} G$ in $(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}$ if and only if it satisfies*

$$\begin{tikzcd}
FA \arrow[r, "s_A"] \arrow[dr, phantom, "\wedge\!\mid"] & GA \arrow[d, "Ga"] \\
FA' \arrow[u, "F^{u}a"] \arrow[r, "s_{A'}"'] & GA'
\end{tikzcd}$$

for any $a\colon A \to A'$ in $\mathbb{A}$.

Proof. Immediate application of the property of adjoint pairs. ∎

Definition 3.5. Let $\mathbb{X}$ be a Pos-enriched category and assume that there is a symmetric monoidal structure $(\otimes, I)$ on $\mathbb{X}$. We say that $(\mathbb{X}, \otimes, I)$ is a **Pos-enriched symmetric monoidal category** if the action of the bifunctor $(- \otimes \star)$ on morphisms is monotone. If $\mathbb{X}$ has a symmetric monoidal closed structure $(\otimes, I, \multimap)$, we say that $(\mathbb{X}, \otimes, I, \multimap)$ is a **Pos-enriched symmetric monoidal closed category** if the same happens also for the bifunctor $(- \multimap \star)$. If $\mathbb{X}$ has a linear category structure with symmetric monoidal comonad $(T, m^{T}, m^{T}_{I}, \varepsilon, \delta)$, we say that it is a **$\mathsf{Pos}$-enriched linear category** if moreover the functor T is Pos-enriched.

Now let us assume that $\mathbb{X}$ is a Pos-enriched *linear category*.

We show how each component of the structure of linear category of $\mathbb{X}$ lifts to an analogous structure for $(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}$, so that the latter is also a linear category.

Among the objects of $(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}$ there is the obvious functor

(unit) $I\colon \mathbb{A} \to \mathbb{X}^{l}$ the constant I functor.

Moreover, for any $F, G\colon \mathbb{A} \to \mathbb{X}^l$ there are the following functors:

(tensor) $F \otimes G\colon \mathbb{A} \to \mathbb{X}^l$ defined by

- **(on objects)** $(F \otimes G)A = (FA \otimes GA)$ for any $A \in |\mathbb{A}|$
- **(on morphisms)** $(F \otimes G)a = (Fa \otimes Ga)$ for any $a \in \mathbb{A}(A, A')$ (the respective upper adjoint is given by $(F^u a \otimes G^u a)$).

(exponential) $F \multimap G\colon \mathbb{A} \to \mathbb{X}^l$ defined by

- **(on objects)** $(F \multimap G)A = (FA \multimap GA)$ for any $A \in |\mathbb{A}|$
- **(on morphisms)** $(F \multimap G)a = (F^u a \multimap Ga)$ for any $a \in \mathbb{A}(A, A')$ (the respective upper adjoint is given by $(Fa \multimap G^u a)$).

Proposition 3.6. *If $\mathbb{X}$ is a* Pos*-enriched symmetric monoidal category, the description of tensor and unit above provides the object part of a symmetric monoidal structure on $(\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}}$ (as a matter of fact, already on $\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}$).*

Proof. **(Tensor)** We want to see that $F \otimes G$ is the object part of a bifunctor

$$(- \otimes \star)\colon (\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}} \times (\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}} \to (\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}}:$$

If $s\colon F \overset{\odot}{\to} F'$ and $t\colon G \overset{\odot}{\to} G'$ are lax transformations we obtain a lax transformation

$$s \otimes t\colon F \otimes G \overset{\odot}{\to} F' \otimes G'$$

defined component-wise by

$$(s \otimes t)_A := s_A \otimes t_A \text{ for any } A \in |\mathbb{A}|\,.$$

That $s \otimes t$ is a lax transformation is an immediate consequence of the fact that the action of $\otimes$ on morphisms preserves the order in each argument.

(unit) The constant I functor is the candidate unit.

(coherence) For $(\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}}$ to be a symmetric monoidal category with respect to the given tensor and unit, we need coherence isomorphisms. These are easily induced by the coherence isomorphisms of $\mathbb{X}$. For functors $F, G, L\colon \mathbb{A} \to \mathbb{X}^l$ consider:

$$\hat{\alpha}_{F,G,L} = \alpha_{F-,G-,L-}\colon F \otimes (G \otimes L) \overset{\cong}{\to} (F \otimes G) \otimes L$$
$$\hat{\lambda}_F = \lambda_{F-}\colon I \otimes F \overset{\cong}{\to} F$$
$$\hat{\rho}_F = \rho_{F-}\colon F \otimes I \overset{\cong}{\to} F$$
$$\hat{\gamma}_{F,G} = \gamma_{F-,G-}\colon F \otimes G \overset{\cong}{\to} G \otimes F$$

Each component $\hat{\alpha}_{F,G,L}$ is the lax transformation $F \otimes (G \otimes L) \overset{\odot}{\to} (F \otimes G) \otimes L$ whose component at $A \in |\mathbb{A}|$ is given by $\alpha_{FA,GA,LA}$. It is therefore in particular a natural isomorphism. Naturality of $\hat{\alpha}$ in each of F, G, L is an immediate consequence of naturality of α. Similarly for $\hat{\lambda}$, $\hat{\rho}$ and $\hat{\gamma}$. It is immediate to verify that these natural isomorphisms satisfy the coherence diagrams for a symmetric monoidal category. □

Proposition 3.7. *If $\mathbb{X}$ is a* Pos*-enriched symmetric monoidal closed category, the given descriptions of tensor and unit and exponential provide the object part of a symmetric monoidal closed structure on* $(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}$.

Proof. We need to see how closedness of $\mathbb{X}$ induces closedness of $(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}$ with respect to the symmetric monoidal structure described in Proposition 3.6. Let

$$\mathrm{ev}^{(Y)}_X \colon ((Y \multimap X) \otimes Y) \to X \quad \text{and}$$
$$\mathrm{ve}^{(Y)}_X \colon X \to (Y \multimap (X \otimes Y))$$

be the components at $X \in |\mathbb{C}|$ of, respectively, counit and unit of the adjunction $(- \otimes Y) \dashv (Y \multimap -) \colon \mathbb{X} \to \mathbb{X}$ for any $Y \in |\mathbb{X}|$.

We need to show that for any given $G \in \left|(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}\right|$ there is an adjunction

$$(- \otimes G) \dashv (G \multimap -) \colon (\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}} \to (\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}.$$

Consider the families indexed by $F \in \left|(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}\right|$:

$$\widehat{\mathrm{ev}}^{(G)}_F \colon ((G \multimap F) \otimes G) \to F \quad \text{where} \quad (\widehat{\mathrm{ev}}^{(G)}_F)_A := \mathrm{ev}^{(GA)}_{FA} \quad \text{and}$$
$$\widehat{\mathrm{ve}}^{(G)}_F \colon F \to (G \multimap (F \otimes G)) \quad \text{where} \quad (\widehat{\mathrm{ve}}^{(G)}_F)_A := \mathrm{ve}^{(GA)}_{FA}$$

We intend to prove that $\widehat{\mathrm{ev}}^{(G)}_F$ and $\widehat{\mathrm{ve}}^{(G)}_F$ are lax transformations.

The family of arrows $\{\mathrm{ev}^{(Y)}_X\}_{Y \in |\mathbb{X}|}$ is not natural in Y, but *extranatural* [31] that is, for any $y \colon Y \to Y'$ in $\mathbb{X}$ the following is a commutative diagram.

$$\begin{tikzcd}
(Y' \multimap X) \otimes Y \arrow[r, "(y \multimap \mathrm{id}_X) \otimes \mathrm{id}_Y"] \arrow[d, "(\mathrm{id}_{Y'} \multimap \mathrm{id}_X) \otimes y"'] & (Y \multimap X) \otimes Y \arrow[d, "\mathrm{ev}^{(Y)}_X"] \\
(Y' \multimap X) \otimes Y' \arrow[r, "\mathrm{ev}^{(Y')}_X"'] & X
\end{tikzcd}$$

Similarly, the family $\{\mathrm{ve}^{(Y)}_X\}_{Y \in |\mathbb{X}|}$ forms an extranatural transformation, that is, for every $y\colon Y \to Y'$ in $\mathbb{X}$ the following is a commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{ve}^{(Y)}_X"] \arrow[d, "\mathrm{ve}^{(Y')}_X"'] & (Y \otimes (X \otimes Y) \arrow[d, "(\mathrm{id}_Y \multimap (\mathrm{id}_X \otimes y))"] \\
Y' \multimap (X \otimes Y') \arrow[r, "y \multimap \mathrm{id}_{X \otimes Y'}"'] & Y \multimap (X \otimes Y')
\end{tikzcd}$$

To prove that the family $\{\mathrm{ev}^{(GA)}_{FA}\}_{A \in |\mathbb{A}|}$ is a lax transformation $(G \multimap F) \otimes G \overset{\odot}{\to} F$ we need to show — recalling Lemma 3.4 as well as the expression of adjoint components of the functor $(G \multimap F) \otimes G)$ — that for any $a\colon A \to A'$ in $\mathbb{A}$ we have

$$Fa \circ \mathrm{ev}^{(GA)}_{FA} \circ ((Ga \multimap F^u a) \otimes G^u a) \leq \mathrm{ev}^{(GA')}_{FA'}\,.$$

We do this with the following calculation:

$$\begin{aligned}
& Fa \circ \mathrm{ev}^{(GA)}_{FA} \circ ((Ga \multimap F^u a) \otimes G^u a) = && \text{(nat. of } \mathrm{ev}^{(GA)}\text{)}\\
& = \mathrm{ev}^{(GA)}_{FA'} \circ ((\mathrm{id}_{GA} \multimap Fa) \otimes \mathrm{id}_{GA}) \circ ((Ga \multimap F^u a) \otimes G^u a) = && \\
& = \mathrm{ev}^{(GA)}_{FA'} \circ ((Ga \multimap (Fa \circ F^u a)) \otimes G^u a) \leq && (Fa \dashv F^u a)\\
& \leq \mathrm{ev}^{(GA)}_{FA'} \circ ((Ga \multimap \mathrm{id}_{FA'}) \otimes G^u a) = && \\
& = \mathrm{ev}^{(GA)}_{FA'} \circ ((Ga \multimap \mathrm{id}_{FA'}) \otimes \mathrm{id}_{GA}) \circ ((\mathrm{id}_{GA'} \multimap \mathrm{id}_{FA'}) \otimes G^u a) = && (\mathrm{ev}^{()}_{FA'} \text{ extranat.})\\
& = \mathrm{ev}^{(GA')}_{FA'} \circ ((\mathrm{id}_{GA'} \multimap \mathrm{id}_{FA'}) \otimes Ga) \circ ((\mathrm{id}_{GA'} \multimap \mathrm{id}_{FA'}) \otimes G^u a) \leq && (Ga \dashv G^u a)\\
& \leq \mathrm{ev}^{(GA')}_{FA'}
\end{aligned}$$

Similarly we obtain that $\{\mathrm{ve}^{(GA)}_{FA}\}_{A \in |\mathbb{A}|}$ is a lax transformation $F \overset{\odot}{\to} (G \multimap (F \otimes G))$. For any $a\colon A \to A'$ in $\mathbb{A}$:

$$\begin{aligned}
& (G^u a \multimap (Fa \otimes Ga)) \circ \mathrm{ve}^{(GA)}_{FA} \circ F^u a = && \\
& = (G^u a \multimap (\mathrm{id}_{FA'} \otimes Ga)) \circ (\mathrm{id}_{GA} \multimap (Fa \otimes \mathrm{id}_{GA})) \circ \mathrm{ve}^{(GA)}_{FA} \circ F^u a = && \text{(nat. of } \mathrm{ve}^{(GA)}\text{)}\\
& = (G^u a \multimap (\mathrm{id}_{FA'} \otimes Ga)) \circ \mathrm{ve}^{(GA)}_{FA'} \circ Fa \circ F^u a \leq && (Fa \dashv F^u a)\\
& \leq (G^u a \multimap (\mathrm{id}_{FA'} \otimes Ga)) \circ \mathrm{ve}^{(GA)}_{FA'} = && \\
& = (G^u a \multimap (\mathrm{id}_{FA'} \otimes \mathrm{id}_{GA'})) \circ (\mathrm{id}_{GA} \multimap (\mathrm{id}_{FA'} \otimes Ga)) \circ \mathrm{ve}^{(GA)}_{FA'} = && (\mathrm{ve}^{()}_{FA'} \text{ extranat.})\\
& = (G^u a \multimap (\mathrm{id}_{FA'} \otimes \mathrm{id}_{GA'})) \circ (Ga \multimap (\mathrm{id}_{FA'} \otimes \mathrm{id}_{GA'})) \circ \mathrm{ve}^{(GA')}_{FA'} \leq && (Ga \dashv G^u a)\\
& \leq \mathrm{ve}^{(GA')}_{FA'}
\end{aligned}$$

The fact that $\widehat{\mathrm{ev}}_{\mathsf{F}}^{(\mathsf{G})}$ and $\widehat{\mathrm{ve}}_{\mathsf{F}}^{(\mathsf{G})}$ are natural in F is an immediate consequence of the naturality of $\mathrm{ev}^{(\mathsf{G}A)}$ and $\mathrm{ve}^{(\mathsf{G}A)}$ for any $A \in |\mathbb{A}|$.

We still need to check that $\widehat{\mathrm{ev}}_{\mathsf{F}}^{(\mathsf{G})}$ and $\widehat{\mathrm{ve}}_{\mathsf{F}}^{(\mathsf{G})}$ satisfy the triangular identities. This is immediate since the equations for triangular identities for any component at $A \in |\mathbb{A}|$ is a particular case of the triangular identity equations for $\mathrm{ev}^{(\mathsf{G}A)}$ and $\mathrm{ve}^{(\mathsf{G}A)}$. ☐

Remark. The proposition gives an abstract account of the closed structure of $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$. It suggests that the choice of lax transformations as morphisms between functors is the correct one to extend the closed structure of $\mathbb{X}$ to $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$, as the obvious candidate counit and unit of $(- \otimes \mathsf{G}) \dashv (\mathsf{G} \multimap -)$ form lax transformations, not natural transformations.

We now turn to see how a linear exponential comonad $(\mathsf{T}, \mathsf{m}^{\mathsf{T}}, \mathsf{m}^{\mathsf{T}}_{\mathrm{I}}, \varepsilon, \delta)$ on $\mathbb{X}$, such that T is a Pos-enriched functor, induces a linear exponential comonad on $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$. Again, most of the machinery needed is fairly general and straightforward.

The following lemma specialises the lifting of Pos-enriched functors and natural transformations (Lemma 3.1 and 3.2) to the case of symmetric monoidal categories and functors.

Lemma 3.8. *Let $(\mathbb{X}, \otimes, \mathrm{I})$ be a Pos-enriched symmetric monoidal category and, moreover, let $(\mathsf{M}, \mathsf{m}^{\mathsf{M}}, \mathsf{m}^{\mathsf{M}}_{\mathrm{I}})$ a Pos-enriched symmetric monoidal endofunctor on $\mathbb{X}$. Then the functor $\mathsf{M}^{\mathrm{lax}\,\mathbb{A}}$ has an obvious structure of symmetric monoidal endofunctor with respect to the symmetric monoidal structure induced on $\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}$ and $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$.*

If $\mu\colon (\mathsf{M}, \mathsf{m}^{\mathsf{M}}, \mathsf{m}^{\mathsf{M}}_{\mathrm{I}}) \dot{\to} (\mathsf{N}, \mathsf{m}^{\mathsf{N}}, \mathsf{m}^{\mathsf{N}}_{\mathrm{I}})$ is a monoidal natural transformation between Pos-enriched symmetric monoidal endofunctors, then the natural transformation $\hat{\mu}$ (with component at $\mathsf{F} \in \left|\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}\right|$ given by μ_{F}) is also monoidal.

Proof. The obvious arrow

$$\mathsf{m}_{\mathrm{I}}^{\mathsf{M}^{\mathrm{lax}\,\mathbb{A}}} \colon \mathrm{I} \overset{\odot}{\to} \mathsf{M}^{\mathrm{lax}\,\mathbb{A}}\mathrm{I} = \mathsf{M}\mathrm{I}$$

is the constant lax transformation $(\mathsf{m}_{\mathrm{I}}^{\mathsf{M}^{\mathrm{lax}\,\mathbb{A}}})_A := \mathsf{m}_{\mathrm{I}}^{\mathsf{M}}$.

For any $\mathsf{F}, \mathsf{G} \in \left|\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}\right|$ the candidate arrows

$$\mathsf{m}_{\mathsf{F},\mathsf{G}}^{\mathsf{M}^{\mathrm{lax}\,\mathbb{A}}} \colon \mathsf{M}\mathsf{F} \otimes \mathsf{M}\mathsf{G} \overset{\odot}{\to} \mathsf{M}(\mathsf{F} \otimes \mathsf{G})$$

are defined component-wise by

$$(\mathsf{m}_{\mathsf{F},\mathsf{G}}^{\mathsf{M}^{\mathrm{lax}\,\mathbb{A}}})_A := \mathsf{m}^{\mathsf{M}}_{\mathsf{F}A,\mathsf{G}A}$$

for any $A \in |\mathbb{A}|$. Each of $\mathsf{m}_{\mathsf{F},\mathsf{G}}^{\mathsf{M}^{\mathrm{lax}\,\mathbb{A}}}$ is in particular a lax transformation since it is natural in A. Naturality in F and G is also an immediate consequence of naturality of m^{M}.

The natural transformation $\hat{\mu}$ is monoidal if the two following diagrams commute:

$$
\begin{tikzcd}
\mathrm{I} \arrow[r, "\mathsf{m}_{\mathrm{I}}^{\mathsf{M}^{\mathrm{lax}\,\mathbb{A}}}"] \arrow[d, equal] & \mathsf{MI} \arrow[d, "\hat{\mu}_{\mathrm{I}} = \mu_{\mathrm{I}}"] \\
\mathrm{I} \arrow[r, "\mathsf{m}_{\mathrm{I}}^{\mathsf{N}^{\mathrm{lax}\,\mathbb{A}}}"'] & \mathsf{NI}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{MF} \otimes \mathsf{MG} \arrow[r, "\mathsf{m}_{\mathsf{F},\mathsf{G}}^{\mathsf{M}^{\mathrm{lax}\,\mathbb{A}}}", "\bullet"'] \arrow[d, "\mu_{\mathsf{F}} \otimes \mu_{\mathsf{G}}"', "\bullet"] & \mathsf{M}(\mathsf{F} \otimes \mathsf{G}) \arrow[d, "\mu_{\mathsf{F}\otimes\mathsf{G}}", "\bullet"'] \\
\mathsf{NF} \otimes \mathsf{NG} \arrow[r, "\bullet", "\mathsf{m}_{\mathsf{F},\mathsf{G}}^{\mathsf{N}^{\mathrm{lax}\,\mathbb{A}}}"'] & \mathsf{N}(\mathsf{F} \otimes \mathsf{G})
\end{tikzcd}
$$

This is guaranteed by the analogous conditions for μ. □

Corollary 3.9. *Let $(\mathbb{X}, \otimes, \mathrm{I})$ be a* Pos*-enriched symmetric monoidal category and let $(\mathsf{T}, \mathsf{m}^{\mathsf{T}}, \mathsf{m}^{\mathsf{T}}_{\mathrm{I}}, \varepsilon, \delta)$ a* Pos*-enriched symmetric monoidal comonad on $\mathbb{X}$. The comonad $(\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}, \hat{\varepsilon}, \hat{\delta})$ induced on $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$ (respectively $\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}$) has an obvious structure of symmetric monoidal comonad.*

Proof. The requirement that T is a Pos-enriched endofunctor immediately implies (as observed in Section 3.1.1) that T lifts to an endofunctor

$$\mathsf{T}^{\mathrm{lax}\,\mathbb{A}} \colon (\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}} \to (\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}} \quad (\text{respectively } \mathbb{X}^{\mathrm{lax}\,\mathbb{A}} \to \mathbb{X}^{\mathrm{lax}\,\mathbb{A}})$$

and that there are natural transformations

$$\hat{\varepsilon} \colon \mathsf{T}^{\mathrm{lax}\,\mathbb{A}} \overset{\bullet}{\to} \mathrm{Id}_{(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}}$$
$$\hat{\delta} \colon (\mathsf{TT})^{\mathrm{lax}\,\mathbb{A}} \overset{\bullet}{\to} \mathsf{T}^{\mathrm{lax}\,\mathbb{A}}$$

which obviously satisfy the equations for a comonad on $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$ (resp. $\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}$).

By Lemma 3.8, the comonad $(\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}, \hat{\varepsilon}, \hat{\delta})$ inherits also the property of being symmetric monoidal from $(\mathsf{T}, \varepsilon, \delta)$. □

To perform the last step we need to have a feel for what the category of coalgebras for $(\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}, \hat{\varepsilon}, \hat{\delta})$ looks like.

A $\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}$-coalgebra in $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$ (respectively $\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}$) is a lax transformation $s \colon \mathsf{F} \overset{\odot}{\to} \mathsf{TF}$, each component of which is a T-coalgebra in $\mathbb{X}$. The unit $\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}$-coalgebra is the constant lax transformation of components $\mathsf{m}^{\mathsf{T}}_{\mathrm{I}} \colon \mathrm{I} \to \mathsf{TI}$. The tensor of two $\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}$-coalgebras $s \colon \mathsf{F} \overset{\odot}{\to} \mathsf{TF}$ and $t \colon \mathsf{G} \overset{\odot}{\to} \mathsf{TG}$ is obtained by taking the tensor of T-coalgebras at each component. A $\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}$-coalgebra morphism $(\mathsf{F}, s) \to (\mathsf{G}, t)$ is a lax transformation $\mathsf{F} \overset{\odot}{\to} \mathsf{G}$ with the property that each of its components is a T-coalgebra morphism.

Proposition 3.10. *Let $\mathbb{X}$ be a* Pos*-enriched linear category. Then the structure of linear category lifts to $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$.*

Proof. Let $(\mathsf{T}, \mathsf{m}^{\mathsf{T}}, \mathsf{m}^{\mathsf{T}}_{\mathrm{I}}, \varepsilon, \delta)$ be the linear exponential comonad of the linear category $\mathbb{X}$. By Proposition 3.7 and Corollary 3.9, $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$ has a symmetric monoidal closed structure induced by that of $\mathbb{X}$, for which $(\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}, \widehat{\varepsilon}, \widehat{\delta})$ is a symmetric monoidal comonad. To see that this is indeed a linear exponential comonad it is convenient to consider the original definition via natural transformations $\mathsf{e}\colon \mathsf{T}\overset{\bullet}{\to}\mathrm{I}$ and $\mathsf{d}\colon \mathsf{T}\overset{\bullet}{\to}(\mathsf{T}-\otimes\mathsf{T}-)$. By Lemma 3.8 these lift to monoidal natural transformations

$$\widehat{\mathsf{e}}\colon \mathsf{T}^{\mathrm{lax}\,\mathbb{A}}\overset{\bullet}{\to}\mathrm{I} \text{ and}$$
$$\widehat{\mathsf{d}}\colon \mathsf{T}^{\mathrm{lax}\,\mathbb{A}}\overset{\bullet}{\to}(\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}-\otimes\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}-)$$

Given the simple description of the category of $\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}$-coalgebras, one has that each component $\widehat{\mathsf{e}}_{\mathsf{F}}$ and $\widehat{\mathsf{d}}_{\mathsf{F}}$, $\mathsf{F} \in \left|\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}\right|$ is a coalgebra morphism. It is also the case that $(\mathsf{F}\mathsf{T}, \widehat{\mathsf{e}}_{\mathsf{F}}, \widehat{\mathsf{d}}_{\mathsf{F}})$ is a commutative comonoid for any $\mathsf{F} \in \left|\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}\right|$. Moreover, since each component of δ is a morphism of commutative comonoids, it is immediate that also each component of $\widehat{\delta}$ is. That is to say, $(\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}, \widehat{\varepsilon}, \widehat{\delta})$ is a linear exponential comonad on $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$. □

3.2 Indexed linear categories

In Section 3.1.2 we have seen that under some requirements on $\mathbb{X}$ the lax functor category $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$ is a linear category. We see now how to use this fact to obtain a (split) indexed category of linear categories, that is a functor

$$\mathcal{B}^{\mathrm{op}} \to \mathsf{Lin}$$

on a category $\mathcal{B}$ of categories and functors.

The first step is extremely simple. It generalises the behaviour of reindexing functors in the split indexed category $\mathrm{Funct}(-, \mathbb{X}) : \mathcal{B}^{\mathrm{op}} \to \mathsf{CAT}$ to categories of lax transformation .

Lemma 3.11. *Let $\mathbb{X}$ be a Pos-enriched category and $\mathsf{N}\colon \mathbb{B} \to \mathbb{A}$ a functor between two categories. Then precomposition with N defines a functor*

$$\mathbb{X}^{\mathrm{lax}\,\mathsf{N}} = \mathsf{N}^*\colon \mathbb{X}^{\mathrm{lax}\,\mathbb{A}} \to \mathbb{X}^{\mathrm{lax}\,\mathbb{B}}.$$

For $\mathcal{B}$ a category of categories and functors, $\mathbb{X}^{\mathrm{lax}\,(-)}\colon \mathcal{B}^{\mathrm{op}} \to \mathsf{CAT}$ is a (split) indexed category.

Proof. For any $\mathsf{F} \in \left|\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}\right|$, define $\mathbb{X}^{\mathrm{lax}\,\mathsf{N}}(\mathsf{F}) = \mathsf{N}^*(\mathsf{F}) := \mathsf{F}\mathsf{N}$. For every $\mathsf{s}\colon \mathsf{F} \overset{\odot}{\to} \mathsf{G}$ in $\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}$, $\mathbb{X}^{\mathrm{lax}\,\mathsf{N}}(\mathsf{s}) = \mathsf{N}^*(\mathsf{s}) := \mathsf{s}_{\mathsf{N}}$ is a lax transformation $\mathsf{F}\mathsf{N} \overset{\odot}{\to} \mathsf{G}\mathsf{N}$, since for any

$b\colon B \to B'$ in $\mathbb{B}$ we have:

$$\begin{tikzcd} FNB \arrow[r, "s_{NB}"] \arrow[d, "FNb"'] \arrow[dr, phantom, "\geq"] & GNB \arrow[d, "GNb"] \\ FNB' \arrow[r, "s_{NB'}"'] & GNB' \end{tikzcd}$$

Precomposition with N is obviously functorial: $(\mathrm{Id}_{\mathbb{A}})^* = \mathrm{Id}_{\mathbb{X}^{\mathrm{lax}\,\mathbb{A}}}$ and $(MN)^* = N^*M^*$ for any $N\colon \mathbb{C} \to \mathbb{B}$, $M\colon \mathbb{B} \to \mathbb{A}$ in $\mathcal{B}$. We therefore have a split indexed category. ☐

With only the assumption that $\mathbb{X}$ is a Pos-enriched category and $\mathcal{B}$ is a category of categories we have a (split) indexed category

$$\mathbb{X}^{\mathrm{lax}\,(-)}\colon \mathcal{B}^{\mathrm{op}} \to \mathsf{CAT}.$$

What can we say of this indexed category when $\mathbb{X}$ is a Pos-enriched linear category? First of all, to be able to handle the closed structure we are interested in the restriction to the indexed category

$$(\mathbb{X}^{l})^{\mathrm{lax}\,(-)}\colon \mathcal{B}^{\mathrm{op}} \to \mathsf{CAT}.$$

For this, we know from Section 3.1.2 that for each $\mathbb{A} \in |\mathcal{B}|$, the image $(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}$ is a linear category.

What about reindexing along $N\colon \mathbb{B} \to \mathbb{A}$?

Proposition 3.12. *Let $\mathbb{X}$ be a Pos-enriched linear category with symmetric monoidal closed structure $(\otimes, I, \multimap)$ and linear exponential comonad $(T, m^T, m^T_I, \varepsilon, \delta)$. For any functor $N\colon \mathbb{B} \to \mathbb{A}$, the functor $N^* = (\mathbb{X}^{l})^{\mathrm{lax}\,N}$ (reindexing along N) preserves the linear category structure of $(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}$, strictly.*

Proof. For any $N\colon \mathbb{B} \to \mathbb{A}$ in $\mathcal{B}$, the functor N^* is:

Strictly symmetric monoidal: For any $F, G \in \left|(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}\right|$, we have

$$N^*(F \otimes G) := (F \otimes G)N = FN \otimes GN = N^*(F) \otimes N^*(G)$$

while for the constant I functor, we have $N^*(I) := IN = I$.

Strictly closed: For any $F, G \in \left|(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}\right|$, we have

$$N^*(F \multimap G) := (F \multimap G)N = (FN \multimap GN) = N^*(F) \multimap N^*(G).$$

Strictly preserving the comonad: Let the symmetric monoidal comonad induced on $(\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}}$ be $(\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}, \hat{\varepsilon}, \hat{\delta})$ (as defined in Section 3.1.2) and $(\mathsf{T}^{\mathrm{lax}\,\mathbb{B}}, \hat{\varepsilon}', \hat{\delta}')$ the one on $(\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{B}}$.

Obviously $\mathsf{T}^{\mathrm{lax}\,\mathbb{B}}\mathsf{N}^* = \mathsf{N}^*\mathsf{T}^{\mathrm{lax}\,\mathbb{A}}$. Also preservation of $\hat{\varepsilon}$ and $\hat{\delta}$ is easy: By definition, the component of $\mathsf{N}^*(\hat{\varepsilon})$ at $\mathsf{F} \in \left|(\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}}\right|$ is given by the natural transformation $\varepsilon_{\mathsf{FN}}$. But $\varepsilon_{\mathsf{FN}}$ is also the component at F of $\hat{\varepsilon}'_{\mathsf{N}^*}$, and similarly for $\hat{\delta}$. That is, $\mathsf{N}^*(\hat{\varepsilon}) = \hat{\varepsilon}'_{\mathsf{N}^*}$ and $\mathsf{N}^*(\hat{\delta}) = \hat{\delta}'_{\mathsf{N}^*}$.

Thus, N^* is a strict linear functor. □

As an immediate consequence of what we have just observed, we get:

Corollary 3.13. *Let $\mathcal{B}$ be a category of categories and functors and $\mathbb{X}$ a Pos-enriched linear category. Then*

$$(\mathbb{X}^l)^{\mathrm{lax}\,(-)} \colon \mathcal{B}^{\mathrm{op}} \to \mathsf{Lin}$$

is a (split) indexed linear category; moreover each reindexing functor preserves the comonad strictly.

3.3 Generic object

We want to use the construction of an indexed linear category given in Section 3.2 to obtain a liner hyperdoctrine (Definition A.1).

We still assume that $\mathbb{X}$ is a Pos-enriched linear category with a Pos-enriched linear exponential comonad. Now we define $\mathcal{B}$ to be the following category:

Objects are finite products (in CAT) of $\mathbb{X}^l$, so there is $(\mathbb{X}^l)^0 = \mathbf{1}$ and $(\mathbb{X}^l)^m$ for $m = 1, 2, \ldots$.

Morphisms from $(\mathbb{X}^l)^m$ to $(\mathbb{X}^l)^n$ are n-tuples of functors $(\mathbb{X}^l)^m \to \mathbb{X}^l$.

In this way we force a *split generic object* for the split indexed linear category

$$(\mathbb{X}^l)^{\mathrm{lax}\,(-)} \colon \mathcal{B}^{\mathrm{op}} \to \mathsf{Lin}.$$

Of course, the category $\mathcal{B}$ has finite products and a distinguished object $\mathsf{U} := \mathbb{X}^l$ that generates every other object by finite products. Moreover by taking $\mathsf{G} := \mathrm{Id}_{\mathbb{X}^l}$ in $(\mathbb{X}^l)^{\mathrm{lax}\,\mathsf{U}}$ we obtain trivially a *split generic object* (U, G).

In such a structure it is possible to interpret type variables and type contexts and every typing rule except universal quantification. For this we need the indexed category to have *simple* U-*products* and this requires most of the effort in the construction of the linear hyperdoctrine.

3.4 Simple U-products and continuity

Consider the first projection in $\mathcal{B}$:

$$\mathsf{fst}_m : (\mathbb{X}^l)^m \times \mathbb{X}^l \to (\mathbb{X}^l)^m$$

To be able to handle universal quantification we need to obtain a right adjoint functor Π_m for any reindexing fst_m^*, $m = 1, 2, \ldots$. These right adjoint functors need to combine to satisfy the *Beck-Chevalley* condition. For a split indexed category, this amounts to the following requirement: For any $N\colon (\mathbb{X}^l)^m \to (\mathbb{X}^l)^n$ in $\mathcal{B}$, the canonical natural transformation $N^* \circ \Pi_n \overset{\bullet}{\to} \Pi_m \circ (N \times \mathrm{Id}_{\mathbb{X}})^*$ is the identity.

Such right adjoint functors can be obtained using the usual tool for the solution of recursive domain equations, that is *bifiniteness* combined with *continuity*.

Since we need bifiniteness, we now leave the generality of Pos-enriched categories and settle for subcategories of the category Dcpo.

Moreover we restrict ourselves to *continuous* lax transformations and *continuous* functors, in the sense that will soon be explained.

We will see how the structure of indexed linear category described so far is preserved by this restriction.

3.4.1 Continuous lax transformations

Definition 3.14. Let $F, G : \mathbb{A} \to \mathbb{X}^l$ be functors from a Pos-enriched category $\mathbb{A}$ to the subcategory of lower adjoint maps of a Dcpo-enriched category $\mathbb{X}$. A **continuous lax transformation** $s : F \overset{\odot}{\to} G$, is a lax transformation satisfying the following additional requirement: Whenever $\{a_i : A_i \to A\}_{i \in \mathbb{I}}$ is a colimiting cone for a directed diagram of *embeddings* $\{f_{ji}\colon A_i \to A_j\}_{i \leq j \in \mathbb{I}}$ in $\mathbb{A}$, then

$$\bigvee^{\uparrow}_{i \in \mathbb{I}} Ga_i \circ s_{A_i} \circ F^u a_i = s_A.$$

Remark. For a lax transformation $s\colon F \overset{\odot}{\to} G$ and any morphism $a_i \in \mathbb{A}(A_i, A)$ it is always the case that $Ga_i \circ s_{A_i} \circ F^u a_i \leq s_A$. The assignment $i \mapsto Ga_i \circ s_{A_i} \circ F^u a_i$ obviously preserves the order in $\mathbb{I}$, so it make sense to consider the supremum over $i \in \mathbb{I}$, having required $\mathbb{X}$ to be Dcpo-enriched.

Remark. We consider a notion of continuity that is different from what one might have expected: We consider only the colimit of a directed diagram of *embeddings* as opposed to the colimit of any directed diagram. The reason is the following. We are interested in functors between categories of lower adjoint maps as opposed to categories of embeddings, because in this way we obtain a smaller model. Continuity of lax transformations needs — as it will soon be shown — to combine with some *bifiniteness* assumptions, and bifiniteness is always expressed in terms of a colimit for a directed diagram of *embeddings* between finite objects.

Definition 3.15. Let $\mathbb{A}$ be a Pos-enriched and $\mathbb{X}$ a Dcpo-enriched category. Define $(\mathbb{X}^{\mathrm{l}})^{\mathrm{clax}\,\mathbb{A}}$ to be the subcategory of $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$ of functors and continuous lax transformations.

Remark. Recall from the definition of $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$ that, even if we consider functors which take values in $\mathbb{X}^{\mathrm{l}}$, the components of a (continuous) lax transformation are not required to be lower adjoint.

The reason for considering continuous lax transformations is to benefit from the assumption (soon to be made) that the objects of $\mathbb{A}$ are *bifinite*, as we see now.

3.4.2 Bifiniteness

The usual notion of bifiniteness is that of an object in Dcpo which is the colimit of a directed diagram of Scott-continuous embeddings (whose relative projections are also Scott-continuous) of finite objects. The bifinite domains form the category BF, a full subcategory of Dcpo. The category SDom of Scott-domains is a full subcategory of BF. Both BF and SDom are closed under directed colimits of embeddings in Dcpo.

But there is also a notion of linearly bifinite complete lattices (which is central in the current construction of a model of $\mathrm{F}_{\mathrm{DILL}}$). In this case one considers embedding-projection pairs in the subcategory Sup of Dcpo. A linearly bifinite complete lattice is the colimit of a directed diagram of linear embeddings (whose relative projection is also linear) of finite objects. These linearly bifinite complete lattices form the full subcategory BFSup of Sup. The category aCD of algebraic completely distributive lattices is an instance of a full subcategory of BFSup that is closed under directed colimits of embeddings in Sup.

We give a definition of subcategory of Dcpo of bifinite objects that includes at once BF, SDom, BFSup, and aCD.

Definition 3.16. Let $\mathbb{A}$ be a subcategory of Dcpo. Consider two objects $A, B \in |\mathbb{A}|$. We say that a map $b : B \to A$ is an **$\mathbb{A}$-embedding** if it is an embedding in $\mathbb{A}$ with the property that the relative projection $b^{\mathrm{p}} \colon A \to B$ is also a map in $\mathbb{A}$. In this case we say that B is a **$\mathbb{A}$-subobject** of A. Similarly we say that $b \colon B \to A$ is a **$\mathbb{A}$-lower adjoint** if both b and its upper adjoint b^{u} are maps in $\mathbb{A}$.

Remark. We use some care in this otherwise obvious definition because it is common use in domain theory to consider embedding-projection pairs in the ambient category Pos. For example since any embedding preserves all existing suprema, an embedding between complete lattices will always be a map of Sup, while its upper adjoint need not to be.

Definition 3.17. Let $\mathbb{A}$ be a subcategory of Dcpo. We say that $\mathbb{A}$ is **closed under directed colimits of embeddings** if whenever $\{a_{ji}\colon A_i \to A_j\}_{i,j\in\mathbb{I}}$ is a directed diagram of $\mathbb{A}$-embeddings then any colimiting cocone in Dcpo is a cocone of $\mathbb{A}$-embeddings. We say that $\mathbb{A}$ is an **admissible** subcategory of Dcpo if it is closed under directed colimits of embeddings and it is such that for any pair of objects $A, B \in |\mathbb{A}|$ the hom-set $\mathbb{A}(A, B) \subseteq \mathsf{Dcpo}(A, B)$ is closed under directed suprema (that is, $\mathbb{A}$ is a Dcpo-enriched subcategory of Dcpo).

Example. The categories BF, SDom, BFSup, aCD are all admissible subcategories of Dcpo.

Remark. Recall that in Dcpo a directed diagram of embeddings $\{a_{ji}\colon A_i \to A_j\}_{i\leq j\in\mathbb{I}}$ is completely determined by any universal cocone $\{a_i\colon A_i \to A\}_{i\in\mathbb{I}}$, since for any $i \leq j$ in $\mathbb{I}$ we have $a_{ji} = a_j^p \circ a_i$ (with relative projection $a_{ij}^p = a_i^p \circ a_j$). Moreover $\{a_i\colon A_i \to A\}_{i\in\mathbb{I}}$ is a universal cocone (in Dcpo and Dcpo^e) if and only if $\bigvee^{\uparrow}_{i\in\mathbb{I}} a_i \circ a_i^p = \mathrm{id}_A$. The same is true in any admissible subcategory. For this reason we say that a universal cocone $\{a_i\colon A_i \to A\}_{i\in\mathbb{I}}$ is a *representation* of A as directed colimit of embeddings.

Definition 3.18. Let $\mathbb{A}$ be an admissible subcategory of Dcpo. We say that an object A of $\mathbb{A}$ is $\mathbb{A}$**-bifinite** if it is the colimit of a directed diagram of $\mathbb{A}$-embeddings between finite objects.

Example. For $\mathbb{A} = \mathsf{Dcpo}$ this coincides with the usual notion of bifinite domain and for $\mathbb{A} = \mathsf{Sup}$ we obtain the notion of linearly bifinite complete lattice. Since Sup is an admissible subcategory of Dcpo, any Sup-bifinite object is also Dcpo-bifinite, while a Dcpo-bifinite object of Sup need not be Sup-bifinite.

Consider the case $\mathbb{A} = \mathsf{SDom}$. This is a full subcategory of Dcpo and it is closed under directed colimits of embeddings, hence a SDom-bifinite object is simply a Dcpo-bifinite object of SDom (and of course each object of SDom is such).

The next result is a well known property of finite domains reformulated in the more general setting of this presentation. It plays a central part in the proof of Theorem 3.23.

Lemma 3.19. *Let $\mathbb{A}$ be an admissible subcategory of Dcpo. Consider a directed colimit of $\mathbb{A}$-embeddings $\{a_i\colon A_i \to A\}_{i\in\mathbb{I}}$. For any $\mathbb{A}$-lower adjoint $b\colon B \to A$ from a finite object B there exists an index $i \in \mathbb{I}$ and an $\mathbb{A}$-lower adjoint $f\colon B \to A_i$ such that $a_i \circ f = b$.*

Proof. For any $y \in B$, we have $b(y) = \bigvee^{\uparrow}_{i\in\mathbb{I}} a_i \circ a_i^p\,(b(y))$. Since $b(y)$ is compact[1] in $\mathbb{A}$ there is an $i_y \in \mathbb{I}$ such that $b(y) = a_{i_y} \circ a_{i_y}^p\,(b(y))$. Take then any $i \geq i_y$, for each

[1]Refer to [3] for the definition of compact elements of a dcpo.

$y \in B$, and define $f := a_i^p \circ b \colon B \to A_i$. This satisfies $a_i \circ f = a_i \circ a_i^p \circ b = b$ because by construction $a_i \circ a_i^p$ fixes any element in the image of b. Moreover it is easy to check that f is an $\mathbb{A}$-lower adjoint with upper adjoint $b^u \circ a_i$. ☐

We now fix a small category of representative objects for finite posets.

Definition 3.20. Let $\mathsf{Pos}_{\mathrm{fin}}$ be the full subcategory of Pos of those posets whose support is a finite subset of the set $\mathbb{N}$ of natural numbers. For any subcategory $\mathbb{A}$ of Pos, define $\mathbb{A}_{\mathrm{fin}} := \mathbb{A} \cap \mathsf{Pos}_{\mathrm{fin}}$.

The following is then obvious:

Lemma 3.21. *Let $\mathbb{A}$ be an admissible subcategory of Dcpo. Any $\mathbb{A}$-bifinite object is the colimit of a directed diagram of embeddings in $\mathbb{A}_{\mathrm{fin}}$.*

Definition 3.22. Let $\mathbb{A}$ be an admissible subcategory of Dcpo and let $\mathbb{X}$ be a Dcpo-enriched category. There is an obvious restriction functor:

$$\mathrm{res}\colon (\mathbb{X}^l)^{\mathrm{clax}\,\mathbb{A}^l} \to (\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}^l_{\mathrm{fin}}}$$

determined by the inclusion $\mathbb{A}^l_{\mathrm{fin}} \subseteq \mathbb{A}^l$.

Next, we are going to prove that when the objects of $\mathbb{A}$ are assumed to be bifinite, the restriction functor just defined is full and faithful. This is the crucial result to handle universal quantification.

Theorem 3.23. *Let $\mathbb{A}$ be an admissible subcategory of Dcpo whose objects are $\mathbb{A}$-bifinite and $\mathbb{X}$ a Dcpo-enriched category. For any pair of functors $F, G \colon \mathbb{A}^l \to \mathbb{X}^l$ there is an order isomorphism between $(\mathbb{X}^l)^{\mathrm{clax}\,\mathbb{A}^l}(F, G)$ (the class of continuous lax transformations $F \stackrel{\odot}{\to} G$) and $(\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}^l_{\mathrm{fin}}}(\mathrm{res}\,F, \mathrm{res}\,G)$ (the set of lax transformations $\mathrm{res}\,F \stackrel{\odot}{\to} \mathrm{res}\,G$).*

Proof. One direction of the order isomorphisms is given by the action of the functor res on the hom-objects

$$\mathrm{res}\colon (\mathbb{X}^l)^{\mathrm{clax}\,\mathbb{A}^l}(F, G) \to (\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}^l_{\mathrm{fin}}}(\mathrm{res}\,F, \mathrm{res}\,G)$$

is the obvious order morphism which assigns to any continuous lax transformation $s \colon F \stackrel{\odot}{\to} G$ the lax transformation $\mathrm{res}(s) \colon \mathrm{res}\,F \stackrel{\odot}{\to} \mathrm{res}\,G$ whose components are $(\mathrm{res}\,s)_B := s_B$ for any $B \in |\mathbb{A}_{\mathrm{fin}}|$.

We now consider how to obtain an inverse morphism

$$\mathrm{ext}\colon (\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}^l_{\mathrm{fin}}}(\mathrm{res}\,F, \mathrm{res}\,G) \to (\mathbb{X}^l)^{\mathrm{clax}\,\mathbb{A}^l}(F, G)\,.$$

If $t\colon \operatorname{res} F \overset{\odot}{\to} \operatorname{res} G$ is a lax transformation, we define a continuous lax transformation $F \overset{\odot}{\to} G$ as follows. For any $A \in |\mathbb{A}|$, since A is bifinite, there is a representation of it as a directed colimit of $\mathbb{A}$-embeddings

$$\{a_i\colon A_i \to A\}_{i \in \mathbb{I}}$$

where $A_i \in |\mathbb{A}_{\mathrm{fin}}|$ for any $i \in \mathbb{I}$.

Therefore, we define the component of $\operatorname{ext} t$ at A to be

$$(\operatorname{ext} t)_A := \bigvee^{\uparrow}_{i \in \mathbb{I}} Ga_i \circ t_{A_i} \circ F^u a_i.$$

First notice that the supremum exists since the family over which is taken is directed. Moreover $(\operatorname{ext} t)_A$ does not depend on the representation of A as directed colimit of embeddings: If $\{b_j\colon B_j \to A\}_{j \in \mathbb{J}}$ is another representation then, using Lemma 3.19, it is easy to see that $\bigvee^{\uparrow}_{j \in \mathbb{J}} Gb_j \circ t_{B_j} \circ F^u b_j = \bigvee^{\uparrow}_{i \in \mathbb{I}} Ga_i \circ t_{A_i} \circ F^u a_i$.

We now check that we obtain a lax transformation. For any $f\colon A \to B$ in $\mathbb{A}^l$, let $\{a_i\colon A_i \to A\}_{i \in \mathbb{I}}$ and $\{b_j\colon B_j \to B\}_{j \in \mathbb{J}}$ be respective representations of A and B as directed colimits of embeddings of finite objects.

For any $i \in \mathbb{I}$, we have an $\mathbb{A}$-lower adjoint $f \circ a_i\colon A_i \to B$ and by Lemma 3.19 an index $j_i \in \mathbb{J}$ and an $\mathbb{A}$-lower adjoint $x^{(i)}\colon A_i \to B_{j_i}$ such that $f \circ a_i = b_{j_i} \circ x^{(i)}$. Using this we obtain

$$\begin{aligned}
&Gf \circ (\operatorname{ext} t)_A \circ F^u f \\
&\quad = Gf \circ \Big(\bigvee^{\uparrow}_{i \in \mathbb{I}} Ga_i \circ t_{A_i} \circ F^u a_i\Big) \circ F^u f && (\mathbb{X}\ \textsf{Dcpo}\text{-enr.})\\
&\quad = \bigvee^{\uparrow}_{i \in \mathbb{I}} G(f \circ a_i) \circ t_{A_i} \circ F^u(f \circ a_i) \\
&\quad = \bigvee^{\uparrow}_{i \in \mathbb{I}} G(b_{j_i} \circ x^{(i)}) \circ t_{A_i} \circ F^u(b_{j_i} \circ x^{(i)}) && (t \text{ lax})\\
&\quad \leq Gb_{j_i} \circ t_{B_{j_i}} \circ F^u b_{j_i} \\
&\quad \leq (\operatorname{ext} t)_B.
\end{aligned}$$

Obviously, $(\operatorname{ext} t)_B = B$ for any $B \in |\mathbb{A}_{\mathrm{fin}}|$. We now check that $\operatorname{ext} t$ is a continuous lax transformation. Assume that $\{a_i\colon A_i \to A\}_{i \in \mathbb{I}}$ is a directed colimit of embeddings. Let $\{b_j\colon B_j \to A\}_{j \in \mathbb{J}}$ be a representation of A and $\left\{c_{k^{(i)}}\colon C_{k^{(i)}} \to A_i\right\}_{k^{(i)} \in \mathbb{K}^{(i)}}$ a representation of A_i as directed colimit of embeddings of finite objects.

We now apply Lemma 3.19 twice: For any $b_j\colon B_j \to A$ there exists an $i_j \in \mathbb{I}$ and a lower adjoint $f_j\colon B_j \to A_{i_j}$ such that $a_{i_j} \circ f_j = b_j$. But there exists also a $k^{(i)}_j \in \mathbb{K}^{(i)}$

and a lower adjoint $g_j \colon B_j \to C_{k_j^{(i)}}$ such that $c_{k_j^{(i)}} \circ g_j = f_j$. We therefore have

$$
\begin{aligned}
(\operatorname{ext} t)_A = & \\
&= \bigvee_{j\in\mathbb{J}}^{\uparrow} Gb_j \circ t_{B_j} \circ F^u b_j \\
&= \bigvee_{j\in\mathbb{J}}^{\uparrow} Ga_{i_j} \circ Gc_{k_j^{(i)}} \circ Gg_j \circ t_{B_j} \circ F^u G g_j \circ F^u c_{k_j^{(i)}} \circ F^u a_{i_j} && (t \text{ lax}) \\
&\leq \bigvee_{j\in\mathbb{J}}^{\uparrow} Ga_{i_j} \circ Gc_{k_j^{(i)}} \circ t_{C_{k_j^{(i)}}} \circ F^u c_{k_j^{(i)}} \circ F^u a_{i_j} \\
&\leq \bigvee_{j\in\mathbb{J}}^{\uparrow} Ga_{i_j} \circ (\operatorname{ext} t)_{A_i} \circ F^u a_{i_j} && (\operatorname{ext} t \text{ lax}) \\
&\leq (\operatorname{ext} t)_A,
\end{aligned}
$$

that is, $(\operatorname{ext} t)_A = \bigvee^{\uparrow}_{i\in\mathbb{I}} Ga_i \circ (\operatorname{ext} t)_{A_i} \circ F^p a_i$ since the other inequality follows by laxness of $\operatorname{ext} t$.

By construction, ext preserves the order between lax transformations. The fact that $(\operatorname{ext} t)_B = t_B$ for any $B \in |\mathbb{A}_{\mathrm{fin}}|$ means that $\mathrm{res} \circ \mathrm{ext}$ is the identity. To check that also $\mathrm{ext} \circ \mathrm{res}$ is the identity, notice that if $s \colon F \overset{\odot}{\to} G$ is a continuous lax transformation, for any representation $\{a_i \colon A_i \to A\}_{i\in\mathbb{I}}$ of A as bifinite object, we have

$$((\mathrm{ext} \circ \mathrm{res})s)_A := \bigvee_{i\in\mathbb{I}}^{\uparrow} Ga_i \circ s_{A_i} \circ F^u a_i = s_A.$$

□

In other words:

Corollary 3.24. *If $\mathbb{A}$ is an admissible subcategory of* Pos *whose objects are $\mathbb{A}$-bifinite and $\mathbb{X}$ is a* Dcpo*-enriched category, then the functor*

$$\mathrm{res} \colon (\mathbb{X}^l)^{\mathrm{clax}\,\mathbb{A}^l} \to (\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}^l_{\mathrm{fin}}}$$

determined by the inclusion $\mathbb{A}_{\mathrm{fin}} \subseteq \mathbb{A}$ is a fully faithful Pos*-functor.*

Continuous lax transformations are determined by the restriction to finite objects. Because of the restriction $\mathrm{res}\, F$ of a functor F to finite objects it is possible to consider the lax thread construction in Dcpo. In most cases this determines the lax limit of $\mathrm{res}\, F$ in $\mathbb{X}$ (we do require that $\mathbb{X}$ has lax limits of this shape; this happens for example for Dcpo, BF, SDom, Sup, BFSup, aCD).

In this way we get a Pos-adjunction

$$\mathrm{laxlim} \dashv \mathrm{diag} \colon (\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}_{\mathrm{fin}}} \to \mathbb{X}.$$

This allows us to model universal quantification. Before we see how in Section 3.5 we need to pause: By restricting ourselves to continuous lax transformations, we no longer have a linear category!

3.4.3 Pruning $(\mathbb{X}^{\mathsf{l}})^{\mathrm{lax}\,\mathbb{A}^{\mathsf{l}}}$ with continuity

In Section 3.1.2 we have seen that whenever $\mathbb{A}$ is a subcategory of Pos and $\mathbb{X}$ a Dcpo-enriched linear category, the category $(\mathbb{X}^{\mathsf{l}})^{\mathrm{lax}\,\mathbb{A}^{\mathsf{l}}}$ of functors and lax transformations inherits the structure of a linear category.

This structure does not restrict to the subcategory $(\mathbb{X}^{\mathsf{l}})^{\mathrm{clax}\,\mathbb{A}^{\mathsf{l}}}$ of continuous lax transformations.

The only difficulty occurs when one checks the adjunction between tensor and function space: the components of unit and counit of this adjunction are not in general continuous lax transformations.

There are two ways of describing the same route out of this problem. The solution is to consider only *continuous* functors, that is — with a well-established abuse of the word *continuous* — functors that preserve directed colimits of embeddings.

The first description is more abstract: Once we consider continuous functors, the full and faithful functor res becomes part of an equivalence of categories. The linear category structure of $(\mathbb{X}^{\mathsf{l}})^{\mathrm{lax}\,\mathbb{A}^{\mathsf{l}}_{\mathrm{fin}}}$ will therefore transfer to the equivalent category of continuous functors $\mathbb{A}^{\mathsf{l}} \to \mathbb{X}^{\mathsf{l}}$ and continuous lax transformations.

But it is also possible to show this directly and without much effort.

Definition 3.25. Let $\mathbb{A}$ and $\mathbb{X}$ be admissible subcategories of Dcpo. A functor $\mathbb{A}^{\mathsf{l}} \to \mathbb{X}^{\mathsf{l}}$ is said to be **continuous** if it carries directed colimits of $\mathbb{A}$-embeddings to directed colimits of $\mathbb{X}$-embeddings. Define $\mathsf{C}\big((\mathbb{X}^{\mathsf{l}})^{\mathrm{clax}\,\mathbb{A}^{\mathsf{l}}}\big)$ to be the category whose objects are continuous functors $\mathbb{A}^{\mathsf{l}} \to \mathbb{X}^{\mathsf{l}}$ and whose morphisms are continuous lax transformations.

Proposition 3.26. *Let $\mathbb{A}$ be an admissible subcategory of* Dcpo *and $\mathbb{X}$ a* Dcpo*-enriched linear category. The category $\mathsf{C}\big((\mathbb{X}^{\mathsf{l}})^{\mathrm{clax}\,\mathbb{A}^{\mathsf{l}}}\big)$ of continuous functors and continuous lax transformations inherits the linear category structure from $\mathbb{X}$.*

Proof. Let $(\otimes, \multimap, \mathsf{T})$ be the tensor, function space and linear exponential comonad of the linear category $\mathbb{X}$. Whenever $\mathsf{F}, \mathsf{G} \colon \mathbb{A}^{\mathsf{l}} \to \mathbb{X}^{\mathsf{l}}$ are continuous functors, so are $\mathsf{F} \otimes \mathsf{G}$, $\mathsf{F} \multimap \mathsf{G}$ and TF, since each of $\otimes$, $\multimap$ and T is Dcpo-enriched. For the same reason, if s and t are continuous lax transformations, also $s \otimes t$, $s \multimap t$ and $\mathsf{T}s$ are continuous.

For any continuous functor $\mathsf{G} \colon \mathbb{A}^{\mathsf{l}} \to \mathbb{X}^{\mathsf{l}}$, consider the adjunction $(- \otimes \mathsf{G}) \dashv (\mathsf{G} \multimap -)$ in the category $(\mathbb{X}^{\mathsf{l}})^{\mathrm{lax}\,\mathbb{A}^{\mathsf{l}}}$. As shown in the proof of Proposition 3.7 this has the natural transformations $\widehat{\mathrm{ev}}^{(\mathsf{G})}$ and $\widehat{\mathrm{ve}}^{(\mathsf{G})}$ as counit and unit.

We need to verify that, for any continuous functor $F\colon \mathbb{A}^l \to \mathbb{X}^l$, the components $\widehat{\mathrm{ev}}^{(G)}_F$ $\widehat{\mathrm{ve}}^{(G)}_F$ are continuous lax transformations $(G \multimap F) \otimes G \overset{\odot}{\to} F$ and $F \overset{\odot}{\to} (G \multimap (F \otimes G)$, respectively.

We know from Proposition 3.7 that these are lax transformations. What we need to check is continuity. Whenever $\{a_i\colon A_i \to A\}_{i\in\mathbb{I}}$ is a directed colimit of $\mathbb{A}$-embeddings in $\mathbb{A}^l$ and F is continuous, we have $\bigvee^{\uparrow}_{i\in\mathbb{I}} Fa_i \circ F^p a_i = \mathrm{id}_{FA}$. [2] Similarly, as G is continuous we get $\bigvee^{\uparrow}_{i\in\mathbb{I}} Ga_i \circ G^p a_i = \mathrm{id}_{GA}$. Therefore

$$
\begin{aligned}
&\textstyle\bigvee^{\uparrow}_{i\in\mathbb{I}} Fa_i \circ \mathrm{ev}^{(GA_i)}_{FA_i} \circ ((Ga_i \multimap F^p a_i) \otimes G^p a_i) \\
&= \textstyle\bigvee^{\uparrow}_{i\in\mathbb{I}} \mathrm{ev}^{(GA_i)}_{FA} \circ ((\mathrm{id}_{GA_i} \multimap Fa_i) \otimes \mathrm{id}_{GA}) \circ ((Ga_i \multimap F^p a_i) \otimes G^p a_i) \\
&= \textstyle\bigvee^{\uparrow}_{i\in\mathbb{I}} \mathrm{ev}^{(GA_i)}_{FA} \circ ((Ga_i \multimap \mathrm{id}_{FA}) \otimes \mathrm{id}_{GA_i}) \circ ((\mathrm{id}_{GA} \multimap (Fa_i \circ F^p a_i)) \otimes G^p a_i) \\
&= \textstyle\bigvee^{\uparrow}_{i\in\mathbb{I}} \mathrm{ev}^{(GA)}_{FA} \circ ((\mathrm{id}_{GA} \multimap \mathrm{id}_{FA}) \otimes Ga_i) \circ ((\mathrm{id}_{GA} \multimap (Fa_i \circ F^p a_i)) \otimes G^p a_i) \\
&= \textstyle\bigvee^{\uparrow}_{i\in\mathbb{I}} \mathrm{ev}^{(GA)}_{FA} \circ ((\mathrm{id}_{GA} \multimap (Fa_i \circ F^p a_i)) \otimes (Ga_i \circ G^p a_i)) \\
&= \mathrm{ev}^{(GA)}_{FA}
\end{aligned}
$$

where the first equality uses naturality of $\mathrm{ev}^{(GA_i)}$ and the third extranaturality of $\mathrm{ev}^{()}_{FA}$. A similar equation proves continuity of $\widehat{\mathrm{ve}}^{(G)}_F$.

The linear exponential comonad induced on $(\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}}$ by the one on $\mathbb{X}$ is such that every natural transformation involved has components which are also natural and are therefore in particular continuous lax transformations, that is, the linear exponential comonad restricts without any trouble to $\mathsf{C}\big((\mathbb{X}^e)^{\mathrm{clax}\,\mathbb{A}^e}\big)$. ☐

Lemma 3.27. *Let $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{X}$ be admissible subcategories of* Dcpo. *If $N\colon \mathbb{B}^l \to \mathbb{A}^l$ is a continuous functor, the reindexing functor $N^*\colon (\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}^l} \to (\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{B}^l}$ restricts and corestricts to a functor $\mathsf{C}\big((\mathbb{X}^l)^{\mathrm{clax}\,\mathbb{A}^l}\big) \to \mathsf{C}\big((\mathbb{X}^l)^{\mathrm{clax}\,\mathbb{B}^l}\big)$.*

Proof. For any continuous functor $F\colon \mathbb{A}^l \to \mathbb{X}^l$ its image under reindexing $N^*(F) = FN$ is obviously continuous. For any continuous lax transformations $s\colon F \overset{\odot}{\to} F'$ in $\mathsf{C}\big((\mathbb{X}^l)^{\mathrm{clax}\,\mathbb{A}^l}\big)$ the lax transformation $N^*(s) = s_N$ is also continuous (if $(a_i\colon A_i \to A)_{i\in\mathbb{I}}$ is a directed colimit of embeddings in $\mathbb{A}^l$, then $(Na_i\colon NA_i \to NA)_{i\in\mathbb{I}}$ is a directed colimit in $\mathbb{B}^l$). ☐

[2]We use the exponent p to remark that $F^p a_i$ is a projection (because of the definition of continuous functor) even if the functor takes values in the cat of lower adjoint maps.

3.5 A linear hyperdoctrine

Let $\mathbb{X}$ be an admissible subcategory of Dcpo, with a structure of Dcpo-enriched linear category $(\otimes, \mathrm{I}, \multimap, \mathsf{T})$. Assume that each object of $\mathbb{X}$ is $\mathbb{X}$-bifinite.

Now we also ask for $\mathbb{X}$ to have lax limits of shape $\mathbb{X}^{\mathfrak{l}}_{\text{fin}}$.

Let $\mathcal{B}$ be the category whose

objects are finite products (in CAT) of $\mathbb{X}^{\mathfrak{l}}$, so there is $(\mathbb{X}^{\mathfrak{l}})^0 = \mathbf{1}$ and $(\mathbb{X}^{\mathfrak{l}})^{\mathfrak{m}}$ for $\mathfrak{m} = 1, 2, \ldots$ and whose

morphisms from $(\mathbb{X}^{\mathfrak{l}})^{\mathfrak{m}}$ to $(\mathbb{X}^{\mathfrak{l}})^{\mathfrak{n}}$ are n-tuples of continuous functors $(\mathbb{X}^{\mathfrak{l}})^{\mathfrak{m}} \to \mathbb{X}^{\mathfrak{l}}$.

As an immediate consequence of what we observed in Section 3.4.3 (and Section 3.3)

$$\mathsf{C}\big((\mathbb{X}^{\mathfrak{l}})^{\mathrm{clax}\,(-)}\big) \colon \mathcal{B}^{\mathrm{op}} \to \mathsf{Lin}$$

is a split indexed linear category with generic object $(\mathsf{U} := \mathbb{X}^{\mathfrak{l}}, \mathsf{G} := \mathrm{Id}_{\mathbb{X}^{\mathfrak{l}}})$.

We now show that this indexed linear category has simple U-products, that is that for any projection

$$\mathrm{fst}_{\mathfrak{m}} \colon (\mathbb{X}^{\mathfrak{l}})^{\mathfrak{m}} \times \mathbb{X}^{\mathfrak{l}} \to (\mathbb{X}^{\mathfrak{l}})^{\mathfrak{m}}$$

the reindexing

$$\mathrm{fst}^*_{\mathfrak{m}} \colon \mathsf{C}\big((\mathbb{X}^{\mathfrak{l}})^{\mathrm{clax}\,(\mathbb{X}^{\mathfrak{l}})^{\mathfrak{m}}}\big) \to \mathsf{C}\big((\mathbb{X}^{\mathfrak{l}})^{\mathrm{clax}\,(\mathbb{X}^{\mathfrak{l}})^{\mathfrak{m}} \times \mathbb{X}^{\mathfrak{l}}}\big)$$

has a right adjoint $\Pi_{\mathfrak{m}}$ and that these combine to satisfy the Beck-Chevalley condition.

Recall from Section 3.4 that for any $\mathbb{B}$ in $\mathcal{B}$ restriction to finite objects determines an equivalence of categories

$$\mathrm{res} \colon \mathsf{C}\big((\mathbb{X}^{\mathfrak{l}})^{\mathrm{clax}\,\mathbb{B}}\big) \to (\mathbb{X}^{\mathfrak{l}})^{\mathrm{lax}\,\mathbb{B}_{\text{fin}}}.$$

So we check the adjunction between the equivalent categories.

3.5.1 Universal quantification

Throughout this section we use the abbreviations $\mathbb{A} = \mathbb{X}^{\mathfrak{l}}_{\text{fin}}$ and $\mathbb{B} = \big(\mathbb{X}^{\mathfrak{l}}_{\text{fin}}\big)^{\mathfrak{m}}$. We look for a right adjoint functor to the reindexing along the first projection:

$$(\mathbb{X}^{\mathfrak{l}})^{\mathrm{lax}\,\mathbb{B}\times\mathbb{A}} \xleftarrow{\mathrm{fst}^*} (\mathbb{X}^{\mathfrak{l}})^{\mathrm{lax}\,\mathbb{B}}$$

We required that $\mathbb{X}$ has lax limits[3] of shape $\mathbb{A}$. Thus we have an adjunction

$$\begin{tikzcd} (\mathbb{X}^{\mathfrak{l}})^{\mathrm{lax}\,\mathbb{A}_{\text{fin}}} \arrow[r, shift left, "\mathrm{laxlim}"] \arrow[r, phantom, "\perp"] & \mathbb{X} \arrow[l, shift left, "\mathrm{diag}"] \end{tikzcd}$$

[3] Recall that in Dcpo these are given by the object of lax threads.

Notice that the codomain of the functor laxlim is $\mathbb{X}$ and not $\mathbb{X}^l$.

Any Pos-adjunction between Pos-enriched categories lifts to a similarly enriched adjunction between categories of lax transformations:

Lemma 3.28. *Let $\mathbb{C}$ be a category and*

$$\begin{tikzcd} \mathbb{Y} \arrow[r, shift left=2, "G"] \arrow[r, phantom, "\top"] & \mathbb{Z} \arrow[l, shift left=2, "F"] \end{tikzcd}$$

a Pos-enriched adjunction. Postcomposing functors and lax transformations with the functors of this adjunction provides a Pos-enriched adjunction:

$$\begin{tikzcd} \mathbb{Y}^{\mathrm{lax}\,\mathbb{C}} \arrow[r, shift left=2, "G^{\mathrm{lax}\,\mathbb{C}}"] \arrow[r, phantom, "\top"] & \mathbb{Z}^{\mathrm{lax}\,\mathbb{C}} \arrow[l, shift left=2, "F^{\mathrm{lax}\,\mathbb{C}}"] \end{tikzcd}$$

Proof. The Pos-adjunction $F \dashv G\colon \mathbb{Z} \to \mathbb{Y}$ amounts to natural Pos-isomorphisms

$$n_{Z,Y}\colon \mathbb{Y}(FZ, Y) \cong \mathbb{Z}(Z, GY)$$

for any $Y \in |\mathbb{Y}|$, $Z \in |\mathbb{Z}|$. Consider functors $N\colon \mathbb{C} \to \mathbb{Z}$ and $M\colon \mathbb{C} \to \mathbb{Y}$ and a lax transformation $s\colon F^{\mathrm{lax}\,\mathbb{C}}N = FN \overset{\odot}{\to} M$. This is a family of $\mathbb{Y}$-morphisms $\{s_C\colon FNC \to MC\}_{C\in|\mathbb{C}|}$ and corresponds to a family of $\mathbb{Z}$-morphisms $\{ns_C\colon NC \to GMC\}_{C\in|\mathbb{C}|}$ under the isomorphisms $n_{NC,MC}$. This is a lax transformation $N \overset{\odot}{\to} GM$ since by naturality of n, for any $c\colon C \to C'$ in $\mathbb{C}$, $GMc \circ ns_C = n(MC \circ s_C)$ while $ns_{C'} \circ Nc = n(s_{C'} \circ FNc)$ and n is an order isomorphism. We therefore have a map

$$\mathbb{Y}^{\mathrm{lax}\,\mathbb{C}}\big(F^{\mathrm{lax}\,\mathbb{C}}N, M\big) \cong \mathbb{X}^{\mathrm{lax}\,\mathbb{C}}\big(N, G^{\mathrm{lax}\,\mathbb{C}}M\big)$$

defined by transposing each component of a lax transformation. This is obviously monotone, as the components of n are monotone. Using the inverse of the isomorphism n we obtain a monotone inverse of this map. That is, there are natural Pos-isomorphism

$$\mathbb{Y}^{\mathrm{lax}\,\mathbb{C}}\big(F^{\mathrm{lax}\,\mathbb{C}}N, M\big) \cong \mathbb{X}^{\mathrm{lax}\,\mathbb{C}}\big(N, G^{\mathrm{lax}\,\mathbb{C}}M\big).$$

□

Using this fact we obtain a Pos-adjunction:

$$\begin{tikzcd} \big((\mathbb{X}^l)^{\mathrm{lax}\,\mathbb{A}}\big)^{\mathrm{lax}\,\mathbb{B}} \arrow[r, shift left=2, "\mathrm{laxlim}^{\mathrm{lax}\,\mathbb{B}}"] \arrow[r, phantom, "\top"] & \mathbb{X}^{\mathrm{lax}\,\mathbb{B}} \arrow[l, shift left=2, "\mathrm{diag}^{\mathrm{lax}\,\mathbb{B}}"] \end{tikzcd}$$

We notice next that the curry functor of the category of functors and natural transformations extends to a functor between categories of lax transformations.

Lemma 3.29. *Currying in the category of functors and natural transformations extends to a fully faithful* Pos*-functor*

$$\mathrm{curry}\colon (\mathbb{X}^{l})^{\mathrm{lax}\,(\mathbb{B}\times\mathbb{A})} \to \big((\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}\big)^{\mathrm{lax}\,\mathbb{B}}.$$

Proof. For any $F\colon \mathbb{B}\times\mathbb{A}\to\mathbb{X}^{l}$, the functor $\mathrm{curry}\,F\colon \mathbb{B}\to(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}$ is exactly what we expect. Notice that it takes values in the subcategory $\mathrm{Funct}(\mathbb{A},\mathbb{X}^{l})$ of functors and natural transformations and that for every $b\in\mathbb{B}(B,B')$ the natural transformation $(\mathrm{curry}\,F)b$ has components which are lower adjoint maps.

If $s\colon F\overset{\odot}{\to}F'$ is a lax transformation between objects of $(\mathbb{X}^{l})^{\mathrm{lax}\,(\mathbb{B}\times\mathbb{A})}$, then we define $\mathrm{curry}\,s\colon\ \mathrm{curry}\,F\overset{\odot}{\to}\mathrm{curry}\,F'$ exactly as if s was a natural transformation: $(\mathrm{curry}\,s)_B := s_{(B,-)}$ for any $B\in|\mathbb{B}|$. It is easy to see that this is a lax transformation, where each component is a lax transformation. Moreover, if $s\le t : F\overset{\odot}{\to}F'$, then obviously $\mathrm{curry}\,s\le\mathrm{curry}\,t$.

We now check that curry is fully faithful as a Pos-enriched functor. Any lax transformation $t\colon\ \mathrm{curry}\,F\overset{\odot}{\to}\mathrm{curry}\,G$ corresponds to a family

$$\big\{(t_B)_A\colon F(B,A)\to G(B,A)\big\}_{(B,A)\in|\mathbb{B}\times\mathbb{A}|}$$

of morphisms in $\mathbb{X}$. This family defines a lax transformation $F\overset{\odot}{\to}G$, since for any $(b,a)\colon(B,A)\to(B',A')$ in $\mathbb{B}\times\mathbb{A}$

$$\begin{tikzcd}
F(B,A) \arrow[r, "(t_B)_A"] \arrow[d, "F(b,\mathrm{id}_A)"'] \arrow[dr, phantom, "\geq"] & G(B,A) \arrow[d, "G(b,\mathrm{id}_A)"] \\
F(B',A) \arrow[r, "(t'_B)_A"] \arrow[d, "F(\mathrm{id}_{B'},a)"'] \arrow[dr, phantom, "\geq"] & G(B',A) \arrow[d, "G(\mathrm{id}_{B'},a)"] \\
F(B',A') \arrow[r, "(t'_B)_{A'}"] & G(B',A')
\end{tikzcd}$$

where the upper inequality is the instance at A of the fact that t is a lax transformation, while the lower inequality expresses that $t_{B'}$ is a lax transformation $(\mathrm{curry}\,F)B'\overset{\odot}{\to}(\mathrm{curry}\,G)B'$. The assignment just described is a monotone map

$$U : \big((\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}\big)^{\mathrm{lax}\,\mathbb{B}}(\mathrm{curry}\,F,\mathrm{curry}\,G)\to(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{B}\times\mathbb{A}}(F,G).$$

If $s : F\overset{\odot}{\to}G$ is a lax transformation in $(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{B}\times\mathbb{A}}$, one obviously has $U(\mathrm{curry}\,s)=s$, and so we have a Pos-isomorphism

$$(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{B}\times\mathbb{A}}(F,G)\cong\big((\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}\big)^{\mathrm{lax}\,\mathbb{B}}(\mathrm{curry}\,F,\mathrm{curry}\,G).$$

$\square$

Corollary 3.30. *The composition* $\mathrm{laxlim}^{\mathrm{lax}\,\mathbb{B}} \circ \mathrm{curry}$ *factors through* $(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{B}} \subseteq \mathbb{X}^{\mathrm{lax}\,\mathbb{B}}$ *considered as an inclusion.*

Proof. For a functor $F \colon \mathbb{B} \times \mathbb{A} \to \mathbb{X}^{l}$ consider $(\mathrm{laxlim}^{\mathrm{lax}\,\mathbb{B}} \circ \mathrm{curry})F = \mathrm{laxlim} \circ (\mathrm{curry}\, F)$. As pointed out in the proof of Lemma 3.29, for any $b \in \mathbb{B}(B, B')$, $(\mathrm{curry}\, F)\, b$ is a natural transformation whose components are lower adjoint maps. It is easy to see that the family of relative upper adjoint $\{F^{u}(b, \mathrm{id}_A)\}_{A \in |\mathbb{A}|}$ forms a lax transformation $(\mathrm{curry}\, F^{u})b \colon (\mathrm{curry}\, F)B' \overset{\odot}{\to} (\mathrm{curry}\, F)B$. But the functor laxlim preserves the order in the hom-objects (Lemma 2.10) so we obtain that $\mathrm{laxlim}\big((\mathrm{curry}\, F)b\big)$ and $\mathrm{laxlim}\big((\mathrm{curry}\, F^{u})b\big)$ are an adjoint pair. $\square$

Next we use the following easy general observation on adjunctions.

Proposition 3.31. *Consider categories, functors and an adjunction as in the given diagram*

$$
\begin{tikzcd}
\mathbb{A} \arrow[d, "H"'] & \mathbb{D} \arrow[l, "L"'] \arrow[d, hook, "H'"] \\
\mathbb{B} \arrow[r, shift left, "G"] \arrow[r, phantom, "\top"] & \mathbb{C} \arrow[l, shift left, "F"]
\end{tikzcd}
$$

Assume that H *is full and faithful,* H' *is a full inclusion,* $HL = FH'$ *and that* GH *takes values in* $\mathbb{D}$*. Then we have* $L \dashv GH$*. The same is true for* Pos*-enriched inclusion, functors and adjunctions.*

Proof. For any $A \in |\mathbb{A}|$ and $D \in |\mathbb{D}|$ we have the following chain of (Pos) isomorphisms natural in A and D:

$$
\begin{aligned}
&\mathbb{A}(LD, A) && \text{(H full and faithful)} \\
&\quad \cong \mathbb{B}(HLD, HA) && \\
&\quad = \mathbb{B}(FD, HA) && (F \dashv G) \\
&\quad \cong \mathbb{C}(D, GHA) && \mathbb{D} \text{ full in } \mathbb{C} \\
&\quad \cong \mathbb{D}(D, GHA) &&
\end{aligned}
$$

$\square$

Notice now that the following diagram is an instance of Proposition 3.31:

$$\begin{tikzcd}
(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{B}\times\mathbb{A}} \arrow[d, "\mathrm{curry}"'] & & (\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{B}} \arrow[ll, "\mathrm{fst}^*"'] \arrow[d, hook] \\
\left((\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{A}}\right)^{\mathrm{lax}\,\mathbb{B}} \arrow[rr, shift left, "\mathrm{laxlim}^{\mathrm{lax}\,\mathbb{B}}"] \arrow[rr, phantom, "\top"] & & \mathbb{X}^{\mathrm{lax}\,\mathbb{B}} \arrow[ll, shift left, "\mathrm{diag}^{\mathrm{lax}\,\mathbb{B}}"]
\end{tikzcd}$$

It is immediate to see that $\mathrm{curry} \circ \mathrm{fst}^* = \mathrm{diag}^{\mathrm{lax}\,\mathbb{B}}$. Lemma 3.29 states that curry is fully faithful and Corollary 3.30 that $\mathrm{laxlim}^{\mathrm{lax}\,\mathbb{B}}$ takes values in $(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{B}}$. We obtain therefore the adjunction we needed:

$$\begin{tikzcd}
(\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{B}\times\mathbb{A}} \arrow[rr, shift left, "\mathrm{laxlim}^{\mathrm{lax}\,\mathbb{B}} \circ \mathrm{curry}"] \arrow[rr, phantom, "\top"] & & (\mathbb{X}^{l})^{\mathrm{lax}\,\mathbb{B}} \arrow[ll, shift left, "\mathrm{fst}^*"]
\end{tikzcd}$$

It is immediate to check that at this level $\Pi_m := \mathrm{laxlim}^{\mathrm{lax}\,\mathbb{B}} \circ \mathrm{curry}$ satisfies the Beck-Chevalley conditions, that is that for any functor $M\colon \mathbb{B} \to \mathbb{B}'$ (where $\mathbb{B} = (\mathbb{X}^{l}_{\mathrm{fin}})^m$ and $\mathbb{B}' = (\mathbb{X}^{l}_{\mathrm{fin}})^n$) one has

$$M^* \circ \Pi_n = \Pi_m \circ (M \times \mathrm{Id}_{\mathbb{A}})^*$$

since this corresponds to the equality $\mathrm{curry}\,(F \circ (M \times \mathrm{Id}_A)) = (\mathrm{curry}\,F) \circ M$ for any functor $F\colon \mathbb{B} \times \mathbb{A} \to \mathbb{X}^{l}$.

The fact that the equivalence of categories

$$\mathrm{res}\colon \mathrm{C}\left((\mathbb{X}^{l})^{\mathrm{clax}\,(\mathbb{X}^{l})^m}\right) \to (\mathbb{X}^{l})^{\mathrm{lax}\,(\mathbb{X}^{l})^m}$$

is "uniform" in m ensures that this adjunction can be transported across in a consistent way to obtain simple U-products for our hyperdoctrine.

Precisely, one obtains

$$\Pi_m\colon (\mathbb{X}^{l})^{\mathrm{lax}\,(\mathbb{X}^{l})^m \times (\mathbb{X}^{l})} \to (\mathbb{X}^{l})^{\mathrm{lax}\,(\mathbb{X}^{l})^m}$$

as the functor that takes any continuous functor $F\colon (\mathbb{X}^{l})^m \times \mathbb{X}^{l} \to \mathbb{X}^{l}$ to

$$\Pi_m(F) = \mathrm{laxlim} \circ \mathrm{res}^{\mathbb{A}} \circ (\mathrm{curry}\,F)$$

and any continuous lax transformation $s\colon F \overset{\odot}{\to} F'$ the continuous lax transformation

$$\mathrm{ext}^{\mathbb{B}}\left(\left(\mathrm{laxlim} \circ \mathrm{res}^{\mathbb{A}}\right)\left(\mathrm{curry}(\mathrm{res}^{\mathbb{B}}\, s)\right)\right)$$

(where the superscript in res and ext refers to the operation of extending from or restricting to $\mathbb{A} = \mathbb{X}^{l}_{\mathrm{fin}}$ and $\mathbb{B} = (\mathbb{X}^{l}_{\mathrm{fin}})^m$).

3.6 Comparison with Coquand-Gunter-Winskel

In [15] Coquand, Gunter and Winskel give a model of *System* F based on the category SDom of Scott-domains and Scott-continuous functions.

The work there was itself based on the Girard model of *System* F with coherence spaces [18] and another similar model with dI-domains [14].

The category SDom is a cartesian closed category which is Dcpo-enriched in the sense that function space and product preserve directed suprema of functions in each component.

SDom is a full subcategory of Dcpo and is closed under directed colimits of embeddings. Moreover each Scott-domain is bifinite (in our terms Dcpo- or equivalently SDom-bifinite). So we have an admissible subcategory of Dcpo which is a Dcpo-enriched cartesian closed category.

In this way we have rephrased the properties of SDom that make the model of [15] work in terms of our assumptions.

When [15] was published the categorical tool of indexed categories with generic object as a way of organising a model of second order logic was just being developed. The exposition of [15] does not try to follow this early indication, although the authors were aware of this possibility: Other references to this model in related works published at the same time [13] as well as later expositions of the model [16] do already organise the model as an indexed category.

The further level of abstraction of our approach is made possible by replacing the notion of *(continuous) section* (of the Grothendieck co-fibration) of a functor of the form $(\mathsf{F} \to \mathsf{G})$ with that of *(continuous) lax transformation* $\mathsf{F} \overset{\odot}{\to} \mathsf{G}$.

Continuous sections and lax transformations. Let $\mathbb{A}$ be a category, $\mathbb{X}$ be a Pos-enriched category and consider the category $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$ of functors and lax transformations. This is, with some care about the size of the sets involved, a Pos-enriched category.

In Section 3.1 we have seen how, whenever $\mathbb{X}$ has a structure of Pos-enriched symmetric monoidal closed category, $(\mathbb{X}^{\mathrm{l}})^{\mathrm{lax}\,\mathbb{A}}$ is also symmetric monoidal closed. A very similar argument replaces "symmetric monoidal closed" with "cartesian closed".

Let I be the unit of the symmetric monoidal closed structure of $\mathbb{X}$ and consider the hom-functor $\mathbb{X}(\mathrm{I}, -)\colon \mathbb{X} \to \mathsf{Pos}$. For any $\mathsf{F}\colon \mathbb{A} \to \mathbb{X}$ we can look at the Grothendieck co-fibration $\Sigma\mathsf{F} \to \mathbb{A}$ of the composition $\mathbb{X}(\mathrm{I}, \mathsf{F}-) : \mathbb{A} \to \mathsf{Pos}$ where Pos is seen as a category of categories.

Lax transformations from the constant unit functor $\mathrm{I} \overset{\odot}{\to} \mathsf{F}$ correspond then exactly to *sections* of $\Sigma\mathsf{F} \to \mathbb{A}$. The same is true for continuous lax transformations and continuous sections when $\mathbb{X}$ is Dcpo-enriched.

When the unit I is a *separator* for $\mathbb{X}$ (that is, when the functor $\mathbb{X}(\mathrm{I}, -)\colon \mathbb{X} \to \mathsf{Set}$

is faithful) and $\mathbb{X}$ is a subcategory of Pos (as it is the case for $\mathbb{X} = \mathsf{SDom}$), the Grothendieck co-fibration of $\mathbb{X}(I, F-) : \mathbb{A} \to \mathsf{Pos}$ corresponds precisely to that of $F : \mathbb{A} \to \mathbb{X}$ as presented in [15].

Lax transformations and universal quantification. A further pleasant aspect of lax transformations is that they allow the justification of the structure to model universal quantification in terms of the functor laxlim, right adjoint to the diagonal (Section 2.1.1 and Section 3.5.1).

Summing up, using categories of functors and lax transformations, we obtained a general description of:

- the cartesian closed structure of each fibre in the model of *System* F of [15] that works also to account for the linear category structure that we need to model F_{DILL} and
- simple U-products in terms of an adjunction diag $\dashv$ laxlim.

3.7 A model of F_{DILL} based on aCD

Now we finally want to describe the structure of the model of F_{DILL} obtained by taking the category $\mathbb{X} = \mathsf{aCD}$ in the linear hyperdoctrine of Section 3.5.

We begin by describing the linear category structure of aCD as well as the rest of the properties that make it suitable to construct a model of F_{DILL}.

To finish, we calculate the interpretation of some universal types in this model.

3.7.1 Algebraic completely distributive lattices

Most of the results presented in this section are taken from [20], or they are well-established results from the theory of lattices and domains. We refer therefore to [20] and [3] for further details and proofs. The only novelty is the observation that aCD has all small lax limits.

The category Sup of complete lattices and linear maps is a $*$-autonomous category, that is a symmetric monoidal closed category equipped with an internal duality [5].

The $*$-autonomous structure in Sup is the following. Let $\mathbf{2}$ be the two element lattice. For lattices A and B, let $(A \multimap B)$ be the lattice of all linear maps from A to B. The duality in Sup is given by $(\_ \multimap \mathbf{2})$. From the theory of $*$-autonomous categories one obtains that $A \otimes B \cong ((A \multimap (B \multimap \mathbf{2})) \multimap \mathbf{2})$, which can also be taken as the definition of the tensor product in Sup.

There is an obvious inclusion $i\colon \mathsf{Sup} \to \mathsf{Dcpo}$ in the category Dcpo of directed complete partial orders and Scott-continuous maps. This inclusion has a left adjoint

$\mathbf{H}$ defined by the following decomposition of Scott-continuous maps to complete lattices.

For any dcpo D consider the set

$$\mathbf{H}(D) := \{X \subseteq D \mid X \text{ is Scott-closed } \}$$

ordered by inclusion, $\mathbf{H}(D)$ is a complete lattice: The supremum of a collection $\{X_i\}_{i \in I}$ of closed subsets of D is given by

$$\bigvee_{i \in I} X_i = \mathsf{cl}\big(\bigcup_{i \in I} X_i\big).$$

where $\mathsf{cl}(\cdot)$ is the closure of a subset in the Scott-topology. Moreover, there are Scott-continuous functions

$$\mathsf{i}_D \colon D \to \mathbf{H}D \quad \text{defined by} \quad d \mapsto \downarrow d.$$

We have the following [20, Lemma 6.1]:

Lemma 3.32. *Let D be a dcpo and B a complete lattice. For every Scott-continuous function $f \colon D \to B$ there is a unique linear function $\hat{f} \colon \mathbf{H}D \multimap B$ such that $f = \hat{f} \circ \mathsf{i}_D$.*

Moreover, the adjunction

$$\begin{tikzcd} \mathsf{Dcpo} \arrow[r, bend left, "\mathbf{H}"] \arrow[r, phantom, "\perp"] & \mathsf{Sup} \arrow[l, hook', bend left, "i"] \end{tikzcd}$$

is a symmetric monoidal adjunction for $(\mathsf{Dcpo}, \times, \mathbf{1})$ and $(\mathsf{Sup}, \otimes, \mathbf{2})$. This is [20, Theorem 6.2]:

Theorem 3.33. *The adjunction $\mathbf{H} \dashv i \colon (\mathsf{Dcpo}, \times, \mathbf{1}) \to (\mathsf{Sup}, \otimes, \mathbf{2})$ is a symmetric monoidal adjunction for linear logic.*

The restriction of $\mathbf{H}$ to an endofunctor $\mathbf{H} \colon \mathsf{Sup} \to \mathsf{Sup}$ is therefore a linear exponential comonad. That is, $(\mathsf{Sup}, \otimes, \mathbf{2}, \multimap)$ with symmetric monoidal comonad $\mathbf{H}$ is a linear category.

The bifunctor $(- \multimap \star)$ (and therefore also $(- \otimes \star)$) is locally linear, that is, it preserves all suprema of linear functions, hence in particular all directed suprema. The comonad $\mathbf{H}$ is instead merely locally continuous, that is, it preserves directed suprema of linear functions (this is an easy consequence of Lemma 3.32). In any case, we have that Sup is a Dcpo-enriched linear category.

The symmetric monoidal adjunction $\mathbf{H} \dashv i$ between Dcpo and Sup restricts and corestricts to an adjunction

$$\begin{tikzcd} \mathsf{aDom} \arrow[r, bend left, "\mathbf{H}"] \arrow[r, phantom, "\perp"] & \mathsf{aCD} \arrow[l, hook', bend left, "i"] \end{tikzcd}$$

between the category aDom of algebraic dcpos and the category aCD: Completely distributive lattices correspond precisely to the Scott-topologies of continuous domains [3, Theorem 7.2.28] and this specialises to algebraic domains and algebraic completely distributive lattices.

The linear category structure on Sup restricts therefore to give a linear category structure on aCD.

Moreover, the category aCD is an admissible subcategory of Dcpo: In Section 2.3.2 we already mentioned the characterisation of algebraic completely distributive lattices as prime-algebraic lattices. Using this fact it is immediate to see that the colimit in Dcpo of a directed diagram of aCD-embeddings is such that its vertex is still prime-algebraic.

In addition, objects of aCD are linearly bifinite, that is, they are colimits of directed diagrams of aCD-embeddings between finite complete lattices (also this result is implicitly contained in [20]; moreover, it is not hard to verify it directly using that objects of aCD are exactly prime-algebraic complete lattices).

Finally, we need aCD to have lax limits of shape $\mathsf{aCD}^{\mathfrak{l}}_{\mathrm{fin}}$. This is immediate from the following fact:

Proposition 3.34. aCD *has all small lax (bi)limits.*

Proof. This is an immediate consequence of the fact that Sup has small lax limits and each lax limit is a lax bilimit (Section 2.2), together with the fact that the lax bilimit of prime-algebraic lattices is still prime-algebraic (Proposition 2.23). ☐

Remark. This argument does not work for the category SDom of Scott-domains. It is still the case that any functor $\mathsf{SDom}^{\mathfrak{l}}_{\mathrm{fin}} \to \mathsf{SDom}^{\mathfrak{l}}$ has a lax limit in SDom, but this is not a lax bilimit. Moreover to prove that the lax limit is still bounded complete it is essential that the functor considered has the category of lower adjoint as target (as lower adjoint maps preserve all existing suprema).

We have given a quick overview of the following properties that make aCD suitable for the construction a model of F_{DILL}. The category aCD with function space given by linear maps and symmetric monoidal comonad **H**

1. is a linear category.
2. Its structure of linear category is Dcpo-enriched.
3. It is an admissible subcategory of Dcpo.
4. Its objects are bifinite with respect to aCD-embeddings.
5. It has small lax limits (of shape $\mathsf{aCD}^{\mathfrak{l}}_{\mathrm{fin}}$).

Notice that also the larger category BFSup of linearly bifinite complete lattices satisfies these requirements and can therefore be used to construct a linear hyperdoctrine.

3.7.2 Some closed types

We calculate the interpretation of some closed universal types in this model. Let B be any element of aCD (the interpretation of a ground type, if ground types are considered). We have:

- $[\![\forall X.\, X]\!] \cong \mathbf{2}$.
- $[\![\forall X.\, B \multimap X]\!] \cong B \multimap \mathbf{2}$.
- $[\![\forall X.\, X \multimap B]\!] \cong B$.
- $[\![\forall X.\, X \multimap X]\!]$ contains the lattice

$$\perp \prec \mathrm{id} = \delta^1 < \delta^2 < \delta^3 < \cdots < \delta^\omega \prec \top$$

  where $\prec$ indicates an immediate neighbour.
- $[\![\forall X.\, !X \multimap !X \multimap X]\!]$ contains at least the lattice

$$\begin{tikzcd}
& \top \arrow[d, equal] & \\
& \top^{(\perp)} \arrow[d, dash] & \\
& \vee \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{fst} \arrow[dr, dash] & & \mathrm{snd} \arrow[dl, dash] \\
& \wedge \arrow[d, equal] & \\
& \perp &
\end{tikzcd}$$

  where the double line indicates an immediate neighbour.
- $[\![\forall X.\, (X \multimap X) \multimap X]\!]$ contains at least a countable chain.

We give a proof for the cases $[\![\forall X.\, X]\!]$, $[\![\forall X.\, X \multimap X]\!]$ and $[\![\forall X.\, !X \multimap !X \multimap X]\!]$, leaving the other ones to the reader.

The interpretation of any closed type $\forall X.\, C(X)$ is the lax limit of the restriction of the functor $[\![C(X)]\!] \colon \mathsf{aCD}^l \to \mathsf{aCD}^l$ to finite lattices, that is, the lattice of lax threads of $F = \mathrm{res}[\![C(X)]\!]$. In each of the examples computed, we guess *some* sensible lax threads. For a candidate lax thread $\langle x_A \rangle = \langle x_A \in FA \rangle_{A \in |\mathsf{aCD}_{\mathrm{fin}}|}$ one needs to check that for any linear adjoint pair $(l, u) : A \multimap B$ the condition $l(x_A) \leq x_B$ is verified. Much more work is required to prove that there are no other lax threads. To give such proofs, we consider a number of linear adjoint pairs that involve very simple lattices and deduce conditions on the components of an arbitrary lax thread.

Construction of linear maps. Recall that for any complete lattice A the set $\mathsf{CP}(A)$ is the subset of all completely prime elements in A. Considered as a poset it carries the induced order. For any poset P we write $\mathrm{dcl}(P)$ for the set of all downward closed subsets of P ordered by inclusion. The following results are useful for the construction of linear functions.

Lemma 3.35. *The forgetful functor from the category* Sup *of complete lattices and linear functions to* Pos*, the category of posets and monotone functions has a left adjoint which is given on objects by* dcl.

For a monotone function $f\colon P \to A$ *from a poset* P *to a complete lattice* A*, the adjoint linear function* $f'\colon \mathrm{dcl}(P) \to A$ *takes lower subsets* $D \subseteq P$ *to* $\bigvee_{x\in D} f(x)$.

Moreover, the left adjoint corestricts to the full subcategory of algebraic completely distributive lattices.

Lemma 3.36. *For any algebraic completely distributive lattice* A *we have* $\mathrm{dcl}\big(\mathsf{CP}(A)\big) \cong A$.

Proof. It is easy to check directly that $\bigvee\colon \mathrm{dcl}\big(\mathsf{CP}(A)\big) \to A$ and $\lambda x. {\downarrow} x \cap \mathsf{CP}(A)\colon A \to \mathrm{dcl}\big(\mathsf{CP}(A)\big)$ are mutually inverse monotone functions. □

Corollary 3.37. *Let* A *be an algebraic completely distributive lattice and* M *a complete lattice. Every monotone function* $f\colon \mathsf{CP}(A) \to M$ *can be uniquely extended to a linear function on* A*, i.e. there is a unique linear function* $\overline{f}$ *such that* $\overline{f}_{|\mathsf{CP}(A)} = f$.

The extension is given by

$$\overline{f} := \lambda x. \bigvee_{a\in\mathsf{CP}(A)} \big\{ f(a) \;\big|\; a \leq x \big\}.$$

Recall from Lemma 2.22 that if $(l,u)\colon A \to B$ is an adjoint pair in Sup, then l preserves the $\lll$ relation. In particular, such a lower adjoint preserves completely prime elements.

Lemma 3.38. *Let* A *be an algebraic completely distributive lattice and* M *a complete lattice. If* $l\colon A \to M$ *is a lower adjoint in* Pos *that preserves completely prime elements, then its upper adjoint* u *preserves arbitrary suprema.*

Proof. For any subset $S \subseteq M$ we have $u(\bigvee S) \geq \bigvee_{s\in S} u(s)$. Now, take a completely prime element $a \lll a \leq u(\bigvee S)$ or equivalently $l(a) \lll l(a) \leq \bigvee S$. Then there is an $s \in S$ that satisfies $l(a) \leq s$ which implies $a \leq u(s)$ and thus $a \leq \bigvee_{s\in S} u(s)$. As every element in A is the supremum of the completely prime elements below it, this implies $u(\bigvee S) \leq \bigvee_{s\in S} u(s)$ and hence that u preserves all suprema. □

We write $\top^{(\perp)}_{A,B} : A \multimap B$ for the largest linear morphism $A \multimap B$ (and $\top^{(\perp)}_{A}$ for $\top^{(\perp)}_{A,A}$):

$$A \xrightarrow{\top^{(\perp)}_{A,B}} B$$

$$\top^{(\perp)}_{A,B}(x) = \begin{cases} \perp, & \text{if } x = \perp \\ \top, & \text{otherwise} \end{cases}$$

For any lattice A there is exactly one linear adjoint pair (l_A, u_A) to $\mathbf{2}$:

lower	upper
$A \xrightarrow{l_A} \mathbf{2}$	$\mathbf{2} \xrightarrow{u_A} A$
$l_A(x) = \begin{cases} \top, & \text{if } x \neq \perp \\ \perp, & \text{otherwise} \end{cases}$	$u_A(x) = \begin{cases} \top, & \text{if } x = \top \\ \perp, & \text{if } x = \perp \end{cases}$

Linear adjoint pairs from $\mathbf{2}$ to A correspond to completely prime elements of A. For any $a \in \mathsf{CP}(A)$:

lower	upper
$\mathbf{2} \xrightarrow{l_a} A$	$A \xrightarrow{u_a} \mathbf{2}$
$l_a(x) = \begin{cases} a, & \text{if } x = \top \\ \perp, & \text{if } x = \perp \end{cases}$	$u_a(x) = \begin{cases} \top, & \text{if } x \geq a \\ \perp, & \text{otherwise} \end{cases}$

$[\![\forall X.\, X]\!]$**:** We need to compute the lattice of lax threads of the functor

$$\mathrm{res}(\mathrm{Id}_{\mathsf{aCD}^l}) : \mathsf{aCD}^l_{\mathrm{fin}} \to \mathsf{aCD}^l.$$

We argue that there are no lax threads apart from $\perp := \langle \perp_A \in A \rangle_{A \in |\mathsf{aCD}_{\mathrm{fin}}|}$ and $\top := \langle \top_A \in A \rangle_{A \in |\mathsf{aCD}_{\mathrm{fin}}|}$. Let $\langle x_A \in A \rangle_{A \in |\mathsf{aCD}_{\mathrm{fin}}|}$ be a lax thread different from $\perp$. Then it must satisfy $x_{\mathbf{2}} = \top_{\mathbf{2}}$ since for a finite complete distributive lattice B such that $x_B \neq \perp_B$, we have $\top_{\mathbf{2}} = l_B(x_B) \leq x_{\mathbf{2}}$. But then, for any $A \in |\mathsf{aCD}_{\mathrm{fin}}|$ and any $a \in \mathsf{CP}(A)$, we have $a = l_a(x_{\mathbf{2}}) \leq x_A$ which forces $x_A = \top_A$.

$[\![\forall X.\, X \multimap X]\!]$**:** We compute the lattice of lax threads of the functor

$$\mathrm{res}(\mathrm{Id}_{\mathsf{aCD}^l} \multimap \mathrm{Id}_{\mathsf{aCD}^l}) : \mathsf{aCD}^l_{\mathrm{fin}} \to \mathsf{aCD}^l.$$

It is immediate to see that it contains at least three elements: the thread of constant $\perp$ maps, $\perp = \langle \perp_A \in A \multimap A \rangle_{A \in |\mathsf{aCD}_{\mathrm{fin}}|}$, the thread of identity maps $\mathrm{id} = \langle \mathrm{id}_A \in A \multimap A \rangle_{A \in |\mathsf{aCD}_{\mathrm{fin}}|}$, and the thread $\top^{(\perp)} = \langle \top^{(\perp)}_A \in A \multimap A \rangle_{A \in |\mathsf{aCD}_{\mathrm{fin}}|}$.

We begin by proving that id is the only atom:

Proposition 3.39. *If $\langle f_A \rangle_A = \langle f_A \colon A \multimap A \rangle_{A \in |\mathsf{aCD}_{\mathrm{fin}}|}$ is a lax thread strictly above $\bot$, then it is larger than or equal to* id.

Proof. We first show that if there exists a lattice B such that $f_B \neq \bot_B$ then necessarily $f_{\mathbf{2}} = \mathrm{id}_{\mathbf{2}}$. Consider the linear adjoint pair $(l_B, u_B) \colon B \to \mathbf{2}$. We have $\top = (l_B \circ f_B \circ u_B)(\top) \leq f_{\mathbf{2}}(\top)$ and, therefore, $f_{\mathbf{2}} = \mathrm{id}_{\mathbf{2}}$. Now we show that $f_{\mathbf{2}} = \mathrm{id}_{\mathbf{2}}$ forces $f_A \geq \mathrm{id}_A$ for any lattice A. Consider a finite distributive lattice A and for any $a \in \mathsf{CP}(A)$, the linear adjoint pair $(l_a, u_a) \colon \mathbf{2} \to A$. We have $l_a \circ \mathrm{id}_{\mathbf{2}} \circ u_a \leq f_A$. Since A is prime-algebraic, $x = \bigvee \{a \in \mathsf{CP}(A) \mid a \leq x\}$ for any $x \in A$. We therefore have $f_A(x) \geq l_a \circ \mathrm{id}_{\mathbf{2}} \circ u_a(x)$ for any completely prime $a \leq x$, which implies $f_A(x) \geq x$. ☐

We now describe a lax thread immediately below $\top^{(\bot)}$.

Component-wise top. For $A \in |\mathsf{aCD}_{\mathrm{fin}}|$, we define two primes $a, a' \in \mathsf{CP}(A)$ to be *connected* if there is a zigzag

$$a = x_0 \geq x_1 \leq x_2 \geq x_3 \leq x_4 \geq \cdots \leq x_n = a',$$

where $x_0, x_1, \ldots, x_n \in \mathsf{CP}(A)$, and we write $a \;\theta_A\; a'$ in this case. The relation θ_A is clearly an equivalence relation which partitions $\mathsf{CP}(A)$ into its connected components. This relation is monotone and hence corresponds to a linear map γ_A given by the following monotone function on $\mathsf{CP}(A)$:

$$\gamma_A(a) := \bigvee [a]_{\theta_A}$$

Note that monotonicity is clear as we even have $a \leq a' \Rightarrow \gamma_A(a) = \gamma_A(a')$. The following observation is crucial to show that this defines a lax thread.

Lemma 3.40. *If $l \colon A \to B$ is a lower adjoint, then $l_| \colon \mathsf{CP}(A) \to \mathsf{CP}(B)$ preserves connectedness, i.e. $a \;\theta_A\; a'$ implies $l(a) \;\theta_B\; l(a')$.*

Proof. This is an immediate consequence of the fact that $l_|$ as a monotone map takes zigzags to zigzags. ☐

Corollary 3.41. *The family $\gamma = \langle \gamma_A \rangle_{A \in |\mathsf{aCD}_{\mathrm{fin}}|}$ is a lax thread.*

Proof. We need to check

$$\begin{tikzcd}
A \arrow[r, "\gamma_A"] \arrow[d, "l"'] \arrow[dr, phantom, "\geq"] & A \arrow[d, "l"] \\
B \arrow[r, "\gamma_B"'] & B
\end{tikzcd}$$

for an arbitrary lower adjoint $\mathfrak{l}\colon A \to B$. The lemma can be restated as saying $\mathfrak{l}\big([a]_{\theta_A}\big) \subseteq \big[\mathfrak{l}(a)\big]_{\theta_B}$ for all $a \in \mathsf{CP}A$. Hence we get

$$\gamma_B\big(\mathfrak{l}(a)\big) = \bigvee\big[\mathfrak{l}(a)\big]_{\theta_B} \geq \bigvee_{a'\theta_A a} \mathfrak{l}(a') = \mathfrak{l}\big(\bigvee[a]_{\theta_A}\big) = \mathfrak{l}\big(\gamma_A(a)\big)$$

for all primes $a \in \mathsf{CP}(A)$ and, consequently, the inequality holds for all $x \in A$. $\square$

We clearly have $\mathrm{id} < \gamma < \top$, but we can even show that γ is the unique co-atom.

Proposition 3.42. *Let $f = \langle f_A \rangle_{A\in|\mathsf{aCD}_{\mathrm{fin}}|}$ be a lax thread different from $\top$. Then we have $f \leq \gamma$.*

Proof. We prove the contrapositive $f \not\leq \gamma \Rightarrow f = \top$ in several steps. If we have $f \not\leq \gamma$, then there is a finite distributive lattice A and a prime $a \in \mathsf{CP}(A)$ such that $f(a) \not\leq \gamma(a) = \bigvee[a]_{\theta_A}$. This implies that there is a prime $a' \in \mathsf{CP}(A) \setminus [a]_{\theta_A}$ with $a' \leq f(a)$. Now we consider the lower adjoint $\mathfrak{l}\colon A \to \mathbf{2}\times\mathbf{2}$ that is given on primes by

$$\mathfrak{l}(b) := \begin{cases} (\top,\bot) & \text{for } a\ \theta_A\ b; \\ (\bot,\top) & \text{otherwise.} \end{cases}$$

Considering the diagram

$$\begin{tikzcd} A \arrow[r, "f_A"] \arrow[d, "\mathfrak{l}"'] \arrow[dr, phantom, "\geq"] & A \arrow[d, "\mathfrak{l}"] \\ \mathbf{2}\times\mathbf{2} \arrow[r, "f_{2\times 2}"'] & \mathbf{2}\times\mathbf{2} \end{tikzcd}$$

we read off the condition

$$f_{\mathbf{2}\times\mathbf{2}}(\top,\bot) = f_{\mathbf{2}\times\mathbf{2}}\big(\mathfrak{l}(a)\big) \geq \mathfrak{l}\big(f_A(a)\big) \geq \mathfrak{l}(a') = (\bot,\top).$$

Now, we show that this implies that $\alpha_A = \top_A$ for all A. To this end, let A be an arbitrary finite distributive lattice and $a, b \in \mathsf{CP}(A)$ two arbitrary primes and consider

$$\begin{tikzcd} \mathbf{2}\times\mathbf{2} \arrow[r, "f_{2\times 2}"] \arrow[d, "\mathfrak{l}_{ab}"'] \arrow[dr, phantom, "\geq"] & \mathbf{2}\times\mathbf{2} \arrow[d, "\mathfrak{l}_{ab}"] \\ A \arrow[r, "f_A"'] & A \end{tikzcd}$$

where l_{ab} denotes lower adjoint that takes $(\top, \bot)$ to a and $(\bot, \top)$ to b. From this diagram we get

$$f_A(a) = f_A\big(l_{ab}(\top, \bot)\big) \geq l_{ab}\big(f_{2\times 2}(\top, \bot)\big) \geq l_{ab}(\bot, \top) = b$$

which implies $f_A(a) = \top$ and hence $f_A = \top$. □

Distance families. Let us come back to Lemma 3.40. Not only do the induced functions $l_|\colon \mathsf{CP}(A) \to \mathsf{CP}(B)$ for lower adjoints l preserve θ, i.e. connectedness, but also the lengths of the zigzags in the following sense: Given a zigzag

$$a = x_0 \geq y_0 \leq x_1 \geq y_1 \leq x_2 \geq y_2 \cdots y_{n-1} \leq x_n \geq y_n = b$$

of primes we say that it has *length* n; i.e. we only count the upward edges $y_i \leq x_{i+1}$. Now we define the (asymmetric) distance $d_A(a, b)$ from one prime to another as the length of the shortest zigzag from a to b; if there is no such zigzag we set the distance to ω. With this definition we have $d_A(a, b) = 0$ if and only if $a \geq b$, and $d_A(a, b)$ is finite if and only if $a\ \theta_L\ b$.

As lower adjoints take zigzags of primes to zigzags of at most the same lengths we immediately get the following refinement of Lemma 3.40:

Lemma 3.43. *If* $l\colon A \to B$ *is a lower adjoint, then we have*

$$d_B\big(l(a), l(b)\big) \leq d_A(a, b)$$

for all primes $a, b \in \mathsf{CP}(A)$.

If we write

$$\mathbb{B}^n_A(a) := \big\{ b \in \mathsf{CP}(A) \;\big|\; d_A(a, b) < n \big\}$$

for the "ball" of primes that have distance less than n from a given prime $a \in \mathsf{CP}(A)$ then we can reformulate the lemma as

$$l\big(\mathbb{B}^n_A(a)\big) \subseteq \mathbb{B}^n_B\big(l(a)\big).$$

Moreover, the distance and the order interact in the following way:

Lemma 3.44. *1. The sets* $\mathbb{B}^n(a)$ *are lower sets.*

2. For $a \leq b$ *we have* $\mathbb{B}^n(a) \subseteq \mathbb{B}^n(b)$.

With these preliminaries we can now construct an ω-chain of lax threads between id and γ: We define δ^n_A by the following function on $\mathsf{CP}(A)$:

$$\delta^n_A(a) := \bigvee \mathbb{B}^n_A(a)$$

It is monotone because of Lemma 3.44.2 and hence can be extended to a linear map on A.

Corollary 3.45. *Each* $\delta^n = \langle \delta^n_A \rangle_{A \in |\mathsf{aCD}_{\mathrm{fin}}|}$, *for* $n = 1, 2, 3, \ldots$, *is a lax thread.*

Proof. We have to check the inequality

$$\begin{tikzcd} A \arrow[r, "\delta^n_A"] \arrow[d, "l"'] \arrow[dr, phantom, "\geq"] & A \arrow[d, "l"] \\ B \arrow[r, "\delta^n_B"'] & B \end{tikzcd}$$

for arbitrary lower adjoints l. But because of the reformulation of Lemma 3.43 we have

$$\delta^n_B\big(l(a)\big) = \bigvee \mathbb{B}^n_B\big(l(a)\big) \geq \bigvee_{a' \in \mathbb{B}^n_A(a)} l(a') = l\big(\bigvee \mathbb{B}^n_A(a)\big) = l\big(\delta^n_A(a)\big)$$

for all primes a and hence for arbitrary arguments. □

Notice that we have $\delta^1 = \mathrm{id}$ as $\mathbb{B}^1_A(a) = {\downarrow} a \cap \mathsf{CP}(A)$. Moreover, we can also consider γ as δ^ω. To summarise, we have seen that $[\![\forall X.\, X \multimap X]\!]$ has the form

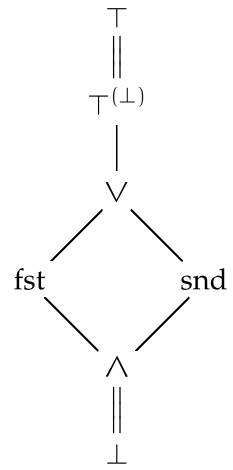
$$\perp \prec \mathrm{id} = \delta^1 < \delta^2 < \delta^3 < \cdots < \delta^\omega \prec \top,$$

but the interval $[\delta^1, \delta^\omega]$ might contain other elements.

$[\![\forall X.\, !X \multimap !X \multimap X]\!]$**:** Instead of working with this directly, we use the isomorphism $\mathsf{Sup}(\mathbf{H}A, B) \cong \mathsf{Dcpo}(A, B)$, for a lattice B, and uncurry and compute the lattice of lax threads of

$$F = \mathrm{res}\,(\mathrm{Id} \times \mathrm{Id} \to \mathrm{Id})\,.$$

Hence, we consider lax threads of Scott-continuous maps $\langle f_A \colon A \times A \to A \rangle_{A \in |\mathsf{aCD}_{\mathrm{fin}}|}$. It is easy to verify that the diagram below is a sublattice of lax threads for F:



Here $\top$ means the lax thread of constant maps $\top_A \colon A \times A \to A$; similarly for $\top^{(\perp)}$ and $\perp$, while the lax threads fst, snd, $\vee$ and $\wedge$ are given by the analogous Scott-continuous morphisms at each component $A \times A \to A$.

In the following let $f_A \colon A \times A \to A$ be the component of a lax thread of F at a finite distributive lattice A. We show that:

- There is no lax thread between $\perp$ and $\wedge$: Assume $f_A \neq \perp_A$ for a lattice A. By checking the laxness condition for the linear adjoint pair $(l_A, u_A) \colon A \multimap \mathbf{2}$ we obtain that $f_{\mathbf{2}} \neq \perp_{\mathbf{2}}$. Consider the adjoint pairs $(l_a, u_a) : \mathbf{2} \multimap A$ for any $a \in \mathsf{CP}(A)$. We have that for any $(x, y) \in A \times A$ the function f_A must satisfy $f_A(x, y) \geq (l_a \circ f_{\mathbf{2}} \circ (u_a \times u_a))(x, y)$. If $(x, y) \in A \times A$ is such that $x \wedge y \neq \perp$ we have therefore

  $$x \wedge y = \bigvee\{a \in \mathsf{CP}(A) \mid a \leq x \wedge y\} = \bigvee_{a \leq x \wedge y} (l_a \circ f_{\mathbf{2}} \circ u_a \times u_a)(x, y) \leq f_A(x, y)$$

  from which we deduce $f_A \geq \wedge_A$ for any lattice A.

- There is no lax thread between $\perp$ and $\wedge$: Suppose $f_A = \top_A$ for a lattice A. By checking the laxness condition with respect to $(l_A, u_A) \colon A \multimap \mathbf{2}$ we obtain that $f_{\mathbf{2}} = \top_{\mathbf{2}}$. Consider the adjoint pairs $(l_a, u_a) : \mathbf{2} \multimap A$ for any $a \in \mathsf{CP}(A)$. We have that for any $(x, y) \in A \times A$ the map f_A must satisfy $f_A(x, y) \geq (l_a \circ f_{\mathbf{2}} \circ (u_a \times u_a))(x, y) = a$. We conclude that $f_A = \top_A$ for any lattice A.

Concluding remarks

To obtain an order-theoretic model of second order linear logic in the form of the term calculus F_{DILL} we addressed two main issues. The first one was how to define a strict functor of categorical models of linear logic. This brought a further understanding of the definition of *linear category* and its relation with symmetric monoidal adjunctions.

The second one was how to construct a *linear hyperdoctrine* starting from a suitable order-theoretic model of linear logic. The idea was to use the same technique that Coquand, Gunter and Winskel [15] used for their model of *System* F based on algebraic Scott-domains. This lead us to an abstract framework for this construction that at once provides a model for second order linear logic and accounts for their model of *System* F.

The key property that makes a category of domains $\mathbb{X}$ (such as SDom or aCD) suitable to handle second order universal quantification (regardless of linearity) is formulated in these terms: The objects of $\mathbb{X}$ need to be *bifinite* and $\mathbb{X}$ needs to be closed under the formation of *lax limits* in Pos for functors of type $\mathbb{X}^{\mathrm{l}}_{\mathrm{fin}} \to \mathbb{X}^{\mathrm{l}}$ (where $\mathbb{X}^{\mathrm{l}}$ indicates the category of adjoint pairs in $\mathbb{X}$ and $\mathbb{X}_{\mathrm{fin}}$ the full subcategory of *finite* objects).

Let us evaluate our model and consider some open questions.

The model obtained using aCD is not very small. The lattice $[\![\forall \mathsf{X}.\mathsf{X}]\!]$ contains only two elements, but $[\![\forall \mathsf{X}.\mathsf{X} \multimap \mathsf{X}]\!]$ contains already (at least!) an ω-chain. Moreover, since aCD is a $*$-autonomous category we have a model of second order *classical* linear logic. In addition if we consider also $\invamp$, it is identified by the semantics with $\otimes$. Also the interpretations of the additives $\&$ and $\oplus$ coincide and correspond to the cartesian product.

We have seen that the category aCD can be used in our linear hyperdoctrine construction. In principle, one could considered also other categories of domains. Nevertheless, here we lack a convincing alternative: The only one that comes to mind is the *larger* category of linearly bifinite complete lattices with the same linear category structure as aCD.

The most obvious question that we have not yet addressed is whether our construction also accounts for the coherence space model of linear logic.

Appendix A

$\mathbf{F}_{\mathrm{DILL}}$

A.1 The type system $\mathbf{F}_{\mathrm{DILL}}$

The second order linear type theory we consider has been around for a while now: It is basically the one outlined by Plotkin in his famous presentation at the Scott Fest in 1993 [37] and mentioned in [38]. While the first presentation had the intuitionistic function space as primitive type constructor (together with linear function space and universal quantification), the first order linear type theory DILL developed in the thesis of Plotkin's student Barber [4] has the linear exponential ! as primitive (together with linear function space and tensor). The linear exponential ! in DILL is seen as a way for translating from an intuitionistic context to a linear one.

The system presented here is therefore DILL augmented with Reynolds-Girard style second order universal quantification and we refer to it as $\mathrm{F}_{\mathrm{DILL}}$.

Types and type contexts. Let TV be a countably infinite set of type variables whose elements are denoted by $X, Y, Z, \ldots$. Type expressions of $\mathrm{F}_{\mathrm{DILL}}$ are given by the grammar

$$A ::= X \mid I \mid A \otimes A \mid A \multimap A \mid !A \mid \forall X.A$$

where $X \in \mathrm{TV}$.

The only binding of a type variable occurs in $\forall X.A$. Type variables occurring unbound in a type expression A are the free type variables of A. A type expression A is closed when it has no free type variable. Type expressions are identified up to α-equivalence, that is up to change of bound type variables. For A, B types, $[X := B]A$ denotes the result (up to α-equivalence) of substituting B for X in A avoiding variable capture. Let TE be the collection of all such type expressions.

A **type context** Θ is a finite list of type variables $X_1, \ldots, X_n$ where no variable occurs twice.

A type expression A is well formed in a given type context Θ if all of its free type variables are listed in Θ. The types of $\mathrm{F}_{\mathrm{DILL}}$ are well formed type expressions in a given type context. These are given formally by judgments $\Theta \vdash A\ \mathsf{Type}$:

$$\frac{}{X_1, \ldots, X_n \vdash X_i \text{ Type}} \text{ for } i = 1, \ldots, n \qquad \frac{}{\vdash I \text{ Type}}$$

$$\frac{\Theta \vdash A \text{ Type}}{\Theta, X \vdash A \text{ Type}}$$

$$\frac{\Theta \vdash A \text{ Type} \qquad \Theta \vdash B \text{ Type}}{\Theta \vdash A \otimes B \text{ Type}}$$

$$\frac{\Theta \vdash A \text{ Type} \qquad \Theta \vdash B \text{ Type}}{\Theta \vdash A \multimap B \text{ Type}} \qquad \frac{\Theta \vdash A \text{ Type}}{\Theta \vdash \;!A \text{ Type}}$$

$$\frac{\Theta, X \vdash A \text{ Type}}{\Theta \vdash \forall X.A \text{ Type}}$$

Pre-terms. Let IV be a countably infinite set of individual variables whose elements are denoted by $x, y, z, \ldots$. Pre-terms of F_{DILL} are given by the grammar

$$\begin{array}{rcl} t, u & ::= & x \mid \bullet \mid \text{let } \bullet : I \text{ be } u \text{ in } t \mid t \otimes t \mid \text{let } x \otimes y : A \otimes B \text{ be } u \text{ in } t \mid \\ & & \lambda x : A.t \mid tt \mid !t \mid \text{let } !x : !A \text{ be } u \text{ in } t \mid \Lambda X.t \mid t(A) \end{array}$$

where $x, y \in$ IV and A and B are type expression.

Binding of individual variables in pre-terms occurs as follows:

- In let $x \otimes y$ be u in t the variables x and y are bound in t.
- In $\lambda x.t$ the variable x is bound in t.
- In let $!x$ be u in t the variable x is bound in t.

An individual variable occurring unbound in a pre-term t is called a free individual variable. A pre-term t is closed when it has no free individual variables. Pre-terms are identified up to α-equivalence, that is up to change of bound individual variables. For t, u pre-terms, $[x := u]t$ denotes the result (up to α-equivalence) of substituting u for x in t avoiding variable capture.

Typing context of F_{DILL}. A typed individual variable is an element of the product IV $\times$ TE, written $x : A$. A finite list of typed individual variables $x_1 : A_1, \ldots, x_n : A_n$ is usually abbreviated by $\Gamma, \Delta, \ldots$. For $\Gamma = x_1 : A_1, \ldots, x_n : A_n$, $\text{fst}(\Gamma) = x_1, \ldots, x_n$ is the list of individual variables from Γ and and $\text{snd}(\Gamma) = A_1, \ldots, A_n$ the relative list of types. The symbol _ indicates the empty list (of individual variables, types or typed individual variables)

A F_{DILL} typing context is a pair of finite lists of typed individual variables $\Gamma;\Delta$ where no individual variable occurs twice (Γ is the intuitionistic part and Δ is the linear part of the typing context).

Given two finite lists of typed individual variables Δ_1 and Δ_2 we write $\Delta = \Delta_1\#\Delta_2$ if Δ is any of the possible interleavings of Δ_1 and Δ_2.

Typing judgements and terms. A typing context $\Gamma;\Delta$ is well defined in the type context Θ if for any type A that occurs in $\mathrm{snd}(\Gamma)$ or $\mathrm{snd}(\Delta)$ one has $\Theta \vdash A$ Type. A typing judgement of F_{DILL} has the form $\Gamma;\Delta \vdash_\Theta t : A$ where $\Gamma;\Delta$ is a typing context well-defined in the type context Θ, t is a pre-term and $\Theta \vdash A$ Type. A term t of type A in typing context $\Gamma;\Delta$ and type context Θ of F_{DILL} is a pre-term such that the judgement $\Gamma;\Delta \vdash_\Theta t : A$ is derived by applying the following rules:

$$\Gamma_1, x : A, \Gamma_2; \_ \vdash_\Theta x : A \quad (\text{Int-Ax}) \qquad \Gamma; x : A \vdash_\Theta x : A \quad (\text{Lin-Ax})$$

$$\Gamma; \_ \vdash_\Theta \bullet : I \quad (\text{I-I}) \qquad \frac{\Gamma;\Delta_1 \vdash_\Theta u : I \quad \Gamma;\Delta_2 \vdash_\Theta t : A}{\Gamma;\Delta_1\#\Delta_2 \vdash_\Theta \mathsf{let}\ \bullet\ \mathsf{be}\ u\ \mathsf{in}\ t : A}\ (\text{I-E})$$

$$\frac{\Gamma;\Delta_1 \vdash_\Theta t : A \quad \Gamma;\Delta_2 \vdash_\Theta u : B}{\Gamma;\Delta_1\#\Delta_2 \vdash_\Theta t \otimes u : A \otimes B}\ (\otimes\text{-I})$$

$$\frac{\Gamma;\Delta_1 \vdash_\Theta u : A \otimes B \quad \Gamma, \Delta_2, x : A, y : B \vdash_\Theta t : C}{\Gamma;\Delta_1\#\Delta_2 \vdash_\Theta \mathsf{let}\ x \otimes y : A \otimes B\ \mathsf{be}\ u\ \mathsf{in}\ t : C}\ (\otimes\text{-E})$$

$$\frac{\Gamma;\Delta, x : A \vdash_\Theta t : B}{\Gamma;\Delta \vdash_\Theta \lambda x : A.t : A \multimap B}\ (\multimap\text{-I})$$

$$\frac{\Gamma;\Delta_1 \vdash_\Theta t : A \multimap B \quad \Gamma;\Delta_2 \vdash_\Theta u : A}{\Gamma;\Delta_1\#\Delta_2 \vdash_\Theta tu : B}\ (\multimap\text{-E})$$

$$\frac{\Gamma; \_ \vdash_\Theta t : A}{\Gamma; \_ \vdash_\Theta !t : !A}\ (!\text{-I}) \qquad \frac{\Gamma;\Delta_1 \vdash_\Theta u : !A \quad \Gamma, x : A; \Delta_2 \vdash_\Theta t : B}{\Gamma;\Delta_1\#\Delta_2 \vdash_\Theta \mathsf{let}\ !x : !A\ \mathsf{be}\ u\ \mathsf{in}\ t : B}\ (!\text{-E})$$

$$\frac{\Gamma;\Delta \vdash_{\Theta,X} t : A}{\Gamma;\Delta \vdash_\Theta \Lambda X.t : \forall X.A}\ (\forall\text{-I}) \qquad \frac{\Gamma;\Delta \vdash_\Theta t : \forall X.A \quad \Theta \vdash B\ \mathsf{Type}}{\Gamma;\Delta \vdash_\Theta t(B) : [X := B]A}\ (\forall\text{-E})$$

In each of the above typing judgements it is intended that any type involved is well formed in the given type context. This means in particular that the variable X must not occur free in any type of $\Gamma;\Delta$ in the $(\forall - I)$ rule.

For a term $t : A$ and a type B, $[X := B]t : [X := B]A$ denotes the result (up to α-equivalence) of substituting A for X in t avoiding variable capture.

Admissible structural rules. The following are admissible structural rules:

$$\frac{\Gamma, x : A, y : B, \Gamma'; \Delta \vdash_\Theta t : C}{\Gamma, y : B, x : A, \Gamma'; \Delta \vdash_\Theta t : C}\,(\text{I-Exch}) \quad \frac{\Gamma; \Delta, x : A, y : B, \Delta' \vdash_\Theta t : C}{\Gamma; \Delta, y : B, x : A, \Delta' \vdash_\Theta t : C}\,(\text{L-Exch})$$

$$\frac{\Gamma; \Delta \vdash_\Theta t : B}{\Gamma, x : A; \Delta \vdash_\Theta t : B}\,(\text{Weak}) \qquad \frac{\Gamma, x : A, y : A; \Delta \vdash_\Theta t : B}{\Gamma, x : A; \Delta \vdash_\Theta t[y := x] : B}\,(\text{Cont})$$

$$\frac{\Gamma, x : A; \Delta \vdash_\Theta t : B \quad \Gamma; \_ \vdash_\Theta u : A}{\Gamma; \Delta \vdash_\Theta [x := u]t : B}\,(\text{I-Cut})$$

$$\frac{\Gamma; \Delta_1, x : A \vdash_\Theta t : B \quad \Gamma; \Delta_2 \vdash_\Theta u : A}{\Gamma; \Delta_1 \# \Delta_2 \vdash_\Theta [x := u]t : B}\,(\text{L-Cut})$$

Weakening and intuitionistic exchange are admissible because of the form of the intuitionistic axiom (Int-Ax), contraction because the rules are given additively in the intuitionistic part of the context, and linear exchange because of the interleaving $\Delta_1\#\Delta_2$ adopted in the rules for the derivation of typing judgements. Any typing judgement $\Gamma; \Delta \vdash_\Theta t : A$ has a unique derivation from the given rules. Intuitionistic and linear cut can also be proved to be admissible by induction on the unique derivation of a typing judgement.

Equality judgements. There are equality judgements $\Gamma; \Delta \vdash_\Theta t = u : A$ that state that two terms t and u are provably equal of type A in a typing context $\Gamma; \Delta$ and type context Θ. These comprise the β- and η-rules for all of the type constructors and some **commuting conversions** which correspond to the trivial proof permutations in linear logic. A **pre-context** is defined by the grammar:

$$\begin{array}{rcl} C[\_] & ::= & \_ \mid \mathsf{let}\ \bullet\ \mathsf{be}\ C[\_]\ \mathsf{in}\ t \mid \mathsf{let}\ \bullet\ \mathsf{be}\ t\ \mathsf{in}\ C[\_] \\ & & \mid t \otimes C[\_] \mid C[\_] \otimes t \mid \mathsf{let}\ x \otimes x\ \mathsf{be}\ C[\_]\ \mathsf{in}\ t \mid \mathsf{let}\ x \otimes x\ \mathsf{be}\ t\ \mathsf{in}\ C[\_] \\ & & \mid \lambda x.C[\_] \mid C[\_]t \mid tC[\_] \mid !C[\_] \mid \mathsf{let}\ !x\ \mathsf{be}\ C[\_]\ \mathsf{in}\ t \\ & & \mid \mathsf{let}\ !x\ \mathsf{be}\ t\ \mathsf{in}\ C[\_] \mid \Lambda X.C[\_] \mid C[\_]A \end{array}$$

The new context constructors compared to DILL are the last two. A pre-context $C[\_]$ can be instantiated with a (pre-)term t to get a (pre-)term $C[t]$ obtained by replacing the the unique occurrence of $\_$ in $C[\_]$ with t.

A $(\Gamma; \Delta, A - \Gamma'; \Delta', B)$-**context** is a pre-context $C[\_]$ such that whenever $\Gamma; \Delta \vdash t : A$ then $\Gamma'; \Delta' \vdash C[t] : B$.

A context is **linear** if it does not contain the clause $!C[\_]$.

The equality judgements of F_{DILL} are

$$\frac{\Gamma; \Delta \vdash_\Theta t : A}{\Gamma; \Delta \vdash_\Theta t = t : A} \qquad \frac{\Gamma; \Delta \vdash_\Theta t = u : A}{\Gamma; \Delta \vdash_\Theta u = t : A}$$

$$\frac{\Gamma; \Delta \vdash_\Theta t = t' : A \quad \Gamma; \Delta \vdash_\Theta t' = t'' : A}{\Gamma; \Delta \vdash_\Theta t = t'' : A}$$

$$\frac{\Gamma; \Delta \vdash_\Theta t = u : A}{\Gamma'; \Delta' \vdash_\Theta C[t] = C[u] : B} \quad \text{where } C[\_] \text{ is a } \Gamma; \Delta, A - \Gamma'; \Delta', B \text{ context}$$

β-rules

$$\Gamma; \Delta \vdash_\Theta \mathsf{let} \ \bullet \ \mathsf{be} \ \bullet \ \mathsf{in}\ t = t : A$$
$$\Gamma; \Delta \vdash_\Theta \mathsf{let}\ x \otimes y\ \mathsf{be}\ u_1 \otimes u_2\ \mathsf{in}\ t = [x, y := u_1, u_2]t : A$$
$$\Gamma; \Delta \vdash_\Theta (\lambda x : A.t)u = [x := u]t$$
$$\Gamma; \Delta \vdash_\Theta \mathsf{let}\ !x : !A\ \mathsf{be}\ !u\ \mathsf{in}\ t = [x := u]t : B$$
$$\Gamma; \Delta \vdash_\Theta (\Lambda X.t)(B) = [X := B]t : [X := B]A$$

η-rules

$$\Gamma; \Delta \vdash_\Theta \mathsf{let} \ \bullet \ \mathsf{be}\ t\ \mathsf{in} \bullet = t : I$$
$$\Gamma; \Delta \vdash_\Theta \mathsf{let}\ x \otimes y\ \mathsf{be}\ t\ \mathsf{in}\ x \otimes y = t : A \otimes B$$
$$\Gamma; \Delta \vdash_\Theta \lambda x : A.(tx) = t : A \multimap B$$
$$\Gamma; \Delta \vdash_\Theta \mathsf{let}\ !x : !A\ \mathsf{be}\ t\ \mathsf{in}\ !x = t : !A$$
$$\Gamma; \Delta \vdash_\Theta (\Lambda X.t(X)) = t : \forall X.A$$

Commuting conversions

For a linear context $C[\_]$:

$$\Gamma; \Delta \vdash \mathsf{let} \bullet \mathsf{be}\ t\ \mathsf{in}\ C[u] = C[\mathsf{let} \bullet \mathsf{be}\ t\ \mathsf{in}\ u : A]$$

$$\Gamma; \Delta \vdash \mathsf{let}\ x \otimes y\ \mathsf{be}\ t\ \mathsf{in}\ C[u] = C[\mathsf{let}\ x \otimes y\ \mathsf{be}\ t\ \mathsf{in}\ u]$$

where $C[\_]$ does not bind x or y or contain either free

$$\Gamma; \Delta \vdash \mathsf{let}\ !x\ \mathsf{be}\ t\ \mathsf{in}\ C[u] = C[\mathsf{let}\ !x\ \mathsf{be}\ t\ \mathsf{in}\ u]$$

where $C[\_]$ does not bind x or contain it free

A.2 Linear hyperdoctrine

To model F_{DILL} we need some suitable kind of *hyperdoctrine*, accordingly to the well-established tools of categorical logic for second order logic (as reference one can take [36] and [27]).

As for the categorical logic machinery, the model that we present in Chapter 3 is quite simple. In particular it is an instance of a *split indexed category* (that is a proper functor) as opposed to a general *indexed category* (that would be given by a *pseudofunctor*). Accordingly, its *generic object* is *split* (for a more general presentation of these notions we refer to [27]). For this reason — and to keep the complexity of notation low — below we only give the simpler definition of categorical model for second order linear logic that we adopt.

Definition A.1. A **linear hyperdoctrine** is given by a functor

$$\Psi : \mathcal{B}^{op} \to \mathsf{Lin}$$

where

(Base) $\mathcal{B}$ is a small category with terminal object $1_\mathcal{B}$ and binary products. There is a distinguished object $\mathsf{U} \in |\mathcal{B}|$ that generates all the other objects by finite products: For any $B \in |\mathcal{B}|$, $B = \mathsf{U}^n$ for some $n \in \mathbb{N}$ ($\mathsf{U}^0 = 1_\mathcal{B}$).

(Split generic object) There is a distinguished object G in the category $\Psi(\mathsf{U})$ such that for any $\mathsf{U}^n \in |\mathcal{B}|$ and $A \in |\Psi(\mathsf{U}^n)|$ there is a unique morphism $F : \mathsf{U}^n \to \mathsf{U}$ in $\mathcal{B}$ for which $\Psi(F)(\mathsf{G}) \cong B$. We refer to the generic object as the pair (U, G).

(Simple U-products) for each object U^n in $\mathcal{B}$, there is a functor $\Pi_n : \Psi(U^n \times U) \to \Psi(U^n)$ right adjoint to $\Psi(\mathrm{fst}_n) : \Psi(U^n) \to \Psi(U^n \times U)$. The functors Π_n are required to be natural in U^n, that is, for any $F \in \mathcal{B}(U^n, U^m)$ one has $\Psi(F) \circ \Pi_m = \Pi_n \circ \Psi(F \times \mathrm{id}_U)$ (Beck-Chevalley condition).

Notation. The functor $\Psi(F)$ is called *reindexing along* F and is often written simply F^*.

Notice that — identifying objects of $\Psi(U^n)$ with $\mathcal{B}$-maps $U^n \to U$ via generic object — there are internal product projection morphisms, that is for any $A : U^n \times U \to U$ and $B : U^n \to U$ there is a map $\mathrm{proj}_B : \Pi_n A \to A \circ \langle \mathrm{Id}_{U^n}, B\rangle$ in $\Psi(U^n)$ given by

$$\mathrm{proj}_B = \langle \mathrm{Id}_{U^n}, B\rangle^*(\mathrm{co}_A)$$

where $\mathrm{co}_A : \mathrm{fst}_n^*(\Pi_n A) \to A$ is the component at A of the counit of the adjunction $\mathrm{fst}_n^* \dashv \Pi_n$.

A.3 Interpretation of F_{DILL}

To give the interpretation function for a F_{DILL} typing judgement $\Gamma; \Delta \vdash_\Theta t : A$ in a linear hyperdoctrine, we need to fix some notation.

Let $(\mathbb{S}, \otimes, I, \multimap)$ be a linear category with a linear exponential comonad $(T, m_I^T, m^T, \varepsilon, \delta)$. Recall that $e : T \overset{\bullet}{\to} I$ and $d : T \overset{\bullet}{\to} T \otimes T$ are the natural transformations induced by the trivial family of commutative comonoids in the category of T-coalgebras. We define the n-ary version of the tensor product and of the natural transformations e, d, ε and δ (which correspond to the context handling maps disc, dupl, der and prom of [4])

If $A_1, \ldots, A_n$ is a (possibly empty) list of objects of $\mathbb{S}$ the n-ary tensor product is given by the identity I for the empty list and by the object obtained by associating to the left (and leaving behind the first occurrence of I) in any other case:

$$\bigotimes(A_1, \ldots, A_n) = \begin{cases} I & \text{if } n = 0 \\ A_1 & \text{if } n = 1 \\ \bigotimes(A_1, \ldots, A_{n-1}) \otimes A_n & \text{otherwise} \end{cases}$$

Accordingly, we give the n-ary version of e, d, ε, δ:

$$e_{A_1,\ldots,A_n} : \bigotimes(TA_1, \ldots, TA_n) \to I$$

$$e_{A_1,\ldots,A_n} = \begin{cases} \mathrm{id}_I & \text{if } n = 0 \\ e_{A_1} & \text{if } n = 1 \\ \lambda_I \circ (e_{A_1,\ldots,A_{n-1}} \otimes e_{A_n}) & \text{otherwise} \end{cases}$$

$$d_{A_1,\ldots,A_n} : \bigotimes(TA_1,\ldots,TA_n) \to \bigotimes(TA_1,\ldots,TA_n) \otimes \bigotimes(TA_1,\ldots,TA_n)$$

$$d_{A_1,\ldots,A_n} = \begin{cases} \lambda_I^{-1} & \text{if } n = 0 \\ d_{A_1} & \text{if } n = 1 \\ \mathsf{sw} \circ (d_{A_1,\ldots,A_{n-1}} \otimes d_{A_n}) & \text{otherwise} \end{cases}$$

$$\varepsilon_{A_1,\ldots,A_n} : \bigotimes(TA_1,\ldots,TA_n) \to \bigotimes(A_1,\ldots,A_n)$$

$$\varepsilon_{A_1,\ldots,A_n} = \begin{cases} \mathrm{id}_I, & \text{if } n = 0 \\ \varepsilon_{A_1}, & \text{if } n = 1 \\ \varepsilon_{A_1,\ldots,A_{n-1}} \otimes \varepsilon_{A_n} & \text{otherwise} \end{cases}$$

$$\delta_{A_1,\ldots,A_n} : \bigotimes(TA_1,\ldots,TA_n) \to T\left(\bigotimes(TA_1,\ldots,TA_n)\right)$$

$$\delta_{A_1,\ldots,A_n} = \begin{cases} \mathsf{m}_I^T, & \text{if } n = 0 \\ \delta_{A_1}, & \text{if } n = 1 \\ \mathsf{m}^T_{\otimes(TA_1,\ldots,TA_{n-1}),TA_n} \circ (\delta_{A_1,\ldots,A_{n-1}} \otimes \delta_{A_n}) & \text{otherwise} \end{cases}$$

A.3.1 The interpretation function

Let

$$\Psi : \mathcal{B}^{op} \to \mathsf{Lin}$$

be a linear hyperdoctrine with generic object (U, G). Types of F_{DILL} are interpreted as objects in a appropriate fibre: The interpretation of a type in a type context $\Theta = X_1,\ldots,X_n$ is an object of the linear category $\Psi(\mathsf{U}^n)$ (with linear exponential comonad T) and it is given by:

For a type $X_1,\ldots,X_n \vdash A$ Type:

$$\begin{aligned} [\![X_i]\!] &= \pi_i^*(\mathsf{G}), \text{where } \pi_i : \mathsf{U}^n \to \mathsf{U} \text{ is the } i\text{-th projection} \\ [\![I]\!] &= I \\ [\![A \otimes B]\!] &= [\![A]\!] \otimes [\![B]\!] \\ [\![A \multimap B]\!] &= [\![A]\!] \multimap [\![B]\!] \\ [\![!A]\!] &= T[\![A]\!] \\ [\![\forall X.A]\!] &= \Pi_n[\![A]\!] \end{aligned}$$

Substitution of types for type variables corresponds to composition in $\mathcal{B}$: Whenever $\Theta, X \vdash A$ Type and $\Theta \vdash B$ Type, one shows by structural induction on A that

$$[\![[X := B]A]\!] = [\![A]\!] \circ \langle \mathrm{Id}, [\![B]\!] \rangle = \langle \mathrm{Id}, [\![B]\!] \rangle^* [\![A]\!].$$

For a typing context

$$\Gamma; \Delta = x_1 : A_1, \ldots, x_m : A_m; y_1 : B_1, \ldots, y_q : B_q$$

that is well defined in the type context $\Theta = X_1, \ldots, X_n$, let $!\Gamma, \Delta$ be the list of types $!A_1, \ldots, !A_m, B_1, \ldots, B_q$. The typing context is interpreted as the object

$$[\![\Gamma; \Delta]\!] = [\![!\Gamma, \Delta]\!] = \bigotimes([\![!A_1]\!], \ldots, [\![!A_n]\!], [\![B_1]\!], \ldots, [\![B_q]\!])$$

in the fibre $\Psi(\mathsf{U}^n)$. The interpretation of a term $\Gamma; \Delta \vdash_\Theta t : A$ is a morphism

$$[\![\Gamma; \Delta]\!] \to [\![A]\!]$$

of the fibre $\Psi(\mathsf{U}^n)$ defined inductively on the unique derivation of the term t as follows. For the typing context $\Gamma; \Delta_1$ and $\Gamma; \Delta_2$ well defined in typing context $\Theta = X_1, \ldots, X_n$ and a $\Delta = \Delta_1 \# \Delta_2$ consider the arrow of $\Psi(\mathsf{U}^n)$

$$\mathsf{split}_{\Gamma,\Delta,\Delta_1,\Delta_2} : [\![\Gamma; \Delta]\!] \to [\![\Gamma; \Delta_1]\!] \otimes [\![\Gamma; \Delta_2]\!]$$

given by the composition

$$[\![\Gamma;\Delta]\!] \xrightarrow{\cong} [\![!\Gamma]\!] \otimes ([\![\Delta_1]\!] \otimes [\![\Delta_2]\!]) \xrightarrow{d_\Gamma \otimes id} ([\![!\Gamma]\!] \otimes [\![!\Gamma]\!]) \otimes ([\![\Delta_1]\!] \otimes [\![\Delta_2]\!]) \xrightarrow{\cong} [\![\Gamma;\Delta_1]\!] \otimes [\![\Gamma;\Delta_2]\!]$$

where the isomorphism arrows are uniquely determined by coherence.

Notice that in the inductive definition of the interpretation of $\Gamma; \Delta \vdash_\Theta t : A$ is we omit the semantic brackets in the subscripts of e, d, ε and δ and avoid to specify any coherent isomorphism.

Intuitionistic Axiom: $[\![\Gamma_1, x : A, \Gamma_2; \_ \vdash_\Theta x : A]\!] : [\![\Gamma_1, A, \Gamma_2; \_]\!] \to [\![A]\!]$ is given by

$$[\![!\Gamma_1, !A, !\Gamma_2]\!] \xrightarrow{\cong} ([\![!\Gamma_1]\!] \otimes T[\![A]\!]) \otimes [\![!\Gamma_2]\!] \xrightarrow{(e_{\Gamma_1} \otimes \varepsilon_A) \otimes e_{\Gamma_2}} (I \otimes [\![A]\!]) \otimes I \xrightarrow{\cong} [\![A]\!].$$

Linear Axiom: $[\![\Gamma; x : A \vdash_\Theta x : A]\!] : [\![\Gamma; A]\!] \to [\![A]\!]$ is given by

$$[\![!\Gamma]\!] \otimes [\![A]\!] \xrightarrow{e_\Gamma \otimes id_A} I \otimes [\![A]\!] \xrightarrow{\cong} [\![A]\!].$$

I**-introduction:** $[\![\Gamma; \_ \vdash_\Theta \bullet : I]\!]$ is given by

$$[\![!\Gamma]\!] \xrightarrow{e_\Gamma} [\![I]\!].$$

I-elimination: Assuming $f = [\![\Gamma; \Delta_1 \vdash_\Theta t : I]\!]$ and $g = [\![\Gamma; \Delta_2 \vdash_\Theta t : A]\!]$, the conclusion of the derivation

$$\frac{\Gamma; \Delta_1 \vdash_\Theta u : I \quad \Gamma; \Delta_2 \vdash_\Theta t : A}{\Gamma; \Delta \vdash_\Theta \mathsf{let} \bullet \mathsf{be}\ u\ \mathsf{in}\ t : A}$$

is interpreted as

$$[\![\Gamma;\Delta]\!] \xrightarrow{\mathsf{split}_{\Gamma,\Delta,\Delta_1,\Delta_2}} [\![\Gamma;\Delta_1]\!] \otimes [\![\Gamma;\Delta_2]\!] \xrightarrow{f \otimes g} I \otimes [\![A]\!] \xrightarrow{\cong} [\![A]\!].$$

$\otimes$-introduction: Assuming $f = [\![\Gamma; \Delta_1 \vdash_\Theta t : A]\!]$ and $g = [\![\Gamma; \Delta_2 \vdash_\Theta u : B]\!]$, the conclusion of the derivation

$$\frac{\Gamma; \Delta_1 \vdash_\Theta t : A \quad \Gamma; \Delta_2 \vdash_\Theta u : B}{\Gamma; \Delta \vdash_\Theta t \otimes u : A \otimes B}$$

is interpreted as

$$[\![\Gamma;\Delta]\!] \xrightarrow{\mathsf{split}_{\Gamma,\Delta,\Delta_1,\Delta_2}} [\![\Gamma;\Delta_1]\!] \otimes [\![\Gamma;\Delta_2]\!] \xrightarrow{f \otimes g} [\![A \otimes B]\!].$$

$\otimes$-elimination: Let $f = [\![\Gamma; \Delta_1 \vdash_\Theta u : A \otimes B]\!]$ and $g = [\![\Gamma; \Delta_2, x : A, y : B \vdash_\Theta t : C]\!]$, then the conclusion of the derivation

$$\frac{\Gamma; \Delta_1 \vdash_\Theta u : A \otimes B \quad \Gamma, \Delta_2, x : A, y : B \vdash_\Theta t : C}{\Gamma; \Delta \vdash_\Theta \mathsf{let}\ x \otimes y : A \otimes B\ \mathsf{be}\ u\ \mathsf{in}\ t : C}$$

is interpreted as

$$[\![\Gamma;\Delta]\!] \xrightarrow{\mathsf{split}_{\Gamma,\Delta,\Delta_1,\Delta_2}} [\![\Gamma;\Delta_1]\!] \otimes [\![\Gamma;\Delta_2]\!] \xrightarrow{f \otimes id} [\![A \otimes B]\!] \otimes [\![\Gamma;\Delta_2]\!] \xrightarrow{\cong} [\![\Gamma;\Delta,A,B]\!] \xrightarrow{g} [\![C]\!].$$

$\multimap$-introduction: Assuming $f = [\![\Gamma; \Delta, x : A \vdash_\Theta t : B]\!]$, the conclusion of the derivation

$$\frac{\Gamma; \Delta, x : A \vdash_\Theta t : B}{\Gamma; \Delta \vdash_\Theta \lambda x : A.t : A \multimap B}$$

is interpreted as

$$[\![\Gamma;\Delta]\!] \xrightarrow{\mathsf{curry}(f)} [\![A \multimap B]\!].$$

$\multimap$-elimination: Assuming $f = [\![\Gamma; \Delta_1 \vdash_\Theta t : A \multimap B]\!]$ and $g = [\![\Gamma; \Delta_2 \vdash_\Theta u : A]\!]$, the conclusion of the derivation

$$\frac{\Gamma; \Delta_1 \vdash_\Theta t : A \multimap B \quad \Gamma; \Delta_2 \vdash_\Theta u : A}{\Gamma; \Delta \vdash_\Theta tu : B}$$

is interpreted as

$$[\![\Gamma;\Delta]\!] \xrightarrow{\mathsf{split}_{\Gamma,\Delta,\Delta_1,\Delta_2}} [\![\Gamma;\Delta_1]\!] \otimes [\![\Gamma;\Delta_2]\!] \xrightarrow{f \otimes g} [\![A \multimap B]\!] \otimes [\![A]\!] \xrightarrow{ev^{([\![B]\!])}_{[\![A]\!]}} [\![B]\!]$$

!-introduction: Assuming $f = [\![\Gamma; \_ \vdash_\Theta t : A]\!]$, the conclusion of the derivation

$$\frac{\Gamma; \_ \vdash_\Theta t : A}{\Gamma; \_ \vdash_\Theta !t : !A}$$

is interpreted as

$$[\![\Gamma; \_]\!] = [\![!\Gamma]\!] \xrightarrow{\delta_\Gamma} T[\![!\Gamma]\!] \xrightarrow{Tf} T[\![A]\!].$$

!-elimination: Assuming $f = [\![\Gamma; \Delta_1 \vdash_\Theta u : !A]\!]$ and $g = [\![\Gamma, x : A; \Delta_2 \vdash_\Theta t : B]\!]$, the conclusion of the derivation

$$\frac{\Gamma; \Delta_1 \vdash_\Theta u : !A \quad \Gamma, x : A; \Delta_2 \vdash_\Theta t : B}{\Gamma; \Delta \vdash_\Theta \mathsf{let}\ !x : !A\ \mathsf{be}\ u\ \mathsf{in}\ t : B}$$

is interpreted as

$$[\![\Gamma;\Delta]\!] \xrightarrow{\mathsf{split}_{\Gamma,\Delta,\Delta_1,\Delta_2}} [\![\Gamma;\Delta_1]\!] \otimes [\![\Gamma;\Delta_2]\!] \xrightarrow{f \otimes id} [\![!A]\!] \otimes [\![\Gamma;\Delta_2]\!] \xrightarrow{\cong} [\![\Gamma,A;\Delta]\!] \xrightarrow{g} [\![B]\!].$$

In the definition of the interpretation of a term given so far, everything takes values in the linear category $\Psi(U^n)$ where n is the length of the type context Θ. In particular, curry and $ev^{()}$ are currying and evaluation for the symmetric monoidal closed structure of $\Psi(U^n)$ while $e, d, \varepsilon, \delta$ are the n-ary versions of the relevant natural transformations in $\Psi(U^n)$.

$\forall$-introduction: Consider the $\Psi\left(U^{n+1}\right)$-morphism

$$f = [\![t]\!] : [\![\Gamma; \Delta]\!] \to [\![A]\!]$$

and notice that if X_{n+1} is not free in any type of $\Gamma; \Delta$ one has that $[\![\Gamma; \Delta]\!]$ in $\Psi(U^{n+1})$ coincides with the image of $[\![\Gamma; \Delta]\!]$ in $\Psi(U^n)$ under the reindexing along the first projection $\mathrm{fst} : U^{n+1} \to U^n$. The conclusion of the derivation

$$\frac{\Gamma; \Delta \vdash_{\Theta,X} t : A}{\Gamma; \Delta \vdash_\Theta \Lambda X.t : \forall X.A}$$

is interpreted as the transpose $\hat{f}$ of f under the adjunction $\mathrm{fst}_n^* \dashv \Pi_n$ in the linear category $\Psi(U^n)$:

$$[\![\Gamma;\Delta]\!] \xrightarrow{\hat{f}} \Pi_n(A)$$

$\forall$-elimination: Assuming $f = [\![t]\!] : [\![\Gamma;\Delta]\!] \to [\![\forall X.A]\!]$ the conclusion of the derivation

$$\frac{\Gamma;\Delta \vdash_\Theta t : \forall X.A \quad \Theta \vdash B \ \mathsf{Type}}{\Gamma;\Delta \vdash_\Theta t(B) : [X := B]A}\ (\forall\text{-E})$$

is interpreted as

$$[\![\Gamma;\Delta]\!] \xrightarrow{f} \Pi_n([\![A]\!]) \xrightarrow{\mathrm{proj}_B} [\![A]\!]\circ\langle \mathrm{Id}_{U^n}, [\![B]\!]\rangle = [\![[X{:=}B]A]\!].$$

By combining soundness of the interpretation of DILL in a linear category [4] and soundness of the interpretation of second order universal quantification in second order hyperdoctrines, we obtain:

Soundness: For every provable equality $\Gamma;\Delta \vdash_\Theta t = u : A$ of F_{DILL} one has $[\![\Gamma;\Delta \vdash_\Theta t : A]\!] = [\![\Gamma;\Delta \vdash_\Theta u : A]\!]$.

Appendix B

Background definitions

B.1 Comonad, coalgebra, category of coalgebras

Let $\mathbb{S}$ be a category. A *comonad* (T, ε, δ) on $\mathbb{S}$ is an endofunctor $T\colon \mathbb{S} \to \mathbb{S}$ together with natural transformations $\varepsilon\colon T \stackrel{\bullet}{\to} \mathrm{Id}_{\mathbb{S}}$ and $\delta\colon T \stackrel{\bullet}{\to} T^2$ such that the following three diagrams of functors and natural transformations commute:

$$\begin{tikzcd} T \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, equal] & T^2 \arrow[d, "T\varepsilon"] \\ T^2 \arrow[r, "\varepsilon_T"'] & TA \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\delta"] \arrow[d, "\delta"'] & T^2 \arrow[d, "\delta_T"] \\ T^2 \arrow[r, "T\delta"'] & T^3A \end{tikzcd}$$

A T-*coalgebra* (A, x) for a comonad (T, ε, δ) is a morphism $x\colon A \to TA$ in $\mathbb{S}$ satisfying:

$$\begin{tikzcd} A \arrow[d, "x"'] \arrow[dr, equal] & \\ TA \arrow[r, "\varepsilon_A"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "x"] \arrow[d, "x"'] & TA \arrow[d, "\delta_A"] \\ TA \arrow[r, "Tx"'] & T^2A \end{tikzcd}$$

Given T-coalgebras (A, x) and (B, y) a *morphism of coalgebras* $f\colon (A, x) \to (B, y)$ is a map $f \in \mathbb{S}(A, B)$ satisfying:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "x"'] & B \arrow[d, "y"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

Given a comonad (T, ε, δ) on $\mathbb{S}$, T-coalgebras and morphisms of coalgebras form a category $\mathbb{S}^T$, the *category of* T-*coalgebras* (Eilenberg-Moore construction).

B.2 Symmetric monoidal categories

B.2.1 Symmetric monoidal categories

A *monoidal category* is a category $\mathbb{S}$ together with a bifunctor $(- \otimes \star)\colon \mathbb{S} \times \mathbb{S} \to \mathbb{S}$, an object $I \in |\mathbb{S}|$ and isomorphisms

$$\begin{aligned} a_{A,B,C}&\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C \\ \lambda_A&\colon I \otimes A \to A \\ \rho_A&\colon A \otimes I \to A \end{aligned}$$

natural in $A, B, C \in |\mathbb{A}|$, such that the following diagrams commute:

$$\begin{tikzcd} A \otimes (B \otimes (C \otimes D)) \arrow[r, "a_{A,B,(C\otimes D)}"] \arrow[d, "\mathrm{id}_A \otimes a_{B,C,D}"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "a_{A\otimes B,C,D}"] & ((A \otimes B) \otimes C) \otimes D \\ A \otimes ((B \otimes C) \otimes D) \arrow[rr, "a_{A,B\otimes C,D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "a_{A,B,C} \otimes \mathrm{id}_D"'] \end{tikzcd}$$

$$\begin{tikzcd} A \otimes (I \otimes B) \arrow[r, "a_{A,I,B}"] \arrow[d, "\mathrm{id}_A \otimes \lambda_B"'] & (A \otimes I) \otimes B \arrow[d, "\rho_A \otimes \mathrm{id}_B"] \\ A \otimes B \arrow[r, equal] & A \otimes B \end{tikzcd} \qquad \begin{tikzcd} I \otimes I \arrow[r, "\lambda_I = \rho_I"] & I \end{tikzcd}$$

A monoidal category $\mathbb{S}$ is *symmetric* monoidal if in addition there is a family of isomorphisms

$$\gamma_{A,B}\colon A \otimes B \to B \otimes A$$

natural in A and $B \in |\mathbb{S}|$ which satisfies the commutativity of the following three diagrams:

$$\begin{tikzcd}
A \otimes I \arrow[r, "\gamma_{A,I}"] \arrow[d, "\rho_A"'] & I \otimes A \arrow[d, "\lambda_A"] \\
A \arrow[r, equal] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[r, "\gamma_{A,B}"] & B \otimes A \arrow[d, "\gamma_{B,A}"] \\
 & A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\mathrm{id}_A \otimes' \gamma_{B,C}"'] & (A \otimes B) \otimes C \arrow[r, "\gamma_{A \otimes B, C}"] & C \otimes (A \otimes B) \arrow[d, "\alpha_{C,A,B}"] \\
A \otimes (C \otimes B) \arrow[r, "\alpha_{A,C,B}"'] & (A \otimes C) \otimes B \arrow[r, "\gamma_{C,A} \otimes \mathrm{id}_B"'] & (C \otimes A) \otimes B
\end{tikzcd}$$

As a consequence of the conditions imposed on the natural isomorphisms α, λ, ρ, γ one has the following (*coherence*) result [31]: Any formal diagram that involves just α, λ, ρ and γ commutes. For any $A, B, C, D \in |\mathbb{S}|$, we denote by $\mathsf{sw}_{A,B,C,D}$ the unique isomorphism:

$$(A \otimes B) \otimes (C \otimes D) \xrightarrow{\mathsf{sw}_{A,B,C,D}} (A \otimes C) \otimes (B \otimes D)$$

B.2.2 Symmetric monoidal functors

Let $(\mathbb{S}, \otimes, I)$ and $(\mathbb{S}', \otimes', I')$ be symmetric monoidal categories. A *symmetric monoidal functor* from $\mathbb{S}$ to $\mathbb{S}'$ is a functor $F\colon \mathbb{S} \to \mathbb{S}'$ together with a morphism

$$\mathsf{m}^F_I\colon I' \to FI$$

and a family of morphisms

$$\mathsf{m}^F_{A,B}\colon FA \otimes' FB \to F(A \otimes B)$$

natural in A and $B \in |\mathbb{S}|$ such that the following four diagrams commute:

$$\begin{tikzcd}
FA \otimes' (FB \otimes' FC) \arrow[r, "\alpha'_{FA,FB,FC}"] \arrow[d, "\mathrm{id}_{FA} \otimes' m^F_{B,C}"'] & (FA \otimes' FB) \otimes' FC \arrow[d, "m^F_{A,B} \otimes' \mathrm{id}_{FC}"] \\
FA \otimes' F(B \otimes C) \arrow[d, "m^F_{A,B\otimes C}"'] & F(A \otimes B) \otimes' FC \arrow[d, "m^F_{A\otimes B,C}"] \\
F(A \otimes (B \otimes C)) \arrow[r, "F\alpha_{A,B,C}"'] & F((A \otimes B) \otimes C)
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes' I' \arrow[r, "\rho'_{FA}"'] \arrow[d, "\mathrm{id}_{FA} \otimes' m^F_I"'] & FA \\
FA \otimes' FI \arrow[r, "m^F_{A,I}"'] & F(A \otimes I) \arrow[u, "F\rho_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I' \otimes' FA \arrow[r, "\lambda'_{FA}"] \arrow[d, "m^F_I \otimes' \mathrm{id}_{FA}"'] & FA \\
FI \otimes' FA \arrow[r, "m^F_{I,A}"'] & F(I \otimes A) \arrow[u, "F\lambda_A"']
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes' FB \arrow[r, "\gamma'_{FA,FB}"] \arrow[d, "m^F_{A,B}"'] & FB \otimes' FA \arrow[d, "m^F_{B,A}"] \\
F(A \otimes B) \arrow[r, "F\gamma_{A,B}"'] & F(B \otimes A)
\end{tikzcd}$$

The functor F is said to be *strong* symmetric monoidal if each of m^F_I and $m^F_{A,B}$ are isomorphisms and *strict* symmetric monoidal if each of m^F_I and $m^F_{A,B}$ are identities.

B.2.3 Monoidal natural transformations

Let $(F, m^F, m^F_I), (G, m^G, m^G_I) \colon (\mathbb{S}, \otimes, I) \to (\mathbb{S}', \otimes', I')$ be symmetric monoidal functors. A *monoidal natural transformation* $\beta \colon (F, m^F, m^F_I) \dot{\to} (G, m^G, m^G_I) \colon \mathbb{S} \to \mathbb{S}'$ is a natural transformation $\beta \colon F \dot{\to} G$ such that the diagrams

$$\begin{tikzcd}
FA \otimes' FB \arrow[r, "\beta_A \otimes' \beta_B"] \arrow[d, "m^F_{A,B}"'] & GA \otimes' GB \arrow[d, "m^G_{A,B}"] \\
F(A \otimes B) \arrow[r, "\beta_{A\otimes B}"'] & G(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
I' \arrow[r, equal] \arrow[d, "m^F_I"'] & I' \arrow[d, "m^G_I"] \\
FI \arrow[r, "\beta_I"] & GI
\end{tikzcd}$$

commute.

B.2.4 The 2-category of symmetric monoidal categories

Symmetric monoidal categories, symmetric monoidal functors and monoidal natural transformations are respectively the 0-cells, 1-cells and 2-cells of a 2-category. Therefore, notice that:

- The composition GF of symmetric monoidal functors $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{B} \to \mathbb{C}$ has a symmetric monoidal structure given by $\mathsf{m}^{(G\circ F)}{}_{A,A'} = G(\mathsf{m}^{F}{}_{A,A'}) \circ \mathsf{m}^{G}{}_{FA,FA'}$ and $\mathsf{m}_{I}^{(G\circ F)} = G(\mathsf{m}_{I}^{F}) \circ \mathsf{m}_{I}^{G}$. So, when asking for a diagram of monoidal functors to commutes, one implicitly also asks for the relative monoidal structures to coincide.

- For $F\colon \mathbb{A} \to \mathbb{B}$, $\alpha\colon H \overset{\bullet}{\to} K\colon \mathbb{B} \to \mathbb{C}$, $G\colon \mathbb{C} \to \mathbb{D}$ symmetric monoidal functors and monoidal natural transformation, $F\alpha$ and αG are monoidal natural transformations. Additionally, horizontal and vertical composition of monoidal natural transformations yields monoidal natural transformations.

Moreover:

- Given symmetric monoidal categories $(\mathbb{A}', \otimes', I', a', \lambda', \rho')$ and $(\mathbb{A}, \otimes, I, a, \lambda, \rho)$ the constant functor $I\colon \mathbb{A}' \to \mathbb{A}$ is (symmetric) monoidal with $\mathsf{m}^{I} = \lambda_I = \rho_I$ and $\mathsf{m}_{I}^{I} = \mathrm{id}_I$.

- For a symmetric monoidal category $(\mathbb{A}, \otimes, I, a, \lambda, \rho, s)$ the functor $(-\otimes-)\colon \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is monoidal with $\mathsf{m}^{(-\otimes-)} = \mathsf{sw}$ and $\mathsf{m}_{I}^{(-\otimes-)} = \lambda_I^{-1} = \rho_I^{-1}$. This defines $(-\otimes-)$ as a monoidal functor because all the diagrams that need to commute do by coherence.

B.2.5 Symmetric monoidal closed categories and functors

A *symmetric monoidal closed category* $(\mathbb{S}, \otimes, I, \multimap)$ is a symmetric monoidal category $(\mathbb{S}, \otimes, I)$ such that for any $B \in |\mathbb{S}|$, the functor $(-\otimes B)\colon \mathbb{S} \to \mathbb{S}$ has a specified right adjoint $(B \multimap -)\colon \mathbb{S} \to \mathbb{S}$. For any $B \in |\mathbb{S}|$, we write $\mathrm{ve}_A^{(B)}$ and $\mathrm{ev}_A^{(B)}$ for the components at $A \in |\mathbb{S}|$ of, respectively, the unit and counit of the adjunction $(-\otimes B) \dashv (B \multimap -)$.

Given a symmetric monoidal functor

$$(F, \mathsf{m}^{F}, \mathsf{m}_{I}^{F})\colon (\mathbb{S}, \otimes, I, \multimap) \to (\mathbb{S}', \otimes', I', \multimap')$$

between symmetric monoidal *closed* categories, there are morphism $k_{A,B} \colon F(A \multimap B) \to (FA \multimap' FB)$ defined by

$$\begin{tikzcd}
F(A \multimap B) \arrow[d, "ve^{(FA)}_{F(A\multimap B)}"'] & & (FA \multimap' FB) \\
(FA \multimap' (F(A \multimap B) \otimes' FA)) \arrow[rr, "(id_{FA} \multimap' m^F_{A\multimap B,A})"] & & (FA \multimap' F((A \multimap B) \otimes A)) \arrow[u, "(id_{FA} \multimap' F ev^{(A)}_B)"']
\end{tikzcd}$$

natural in A and $B \in |\mathbb{S}|$. A *symmetric monoidal closed functor* $\mathbb{S} \to \mathbb{S}'$ is simply a symmetric monoidal functor F. A *strong symmetric monoidal closed* functor is a strong symmetric monoidal functor F for which the $k_{A,B}$ are isomorphisms. A *strict symmetric monoidal closed* functor is a strict symmetric monoidal functor F for which $k_{A,B}$ are identities.

B.3 Commutative comonoids and diagonal structure

Let $(\mathbb{A}, \otimes, I, a, \lambda, \rho, \gamma)$ be a symmetric monoidal category. A *commutative comonoid* (A, t, Δ) in $\mathbb{A}$ is an object $A \in \mathbb{A}$ together with morphisms $t \colon A \to I$ and $\Delta \colon A \to A \otimes A$ such that the following four diagrams commute:

$$\begin{tikzcd}
 & A \arrow[dl, "\cong"'] \arrow[d, "\Delta"'] \arrow[dr, "\cong"] & \\
A \otimes I & A \otimes A \arrow[l, "id_A \otimes t"] \arrow[r, "t \otimes id_A"'] & I \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
A \otimes A \arrow[rr, "\gamma_{A,A}"] & & A \otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[r, "id_A \otimes \Delta"] & A \otimes (A \otimes A) \arrow[d, "\cong"] \\
A \otimes A \arrow[rr, "\Delta \otimes id_A"'] & & (A \otimes A) \otimes A
\end{tikzcd}$$

Let (A, t_A, Δ_A) and (B, t_B, Δ_B) be two commutative comonoids in $\mathbb{A}$. A *morphism of commutative comonoids* $(A, t_A, \Delta_A) \to (B, t_B, \Delta_B)$ is a morphism $f \colon A \to B$ in $\mathbb{A}$ such that the two following diagrams commute:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "t_A"'] & B \arrow[d, "t_B"] \\
I \arrow[r, equal] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\Delta_A"'] & B \arrow[d, "\Delta_B"] \\
A \otimes A \arrow[r, "f \otimes f"'] & B \otimes B
\end{tikzcd}
$$

A *diagonal structure* on $\mathbb{A}$ is a family $\{(A, t_A, \Delta_A)\}_{A \in |\mathbb{A}|}$ of commutative comonoids, indexed by the objects of $\mathbb{A}$, that respects the symmetric monoidal structure of $\mathbb{A}$, that is, such that the following three diagrams commute:

$$
\begin{tikzcd}
& A \otimes B \arrow[dl, "t_A \oplus t_B"'] \arrow[dr, "t_{A \oplus B}"] & \\
I \otimes I \arrow[rr, "\cong"'] & & I
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, equal, "t_I"] & I
\end{tikzcd}
$$

$$
\begin{tikzcd}
& A \otimes B \arrow[dl, "\Delta_A \oplus \Delta_B"'] \arrow[dr, "\Delta_{A \oplus B}"] & \\
(A \otimes A) \otimes (B \otimes B) \arrow[rr, "\cong"'] & & (A \otimes B) \otimes (A \otimes B)
\end{tikzcd}
$$

B.4 Enriched categories

We consider $\mathcal{V}$-enriched categories, functors and natural transformations only for $\mathcal{V}$=Pos,Dcpo,Sup. For the general theory of enriched category we refer to [30].

Definition B.1. A Pos-*enriched* category (or a Pos-category) is a category $\mathbb{X}$ such that $\mathbb{X}(A, B)$ is a poset for any pair of $\mathbb{X}$-objects A and B, and composition in $\mathbb{X}$ is monotone.

A functor $F\colon \mathbb{X} \to \mathbb{Y}$ between Pos-categories is a Pos-*enriched* functor (or a Pos-functor) if it preserves the order relation in the hom-objects.

Similarly:

Definition B.2. A Pos-category $\mathbb{X}$ is Dcpo-*enriched* (or a Dcpo-category) if for any pair of $\mathbb{X}$-objects A and B, $\mathbb{X}(A, B)$ is a directed complete poset and composition in $\mathbb{X}$ is Scott-continuous.

A functor $F\colon \mathbb{X} \to \mathbb{Y}$ between Dcpo-categories is a Dcpo-*enriched* functor (or a Dcpo-functor) if it preserves directed suprema in the hom-objects.

Definition B.3. A Pos-category $\mathbb{X}$ is Sup-*enriched* (or a Sup-category) if for any pair of $\mathbb{X}$-objects A and B, $\mathbb{X}(A, B)$ is a complete lattice and composition in $\mathbb{X}$ preserves arbitrary suprema in each component.

A functor $F\colon \mathbb{X} \to \mathbb{Y}$ between Sup-categories is a Sup-*enriched* functor (or a Sup-functor) if it preserves arbitrary suprema in the hom-objects.

In each of the cases $\mathcal{V}$=Pos, Dcpo, Sup, a $\mathcal{V}$-natural transformation $\alpha\colon F \to G\colon \mathbb{X} \to \mathbb{Y}$ is nothing but a natural transformation.

Pos-categories and foundational issues. Let $\mathbb{X}$ be a Pos-category, and consider functors $\mathbb{I} \to \mathbb{X}$ (with no enrichment requirement on $\mathbb{I}$ or the functors). The *class* of natural transformations between two functors inherits a point-wise structure of partial order. Vertical composition of natural transformations preserves this structure. That is to say, the category Funct$(\mathbb{I}, \mathbb{X})$ of functors and natural transformations is essentially Pos-enriched; the problem is that the hom-objects are not (small) sets. Foundationally, this can be mended by working in a set theory with Grothendieck universes. The hom-objects then live in the next universe.

The same is true if we consider the class of *lax transformations* (Definition 2.7) $F \overset{\odot}{\to} G\colon \mathbb{I} \to \mathbb{X}$: We obtain a category $\mathbb{X}^{\mathrm{lax}\,\mathbb{I}}$ of functors and lax transformations with respect to componentwise composition. The category $\mathbb{X}^{\mathrm{lax}\,\mathbb{I}}$ has then an obvious structure of Pos-category.

As far as enrichments are concerned in this thesis we do not care about these foundational issues and just call any category of functors and lax (or natural) transformations into a Pos-category again Pos-enriched.

This is justified by the fact that for all the constructions that lead to a problem of *size* we can just use the legitimate embedding of one universe into the next level.

Fully faithful Pos-functors. Let $\mathbb{A}$, $\mathbb{B}$ be two Pos-categories and $F\colon \mathbb{A} \to \mathbb{B}$ a Pos-functor. The functor F is said to be *fully faithful* if for any pair of objects A, A' in $\mathbb{A}$, the monotone map $\mathbb{A}(A, A') \to \mathbb{B}(FA, FA')$ determined by F is an isomorphism of posets. This property is strictly stronger than asking for $\mathbb{A}(A, A') \to \mathbb{B}(FA, FA')$ to be a monotone map with an inverse function. That is, a fully faithful Pos-functor is more than a Pos-functor whose underlying (Set-)functor is full and faithful.

Pos-adjunctions. (Kelly. [30], 1.6–1.11) Let $\mathbb{A}$ and $\mathbb{B}$ be two Pos-categories. For a pair of functors

$$\begin{tikzcd} \mathbb{A} \arrow[r, shift left, "F"] & \mathbb{B} \arrow[l, shift left, "G"] \end{tikzcd}$$

and natural isomorphisms

$$n_{A,B} \colon \mathbb{A}(A, GB) \cong \mathbb{B}(FA, B)$$

the following is true:

$$\begin{aligned} F \text{ is a } \mathsf{Pos}\text{-functor} &\iff n_{A,B} \text{ is monotone (for any } A \in |\mathbb{A}|,\ B \in |\mathbb{B}|); \\ G \text{ is a } \mathsf{Pos}\text{-functor} &\iff n_{A,B}^{-1} \text{ is monotone (for any } A \in |\mathbb{A}|,\ B \in |\mathbb{B}|). \end{aligned}$$

Therefore a Pos-adjunction between Pos-functors corresponds to natural order isomorphisms $\mathbb{A}(A, GB) \cong \mathbb{B}(FA, B)$.

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