Stably compact spaces and the probabilistic powerspace construction

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published 2004 (last update: 11 May 2020)

Abstract

We put forward a revised definition of *stably compact spaces* which allows us to show their equivalence with Nachbin's *compact ordered spaces* in an entirely elementary fashion. We then exhibit some constructions for stably compact spaces which apparently have not appeared in the literature before. These constructions allow us to show that the set of (sub-)probability valuations can be equipped with a topology which turns this set into another stably compact space. The topology chosen is not random; it is the weakest topology which makes integration of lower semicontinuous functions a continuous operation.

1   Introduction

This text contains the notes of a talk given at the Bellairs Research Centre in Barbados in April 2003. It is intended to explain the *topological* background to our work

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with Mathias Kegelmann, M. Andrew Moshier, and Philipp Sünderhauf on a *continuous* version of Domain Theory in Logical Form (a programme introduced in [Abr87]), published under the heading *Multi-lingual Sequent Calculus*, [JS96, JKM97, JKM99, Keg02, JKM01, MJ02, Mos]. From a classical perspective, the Multi-lingual Sequent Calculus provides logical descriptions for *stably compact spaces*, and it is this concept which is here developed to some extent. A lot of the material below is not new but the fundamental facts are somewhat scattered in the literature. It is hoped that these notes, over time, will be developed into a more comprehensive account of stable compactness.

One reason why a directly accessible source for information on these topological spaces is still missing, perhaps, is the fact that they are commonly defined as a certain class of *sober* spaces. This points to their prominent role in Stone Duality but unfortunately makes the subject somewhat specialised for mainstream mathematicians and computer scientists. Below we will work with an equivalent definition which only uses the basic concepts of "open" and "compact". The possibility of such a definition was recently pointed out in the presentation of the subject in [GHK+03, Section VI-6]. Besides only employing elementary topological concepts, the new definition makes it clear that stably compact spaces are precisely the T_0 analogues of compact Hausdorff spaces, in particular, they are those T_0 spaces in which (saturated) compact subsets behave as one is used to: They can be intersected arbitrarily, and whenever an intersection belongs to an open set so does a finite intersection already. Finally, it is possible to explain in an entirely elementary fashion the precise relationship between Nachbin's *compact ordered spaces*, [Nac65], and stably compact spaces. This first appeared in [GHK+80, Exercises VII-1.16-19] and is here carried out in sections 2.1 to 2.3.

In Section 2.5 we present some constructions for stably compact spaces which in some form or other have surely been observed before but again, it may be helpful to have them collected together in one place. Our own motivation for studying them is related to the *probabilistic powerspace construction*, introduced into the world of Semantics by Saheb-Djahromi, [SD80], and studied from a domain theoretic viewpoint by a number of authors, [JP89, Jon90, Kir93, Tix95, Hec96, JT98, AM01]. The elements of the probabilistic powerspace are *valuations*, which are functions assigning a "measure" to every open set of a topological space. This in itself is sufficient to define integration of (semi-)continuous functions (for a survey, see [Law]) but the relationship with measures has also been explored, see [AM01] and the article by Klaus Keimel in this volume.

It was a bothersome aspect of the probabilistic powerspace construction that it could not be restricted to any of the well-known classes of continuous domains, [JT98], and this issue remains unresolved. However, the class of stably compact spaces *does* support it, and in the second part of this note we present a proof of this fact. It requires us to define a topology for the set of (probability) valuations which is again stably compact. Of course, the topology must be meaningful, and indeed, we can show that it is the weakest topology which makes integration of semicontinuous functions a continuous operation.

The author is preparing a paper together with Klaus Keimel which will put the results presented here in a wider context. This joint work has already influenced the present text and Klaus Keimel's suggestions and comments are gratefully acknowledged. Comments from Reinhold Heckmann and Martín Escardó have helped to iron out some unevenness in presentation in an earlier version.

2 Compact ordered and stably compact spaces

2.1 Compact ordered spaces

A *partially ordered topological space* (or *ordered space*, for short) in the sense of Nachbin [Nac65] is a set X with a topology $\mathcal{O}$ and a partial order $\leq$ such that the graph of the order is closed in $X \times X$. This captures the (reasonable) assumption that, for two converging nets $x_i \to x$ and $y_i \to y$, the property $x_i \leq y_i$ for all $i \in I$ implies $x \leq y$. In terms of open sets, this is equivalent to saying that for any two points $x \not\leq y$ in X there are open sets U containing x and V containing y such that for every $x' \in U$ and $y' \in V$, $x' \not\leq y'$ holds. Since $x \neq y$ is equivalent to "$x \not\leq y$ or $y \not\leq x$", it follows that ordered spaces are Hausdorff.

A subset U of X is called an *upper* (*lower*) set, if $x \in U$ implies $y \in U$ for all $y \geq x$ (resp., $y \leq x$). The smallest upper (lower) set containing a subset A is denoted $\uparrow A$ (resp., $\downarrow A$). In an ordered space sets of the form $\uparrow x = \uparrow\{x\}$ or $\downarrow x = \downarrow\{x\}$ are always closed. More generally, one has:

Lemma 1. *If A is a compact subset of a partially ordered space $(X, \mathcal{O}, \leq)$ then $\uparrow A$ and $\downarrow A$ are closed.*

Proof. Consider $x \notin \uparrow A$. For every $y \in A$ we have $y \not\leq x$, so we find open sets $y \in U_y, x \in V_y$ where no element of V_y is above any element of U_y. The collection of all $U_y, y \in A$ covers A and by compactness a finite subcollection $U_{y_1}, \ldots, U_{y_n}$ does so as well. We form the intersection of the corresponding V_{y_i} and obtain a neighbourhood of x, no element of which is above any element of A. In other words, $\bigcap_{i=1}^{n} V_{y_i}$ is disjoint from $\uparrow A$.

The claim for $\downarrow A$ is proved analogously. $\square$

Our emphasis in this note is on partially ordered spaces which are compact. In this case the preceding observation has strong consequences as was first noted by Leopoldo Nachbin [Nac65]:

Lemma 2. *Let $(X, \mathcal{O}, \leq)$ be a compact ordered space.*

(i) (Order normality) *Let A and B be disjoint closed subsets of X, where A is an upper and B is a lower set. Then there exist disjoint open neighbourhoods $U \supseteq A$ and $V \supseteq B$ where again U is an upper and V is a lower set.*

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(ii) (Order separation) *Whenever $x \nleq y$ there exist an open upper set U containing x and an open lower set V containing y which are disjoint.*

(iii) (Order Urysohn property) *For every pair A, B of disjoint closed subsets, where A is an upper and B is a lower set, there exists a continuous order-preserving function into the unit interval which has value 1 on A and 0 on B.*

Proof. By normality of compact Hausdorff spaces, A and B have disjoint open neighbourhoods U' and V'. Observe that $X \setminus U'$ and $X \setminus V'$ are compact subsets of X, so we can employ Lemma 1 and set $U = X \setminus {\downarrow}(X \setminus U')$ and $V = X \setminus {\uparrow}(X \setminus V')$. Order separation is a special case of order normality, and the order preserving version of Urysohn's Lemma follows, as usual, by repeated application of order normality. $\qquad\square$

2.2   The upwards topology of a compact ordered space

One way to interpret Lemma 2 is to say that there is an abundance of open *upper* sets in a compact ordered space. For any ordered space, the set

$$\mathcal{U} := \{ U \in \mathcal{O} \mid U = {\uparrow}U \}$$

of open upper sets is a topology coarser than the original one; we call it the *topology of convergence from below* or *upwards topology* for short. The resulting topological space $(X, \mathcal{U})$ we denote by $X^{\uparrow}$.

Sets of the form $X \setminus {\downarrow}x$ always belong to $\mathcal{U}$ and therefore every upper set is equal to the intersection of its $\mathcal{U}$-open neighbourhoods, that is, it is $\mathcal{U}$-*saturated*. The converse direction being trivial (i.e., intersections of upper sets are always upper), we thus have:

Proposition 3. *In an ordered space the upper sets are precisely the $\mathcal{U}$-saturated ones.*

For a general topological space $(X, \mathcal{G})$ one sets $x \leq_{\mathcal{G}} y$ if every neighbourhood of x also contains y. This is always a preorder and it is anti-symmetric if and only if the space is T_0. It is called the *specialisation order associated with* $\mathcal{G}$. The preceding proposition tells us that $\leq_{\mathcal{U}}$ is precisely the original order $\leq$ in any ordered space.

In order to analyse the properties of $\mathcal{U}$ further in the case where $(X, \mathcal{O}, \leq)$ is compact, we also consider the set of compact saturated sets:

$$\mathcal{K}_{\mathcal{U}} := \{ K \subset X \mid K \text{ is } \mathcal{U}\text{-saturated and } \mathcal{U}\text{-compact} \}$$

Lemma 4. *Let $(X, \mathcal{O}, \leq)$ be a compact ordered space. The elements of $\mathcal{K}_{\mathcal{U}}$ are precisely those subsets of X which are upper and closed with respect to $\mathcal{O}$.*

Proof. The upper closed sets of X are $\mathcal{U}$-compact because the topology $\mathcal{U}$ is weaker than $\mathcal{O}$. For the converse we use order separation: Let $x \notin A \in \mathcal{K}_{\mathcal{U}}$. For every $y \in A$ we have $y \nleq x$ and hence find an open upper set $y \in U_y$ and an open lower set $x \in V_y$ which are disjoint. By compactness, finitely many U_y cover A and the intersection of the corresponding V_y will provide the open neighbourhood of x disjoint from A. $\qquad\square$

We now have enough information to show that from $\mathcal{U}$ alone we can reconstruct the original compact ordered space. In general, one considers the *patch topology* $\mathcal{G}_p$ of a topological space $(X, \mathcal{G})$ by augmenting $\mathcal{G}$ with complements of compact saturated sets. With this terminology we can formulate the following:

Theorem 5. *Let $(X, \mathcal{O}, \leq)$ be a compact ordered space. Then $\mathcal{O} = \mathcal{U}_p$ and $\leq\, = \,\leq_{\mathcal{U}}$.*

Proof. Because of Lemma 4, $\mathcal{U}_p$ is contained in $\mathcal{O}$. It is Hausdorff because of order separation and therefore the identity map $i\colon (X, \mathcal{O}) \to (X, \mathcal{U}_p)$ is a homeomorphism.

The possibility to reconstruct the order out of the upwards topology has been remarked before. $\qquad\square$

Since with $(X, \mathcal{O}, \leq)$, the "upside-down" space $(X, \mathcal{O}, \geq)$ is also compact ordered, the results in this section hold equally well for the *topology $\mathcal{D}$ of convergence from above* or *downwards topology*. By Lemma 4, its open sets are precisely the complements of the compact saturated sets of $\mathcal{U}$.

2.3 Stably compact spaces

As it turns out, topologies which arise as upwards topologies in compact ordered spaces can be characterised intrinsically. We begin with the following observations:

Proposition 6. *For a compact ordered space $(X, \mathcal{O}, \leq)$ the upwards topology $\mathcal{U}$ is*

(i) *T_0;*

(ii) *compact;*

(iii) *locally compact;*

(iv) *coherent, that is, pairs of compact saturated sets have compact intersection;*

(v) *well-filtered, that is, for any filter base $(A_i)_{i \in I}$ of compact saturated sets, for which $\bigcap_i A_i$ is contained in an open upper set U, there is an index i_0 such that A_{i_0} is contained in U already.*

Proof. The T_0 separation property follows from order separation, (ii) is trivially true because $\mathcal{U}$ is weaker than $\mathcal{O}$, and (iii) is a reformulation of order normality. Coherence and well-filteredness follow from Lemma 4 which says that the compact saturated sets of $(X, \mathcal{U})$ are compact subsets in the original Hausdorff space $(X, \mathcal{O}, \leq)$. $\qquad\square$

Definition 7. *A T_0 space which is compact, locally compact, coherent, and well-filtered is called* stably compact.

In recent literature it has been customary to use "sober" instead of "well-filtered" in the definition of stably compact spaces. However, in the presence of local compactness these two properties are equivalent, [GHK$^+$03, Theorem II-1.21]. With this note we would like to make a case for the revised definition, because it makes it apparent that *stably compact spaces are the T_0-analogue of compact Hausdorff spaces*, in the sense that compact saturated sets in the former have the same properties as compact subsets in the latter.[1] The following lemma illustrates this:

Lemma 8. *Let $(X, \mathcal{U})$ be a stably compact space. Then any collection of compact saturated subsets has compact intersection.*

Proof. Finite intersections leading again to compact saturated subsets, we can assume the collection to be filtered. By well-filteredness, an open cover of the intersection will contain an element of the filter base already. This being compact, a finite subcover will suffice. □

This result justifies the following definition.

Definition 9. *Let $(X, \mathcal{U})$ be a stably compact space. The* co-compact topology $\mathcal{U}_\kappa$ *on X is given by the complements of compact saturated sets.*

The reader is correct in suspecting that the passage to the co-compact topology is an involution for stably compact spaces. This will follow easily from Theorem 12 and is recorded as Corollary 13 below. For the moment we consider the following proposition which is reminiscent of the well-known fact that a compact Hausdorff-topology cannot be weakened without losing separation.

Proposition 10. *Let $(X, \mathcal{U})$ be a stably compact space. Let further $\mathcal{B}$ be a subset of $\mathcal{U}$ and $\mathcal{C}$ a subset of the co-compact topology $\mathcal{U}_\kappa$, such that the following property holds:*

$$\forall x, y \in X. \; x \not\leq_\mathcal{U} y \Rightarrow \exists U \in \mathcal{B}, L \in \mathcal{C}. \; x \in U, y \in L, L \cap U = \emptyset \, .$$

Then $\mathcal{B}$ is a subbasis for $\mathcal{U}$.

Proof. Let x be an element of an open set $O \in \mathcal{U}$. Then by assumption for every y in $X \setminus O$ there exist disjoint sets $U_y \in \mathcal{B}$ and $L_y \in \mathcal{C}$ which contain x and y, respectively. The complements of the L_y are compact saturated by definition and their intersection is contained in O. Well-filteredness tells us that the same is true for a finite subcollection of L_y's. The intersection of the corresponding U_y is a neighbourhood of x contained in O. □

Corollary 11. *Let $\mathcal{U}$ and $\mathcal{U}'$ be stably compact topologies on a set X such that $\leq_\mathcal{U} = \leq_{\mathcal{U}'}$, $\mathcal{U} \subseteq \mathcal{U}'$ and $\mathcal{K}_\mathcal{U} \subseteq \mathcal{K}_{\mathcal{U}'}$. Then $\mathcal{U} = \mathcal{U}'$.*

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[1] I am indebted to Paweł Waszkiewicz, whose interest in the topic forced me to reconsider the definition of stable compactness.

We are now ready to complete the link with compact ordered spaces.

Theorem 12. *Let $(X, \mathcal{U})$ be a stably compact space. Consider its patch topology $\mathcal{U}_p$ and specialisation order $\leq_{\mathcal{U}}$. Then $(X, \mathcal{U}_p, \leq_{\mathcal{U}})$ is a compact ordered space. Furthermore, the upwards topology arising from $\mathcal{U}_p$ and $\leq_{\mathcal{U}}$ is equal to $\mathcal{U}$, and the co-compact topology $\mathcal{U}_\kappa$ is equal to the topology of convergence from above derived from $\mathcal{U}_p$ and $\leq_{\mathcal{U}}$.*

Proof. The Hausdorff separation property and the closedness of $\leq_{\mathcal{U}}$ follow from T_0 and local compactness. Compactness of the patch topology requires the Axiom of Choice in the form of Alexander's Subbase Lemma: Let $\mathcal{B} \cup \mathcal{C}$ be a covering of X where the open sets in $\mathcal{B}$ are chosen from $\mathcal{U}$ and the ones in $\mathcal{C}$ are complements of compact saturated sets. The points not covered by the elements of $\mathcal{C}$ form a compact saturated set by Lemma 8 and must be covered by elements of $\mathcal{B}$. A finite subcollection $\mathcal{B}' \subseteq_{\text{fin}} \mathcal{B}$ will suffice for the purpose. By well-filteredness, then, a finite intersection of complements of elements of $\mathcal{C}$ will be contained in $\bigcup \mathcal{B}'$ already. This completes the selection of a finite subcover, and we have shown that $(X, \mathcal{U}_p, \leq_{\mathcal{U}})$ is a compact ordered space.

The same argument shows that every compact saturated set in $(X, \mathcal{U})$ is also compact in the patch topology.

The specialisation order that one derives from the topology of convergence from below on the space $(X, \mathcal{U}_p, \leq_{\mathcal{U}})$ is the same as $\leq_{\mathcal{U}}$ by Theorem 5.

We are therefore in the situation described by Corollary 11 and can conclude that no new open upper sets arise in the patch construction. Lemma 4, then, tells us that the closed upper sets in $(X, \mathcal{U}_p, \leq_{\mathcal{U}})$ are precisely the compact saturated sets of $\mathcal{U}$. Hence the co-compact topology with respect to $\mathcal{U}$ is equal to the topology of convergence from below on $(X, \mathcal{U}_p, \leq_{\mathcal{U}})$. $\qquad\square$

Corollary 13. *Let $(X, \mathcal{U})$ be a stably compact space.*

(i) The co-compact topology $\mathcal{U}_\kappa$ is also stably compact.

(ii) $(\mathcal{U}_\kappa)_\kappa = \mathcal{U}$

2.4 Examples

The prime example of an ordered space is given by the real line with the usual topology and the usual order. The upwards topology in this case consists of sets of the form $]r, \infty[$ (plus $\mathbb{R}$ and $\emptyset$, of course), and non-empty compact saturated sets associated to this, in turn, are the sets of the form $[r, \infty[$. We denote the real line with the upwards topology by $\mathbb{R}^\uparrow$. Also of interest to us is the non-negative part of this, denoted by $\mathbb{R}_+^\uparrow$. One obtains a *compact* ordered space by either restricting to a compact subset, such as the unit interval, or by extending the real line with elements at infinity in the usual way, denoted here by $\overline{\mathbb{R}} = [-\infty, \infty]$ and $\overline{\mathbb{R}}_+ = [0, \infty]$.

In general, one cannot expect a compact ordered space to be fully determined by its order alone, after all, every compact Hausdorff space can be equipped with a trivial closed order, namely, the identity relation. The before-mentioned semantic domains, however, do provide examples where the order structure is rich enough to determine a non-trivial stably compact topology. We review the definitions: A *dcpo* (for *directed-complete partial order*) is an ordered set in which every directed subset has a supremum. The closed sets of the *Scott-topology* σ_D of a dcpo D are those lower sets which are closed under formation of directed suprema. It follows that a function between dcpos is continuous with respect to the two Scott-topologies if and only if it preserves the order and suprema of directed sets. In order to emphasise the dcpo context, such functions are usually called *Scott-continuous*.

The specialisation order associated with the Scott-topology, which is always T_0, will give back the original order of the dcpo. An element x of a dcpo D is *way-below* an element y (written $x \ll y$) if whenever y is below the supremum of a directed set $A \subseteq D$, then x is below some element of A. A dcpo D is *continuous* or a *domain* if every element equals the directed supremum of its way-below approximants.

The Scott-topology of a domain is always well-filtered, [Jun89, Lemma 4.12], and coherence can be characterised in an order-theoretic fashion as well, see [Jun89, Lemma 4.18], [GHK$^+$03, Proposition III-5.12]. As a special case, coherence holds in every continuous complete lattice (known as *continuous lattice* for short). Two examples are of interest here: The unit interval $[0, 1]$ (or $\overline{\mathbb{R}}$ or $\overline{\mathbb{R}}_+$) is a continuous lattice and the Scott-topology is precisely the topology of convergence from below, discussed before. An element x of $[0, 1]$ is way-below y if $x = 0$ or $x < y$. The other class of examples is given by open set lattices of locally compact spaces. Here, the way-below relation is characterised by $U \ll V$ if and only if there exists a compact saturated set K such that $U \subseteq K \subseteq V$. Stably compact spaces qualify, and their open set lattices have the additional property (not true in general) that $U \ll V_1$ and $U \ll V_2$ imply $U \ll V_1 \cap V_2$.

The topic of this note is connected to Domain Theory via the *Lawson-topology* λ, which is defined as the extension of the Scott-topology with complements of principal upper sets $\uparrow x$. It is easy to see that in a domain every compact saturated set (with respect to σ_D) is the intersection of finite unions of principal upper sets, and so in this context the Lawson-topology is precisely the patch topology derived from σ_D.

Furthermore, a domain is λ-compact if and only if σ_D is stably compact. Since σ_D always satisfies requirements (i)–(iii) and (v) for a stably compact space as listed in Proposition 6, it is reasonable to call λ-compact domains *coherent*. This is the terminology introduced in [JS96] and adopted in [GHK$^+$03].

Coherent domains have played a significant role in the development of Domain Theory. Without attempting completeness, we remind the reader of the following appearances of the concept in the literature. In [Plo76] a very large class of (algebraic) domains is introduced, called *SFP-objects* (and later also referred to as *bifinite domains*). Plotkin's "2/3 SFP Theorem", [Plo81], states that coherence accounts for two of the three defining properties of SFP-domains. In [Jun90], a maximality result is

shown for cartesian closed categories of domains; coherence is the crucial concept there. Finally, in [JT98] we showed that the probabilistic powerdomain of a coherent domain is again coherent. The results in Section 3 below are a direct generalisation of this but the proof is much simpler.

2.5   Morphisms and constructions

Although theorems 5 and 12 suggest that we can switch freely between compact ordered and stably compact spaces, a difference between the two standpoints does become apparent when one considers the corresponding morphisms: neither is a continuous map between stably compact spaces patch continuous, nor is every patch continuous function continuous with respect to the original topologies. Indeed, it is the fact that T_0-continuous maps arise in applications to Denotational Semantics which motivates our interest in stably compact spaces.

Nevertheless, a connection between *subclasses* of continuous maps can be made. A continuous map $f\colon X \to X'$ between locally compact spaces is called *perfect* if the preimage $f^{-1}(K)$ of every compact saturated set $K \subseteq X'$ is compact in X. The following is true:

Proposition 14. *For locally compact spaces $(X, \mathcal{U})$ and $(X', \mathcal{U}')$ a map $f\colon X \to X'$ is perfect, if and only if it is continuous with respect to the patch topologies on X and X' and monotone (i.e., order preserving) with respect to the specialisation orders.*

In the remainder of this section we study some constructions on spaces and how they interact with the translations given in theorems 5 and 12.

Proposition 15. *Arbitrary products of stably compact spaces are stably compact, and the product topology equals the upwards topology of the product of the corresponding compact ordered spaces.*

Proof. Let $(X_i, \mathcal{U}_i)_{i \in I}$ be any family of stably compact spaces and let $(X_i, \mathcal{O}_i, \leq_i)$ be the corresponding compact ordered spaces. We prove the second claim because it entails the first. By Tychonoff's Theorem the product $\mathcal{O}$ of the patch topologies $\mathcal{O}_i$ is again compact Hausdorff, and the shape of basic open sets in the product gives immediately that the coordinatewise order $\leq$ is closed. So $(\prod_{i \in I} X_i, \mathcal{O}, \leq)$ is a compact ordered space.

A basic open set from the product of the $\mathcal{U}_i$ is also open in $\mathcal{O}$. For the converse we employ Proposition 10, where the product of the $\mathcal{U}_i$ plays the role of $\mathcal{B}$ and the product of the respective co-compact topologies $(\mathcal{U}_i)_\kappa$ plays the role of $\mathcal{C}$ in the stably compact space derived from $(\prod_{i \in I} X_i, \mathcal{O}, \leq)$. The separation property is obviously satisfied because $x \not\leq y$ means $x_i \not\leq y_i$ for some index i. $\square$

Subspaces do not, in general, inherit any of the properties under consideration, except that the order remains closed. However, we have the following:

Proposition 16. *Let Y be a patch-closed subset of a stably compact space $(X, \mathcal{U})$. Then Y is stably compact when equipped with the subspace topology $\mathcal{U}\!\restriction_Y$, and $(\mathcal{U}\!\restriction_Y)_p = \mathcal{U}_p\!\restriction_Y$.*

Proof. The subspace $(Y, \mathcal{U}_p\!\restriction_Y, \leq\!\restriction_{Y \times Y})$ is of course again a compact ordered space. If A is a closed lower set in Y, then its lower closure $\downarrow\!A$ in X is again closed as A is compact in X. This shows that the upper opens of $(Y, \mathcal{U}_p\!\restriction_Y, \leq\!\restriction_{Y \times Y})$ belong to $\mathcal{U}\!\restriction_Y$. The converse inclusion is trivial. $\qquad\square$

The second case where we know something about the stable compactness of a subspace is related to continuous retractions. This fact is mentioned in [Law88] already but the proof uses a different characterisation of stable compactness.

Proposition 17. *Let Y be a continuous retract of a stably compact space X. Then Y is stably compact.*

Proof. Let $e\colon Y \to X$ be the section and $r\colon X \to Y$ the retraction map (both continuous). We check the defining properties for stable compactness. First of all, Y is a T_0-space because e is injective. The compactness of Y follows from the continuity of the (surjective) map r. If $x \in O \subseteq Y$, with O open in Y, then $r^{-1}(O)$ is an open neighbourhood of $e(x)$. Hence there is an open set U and a compact saturated set L in X such that $e(x) \in U \subseteq L \subseteq r^{-1}(O)$. The image of L under r is compact in Y, is contained in O, and contains the open set $e^{-1}(U)$ which contains x. This proves that Y is locally compact.

For stability, let K_1, K_2 be compact saturated sets in Y. We get that $e(K_1)$ and $e(K_2)$ are compact in X and hence $\uparrow\!e(K_i)$ is compact saturated in X. By the stability of X the intersection $(\uparrow\!e(K_1)) \cap (\uparrow\!e(K_2))$ is compact again. Its image under r is precisely $K_1 \cap K_2$; it is compact in Y by the continuity of r. Well-filteredness is shown in the same way. $\qquad\square$

Note that e does not need to be a perfect map in general, so the result is not subsumed by Proposition 16 already.[2]

3 The probabilistic powerspace

3.1 Valuations

For a topological space $(X, \mathcal{G})$ we consider maps $\mu\colon \mathcal{G} \to \mathbb{R}$ with the following properties

- $\mu(\emptyset) = 0$ (strict);

- $\forall O, O' \in \mathcal{G}.\ \mu(O) + \mu(O') = \mu(O \cap O') + \mu(O \cup O')$ (modular);

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[2]Perfectness of e is guaranteed if e is an upper adjoint. This situation is called an *insertion-closure pair* in [AJ94, Section 3.1.5].

- $\forall O, O' \in \mathcal{G}.\ O \subseteq O' \Rightarrow \mu(O) \leq \mu(O')$ (monotone).

Functions of this kind are called (*monotone*) *valuations*; they are the elements of $\mathfrak{m}\mathfrak{V}(X)$. We are interested in the case where the following additional property is satisfied

- $\mu(\bigcup_{i \in I} O_i) = \sup_{i \in I} \mu(O_i)$ (Scott-continuous)

for all *directed* families $(O_i)_{i \in I}$ of open sets. If we equip the complete lattice $(\mathcal{G}, \subseteq)$ with its Scott-topology, and likewise $(\mathbb{R}, \leq)$ with the topology of convergence from below, then the continuity of a valuation is just the topological one. We restrict further by stipulating

- $\mu(X) = 1$ (normalised)

which brings us to continuous *probability valuations*. We denote the set of all continuous probability valuations by $\mathfrak{V}_1(X)$. Likewise, the condition $\mu(X) \leq 1$ gives rise to the set of *sub-probability valuations*, denoted by $\mathfrak{V}_{\leq 1}(X)$.

We equip $\mathfrak{V}_1(X)$ with the topology inherited from the product topology on $[0,1]^{\mathcal{G}}$, where $[0,1]$ carries the upwards topology. For simplicity, we call it the *product topology* and we denote it by $\mathcal{P}$. Likewise, we consider the *product order* $\leq_{\mathcal{P}}$ which is inherited from the natural order on $\mathbb{R}$.

On stably compact spaces there is a close relationship between monotone and continuous valuations, a fact which we will make crucial use of in the next section.

Proposition 18. *Let $(X, \mathcal{U})$ be a stably compact space and $\mu \colon \mathcal{U} \to \overline{\mathbb{R}}_+$ be a valuation. The following defines the largest continuous valuation below μ in the pointwise order:*

$$\Phi(\mu)(O) := \sup\{\mu(V) \mid V \ll O\}$$

where $V \ll O$ means that there is a compact saturated set K such that $V \subseteq K \subseteq O$. Furthermore, the operation $\Phi \colon \mathfrak{m}\mathfrak{V}(X) \to \mathfrak{m}\mathfrak{V}(X)$ is idempotent and continuous with respect to the product topology, and maps (sub-)probability valuations to (sub-)probability valuations.

Proof. It is clear that $\Phi(\mu)(\emptyset) = 0$ holds, and that $\Phi(\mu)$ is monotone. For the modular law, we exploit stable compactness which gives us that $O \cap O'$ is approximated by sets of the form $V \cap V'$ where $V \ll O$ and $V' \ll O'$. The continuity of $\Phi(\mu)$ follows from its definition.

If ν is any other *continuous* valuation below μ, then for every $O \in \mathcal{U}$ we have $\nu(O) = \sup\{\nu(V) \mid V \ll O\}$ by local compactness and continuity, hence $\nu \leq \Phi(\mu)$.

A continuous valuation is kept fixed by Φ because every open set equals the directed union of those open sets way-below it.

In order to see that the operation of making a valuation continuous is itself continuous with respect to the product topology on $\mathfrak{m}\mathfrak{V}(X)$, observe that $\Phi(\mu)(O)$ is greater than a real number r, if and only if $\mu(V) > r$ for some $V \subseteq K \subseteq O$. Hence the preimage of the subbasic open set $\{\mu \in \mathfrak{m}\mathfrak{V}(X) \mid \mu(O) > r\}$ equals $\bigcup_{V \ll O}\{\mu \in \mathfrak{m}\mathfrak{V}(X) \mid \mu(V) > r\}$.

The last statement follows immediately from the fact that the whole space X is compact and open at the same time. $\square$

3.2 The product topology on spaces of valuations

We now have all the tools to show that the product topology restricted to $\mathfrak{V}_1(X)$ is stably compact. We start with the stably compact space $Y = \prod_{O \in \mathcal{U}}[0,1]$, where each copy of the unit interval is equipped with the upwards topology. The corresponding patch topology is just the product topology of the usual metric topology. The projection $\pi_\emptyset \colon Y \to [0,1]$ is patch-continuous, and hence the preimage $\pi_\emptyset^{-1}(0)$ is patch-closed in Y. Likewise for the preimage $\pi_X^{-1}(1)$. Assume $O \subseteq O' \in \mathcal{U}$; then we can use the projections π_O and $\pi_{O'}$ to excise the set of tuples μ for which $\mu(O) \leq \mu(O')$. Again, this is a patch-closed subset of Y. Exploiting the continuity of addition in the usual topology on $\mathbb{R}$ we can restrict further to a patch-closed subset of tuples satisfying the modular law. By invoking Proposition 16 we have thus shown that the set $\mathfrak{m}\mathfrak{V}_1(X)$ of monotone sub-probability valuations on a stably compact space X is stably compact when equipped with the product topology $\mathcal{P}$. The same is obviously true of $\mathfrak{m}\mathfrak{V}_{\leq 1}(X)$.

Now we can apply the continuous retraction Φ from Proposition 18 to conclude with the help of Proposition 17:

Theorem 19. *The set $\mathfrak{V}_1(X)$ of continuous probability valuations on a stably compact space X is stably compact when equipped with the product topology $\mathcal{P}$. The same holds for $\mathfrak{V}_{\leq 1}(X)$.*

For the remainder of this paper we restrict ourselves to continuous valuations and we ask whether the product topology can be characterised in other ways. Specifically, we will compare $\mathcal{P}$ with *weak topologies* which arise as the initial topologies making certain *test functions* $\gamma \colon \mathfrak{V}(X) \to \overline{\mathbb{R}}$ continuous. For example, every open set $O \in \mathcal{U}$ gives rise to a test function $\gamma_O \colon \mathfrak{V}(X) \to \overline{\mathbb{R}}_+^\uparrow$ which evaluates a valuation at O. Of course, these maps are just the projections π_O, which we considered above, and the product topology is the initial topology making all γ_O, $O \in \mathcal{U}$, continuous.

More interestingly, let $\mathrm{LSC}(X)$ denote the set of lower continuous functions on $(X, \mathcal{U})$ into the extended non-negative reals. A number of authors, [Jon90, Kir93, Tix95, Hec96], have given definitions (of increasing generality) of an integral for lower semicontinuous functions with respect to continuous valuations. The definitions by Tix and Heckmann are directly applicable to stably compact spaces, and give the same result, [Hec96, page 197] (see also [AM01, Section 6.3]). Thus every lower semicontinuous function $f \colon X \to \overline{\mathbb{R}}_+^\uparrow$ gives rise to a test function $\gamma_f \colon \mathfrak{V}(X) \to \overline{\mathbb{R}}_+^\uparrow$ by setting $\gamma_f(\mu) := \int f d\mu$. The following is now entirely straightforward to show ([Kir93, Satz 8.5], [Tix95, Lemma 4.9], [Hec96, Theorem 8.3]):

Theorem 20. *The product topology restricted to $\mathfrak{V}_1(X)$ and $\mathfrak{V}_{\leq 1}(X)$, respectively, is equal to the weak topology arising from the test functions γ_f for $f \in \mathrm{LSC}(X)$.*

In other words, a net $(\mu_i)_{i \in I}$ of valuations converges to a valuation μ with respect to $\mathcal{P}$, if and only if the values of the integrals $\int f d\mu_i$ converge to $\int f d\mu$ in $\overline{\mathbb{R}}_+^\uparrow$ for every lower semicontinuous function $f \colon X \to \overline{\mathbb{R}}_+^\uparrow$.

A subbasis for the product topology on $\mathfrak{V}_1(X)$ is given by sets of the form

$$[O > r] := \{\mu \in \mathfrak{V}_1(X) \mid \mu(O) > r\}$$

where $O \in \mathcal{U}$ and $r \in \mathbb{R}$. This is the starting point for the logical description of the probabilistic powerspace construction carried out in [MJ02]. The fact that the sets

$$[K \geq r] := \{\mu \in \mathfrak{V}_1(X) \mid \forall O \in \mathcal{U}.\ K \subseteq O \Rightarrow \mu(O) \geq r\}$$

where $K \in \mathcal{K}_{\mathcal{U}}$ and $r \in \mathbb{R}$, are a subbasis for the set of compact saturated sets in $(\mathfrak{V}_1(X), \mathcal{P})$, is also important for the smooth working of the logical framework. Of course, these two observations are also true for $\mathfrak{V}_{\leq 1}(X)$.

The natural order between valuations, $\mu \preccurlyeq \nu$ iff $\forall O \in \mathcal{G}.\ \mu(O) \leq \nu(O)$, can easily be shown to be directed-complete, and it therefore makes sense to study the Scott-topology on $\mathfrak{V}_1(X)$. It is shown in [Tix95] that for domains D with their Scott-topology this, too, yields the weak topology in the sense above. It follows that for coherent domains the product topology is given order-theoretically as the Scott-topology. On a general stably compact space this need not be so:

Example 21. *For $X = \{0, 1\}$, equipped with the discrete topology, the set of probability valuations is in one-to-one correspondence with the unit interval. The product topology is the same as the usual metric topology on $[0, 1]$ but the order is identity and so the Scott-topology is discrete.*

For sub-probability measures the order on $\mathfrak{V}_{\leq 1}(X)$ is never trivial, and so equality between product and Scott-topology may hold in general. We leave this as an open problem.[3]

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[3]**Added 11 May 2020.** Jean Goubault-Larrecq has given an example which shows that the two topologies are different in general. The space X is the one-point compactification of $\mathbb{N}$ where $\mathbb{N}$ is equipped with the discrete topology.

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