

MEET-CONTINUITY AND
LOCALLY COMPACT SOBER DCPOS

by

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Abstract

In this thesis, we investigate meet-continuity over dcpos. We give different equivalent descriptions of meet-continuous dcpos, among which an important characterisation is given via forbidden substructures. By checking the function space of such substructures we prove, as a central contribution, that any dcpo with a core-compact function space must be meet-continuous. As an application, this result entails that any cartesian closed full subcategory of quasicontinuous domains consists of continuous domains entirely. That is to say, both the category of continuous domains and that of quasicontinuous domains share the same cartesian closed full subcategories.

Our new characterisation of meet-continuous dcpos also allows us to say more about full subcategories of locally compact sober dcpos which are generalisations of quasicontinuous domains. After developing some theory of characterising coherence and bicompleteness of dcpos, we conclude that any cartesian closed full subcategory of pointed locally compact sober dcpos is entirely contained in the category of stably compact dcpos or that of L-dcpo.

As a by-product, our study of coherence of dcpos enables us to characterise Lawson-compactness over arbitrary dcpos.

Dedicated to my parents

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Chapter 1

Introduction

Domain theory, initially introduced by Dana Scott [Sco70], serves as a mathematical universe within which people can interpret *higher-order functional programming languages* (see [Str06] for a systematic explanation). In this universe, types of programs are interpreted as *domains* (algebraic or continuous ones), and programs themselves are then viewed as *Scott-continuous functions* between domains. Such a translation leads one to a semantic category. Within this category, higher-order types are naturally interpreted as hom-sets. Since types correspond to domains in this interpretation, this means a suitable semantic category of a higher-order functional programming language should be domain-enriched. Moreover, it should be *cartesian closed* in order to model the currying and uncurrying processes in the language.

Indeed, there exists a quite satisfactory theory which deals with the cartesian closedness of domains. Gordon Plotkin [Plo76], Michael Smyth [Smy83a], and Achim Jung [Jun89, Jun90b] have made essential contributions to this theory. In the category of pointed algebraic domains, *bifinite domains* and *algebraic L-domains* form exactly the two maximal full subcategories in the sense that any cartesian closed

full subcategory of pointed algebraic domains is entirely contained in one of them, [Jun90a, Corollary 3.8]. In the case of pointed continuous domains, we have a similar result with *FS-domains* in lieu of bifinite domains and *continuous L-domains* in lieu of algebraic L-domains, respectively, [Jun90b, Corollary 10]. The category of bifinite domains, in particular, has really nice mathematical properties. For example, it is cartesian closed and closed under lifting, coalesced sum, bilimits and the Smyth, Hoare and Plotkin powerdomain constructions, et cetera. The category of bifinite domains also has a logical duality, the celebrated *Domain Theory in Logical Form*, by Samson Abramsky [Abr91]. All those properties have their counterparts in program constructs and have made this category a nice semantic universe to work within.

In 1980, [SD80] Nasser Saheb-Djahromi considered programs with a *probabilistic choice* operator and in order to accommodate this in the semantics, introduced the set of *probability measures* on *cpos*. Following Saheb-Djahromi's work, probabilistic non-determinism was studied more deeply by Claire Jones and Gordon Plotkin in [JP89, Jon90], where they introduced the *probabilistic powerdomain construction*, and showed that the behaviour of such a construction is much more easily understood in the context of continuous domains rather than algebraic domains and in fact, the probabilistic powerdomain is never algebraic. This fact forces us to leave the cosy zone of bifinite domains and search for semantic domains in the continuous setting, especially in the categories of FS-domains and continuous L-domains. Indeed, continuous domains have their own advantages of making their entrance through the real numbers over which probability values range, and being stable under *retractions*, a crucial property with which certain mathematical techniques can be applied. However, the probabilistic powerdomain construction, as shown in [Jon90], destroys the structure of continuous L-domains, and behaves opaquely over both FS-domains and

RB-domains (retracts of bifinite domains), [JT98]. Actually, the question whether the probabilistic powerdomain of an FS-domain is again an FS-domain has been open for decades.

Facing such a difficulty, naturally one wonders whether we could extend our mathematics out of the scope of continuity to find a semantic category which is indeed cartesian closed and closed under the probabilistic powerdomain construction. A natural and more liberal setting is the category of *quasicontinuous domains* which was introduced in the early eighties by Gerhard Gierz, Jimmie Lawson and Albert Stralka, [GLS83], as a generalisation of classical continuous domains. Indeed, Jean Goubault-Larrecq, [GL12], was able to show that the category of ω *QRB-domains* (quasi-version of countably-based RB-domains) is closed under the probabilistic powerdomain construction, adding to what is a very small set of such closure results, and later this result was generalised to the category of QRB-domains in [GLJ14]. This led many researchers to re-examine quasicontinuous domains [GLJ14, HK13, LX14, LX13, ZK12] and many pleasing properties were established. For example, it was proved by Goubault-Larrecq and Jung [GLJ14], and independently by Lawson and Xiaoyong Xi, [LX14], that *QFS-domains* (quasi-version of FS-domains) and QRB-domains are the same class and that they can be characterised as being precisely the *Lawson-compact* quasicontinuous domains, while, in the classical case, whether FS-domains and RB-domains are the same is one of the oldest and best-known open problems in domain theory.

However, unlike the category of FS-domains, that of QFS-domains is not cartesian closed. This raises the question whether there are any new cartesian closed categories consisting of quasicontinuous domains at all. As a central contribution of this thesis, we show that this is not the case. To be more precise, we show that all cartesian closed categories of quasicontinuous domains actually consist of domains

entirely [JJK⁺15]. This fact pulls us back into the classical setting when we consider modelling a higher-order functional programming language with probabilistic features and also motivates us to extend our search scope beyond quasicontinuity. As a common generalisation of both continuous and quasicontinuous domains, we consider *core-compact and sober dcpos*: dcpos which are core-compact and sober in the Scott topology. This gentle extension keeps us from stepping too far from the central theory. As a justification, core-compactness is the essential property of locating exponentiable objects in the category of topological spaces, and sobriety, meanwhile, is the key fact that links domains to their logical counterparts via the Stone duality, along the lines of *Abramsky's Domain Theory in Logical Form*. So our new question asks whether there are any new cartesian closed categories of core-compact and sober dcpos. We give a partial answer to this question by showing that any cartesian closed full subcategory of pointed core-compact sober dcpos is contained in that of *stably compact dcpos* or consists of *L-dcpo*s entirely, [JLL16a]. This implies that stable compactness is an indispensable condition that one should take into account when interpreting a probabilistic higher-order programming language in the category of core-compact sober dcpos.

We organise this thesis as follows:

In the following chapter, we collect preliminary results that we need for our further discussion. Basically we try to explain that both continuous and quasicontinuous domains are core-compact (equivalent to local compactness when sobriety is present) and sober in the Scott topology. Most of the material is not original and can be found in [GHK⁺03, AJ94, GL13]. In Section 2.6, however, we give a new *well-filtered dcpo* which is not sober in the Scott topology, which seems to us simpler than the first such example given by Kou in [Kou01]. Also, we give a direct proof of Theorem 2.5.7 which originally appears as [GHK⁺80, Theorem II-4.11] and is proved there by using

the fact that core-compact spaces are exponentiable and that the Scott topology equals the Isbell topology on continuous open-set lattices (viewed as function spaces targeted to the Sierpiński space). At the end of this chapter, we explain the main motivation to our research, the so-called *Jung-Tix problem*, and the works that are meant to attack it.

Chapter 3 contains the central contribution of this thesis, where *meet-continuity* on dcpos is investigated. We start with explaining how meet-continuity can be defined on arbitrary dcpos, following which we deliver the result that meet-continuity fills the gap between quasicontinuity and continuity over dcpos, a remarkable result from [KLL03]. In Section 3.2, we study meet-continuity from a topological point of view. Many characterisations of meet-continuous dcpos are given here. In particular, we show that meet-continuous dcpos are exactly those dcpos which are *locally compatible* in the Scott topology. Section 3.3 contains our new order-theoretical characterisation of meet-continuity. Roughly speaking, we show that a dcpo is meet-continuous if and only if certain order structures are not occurring in the dcpo as a retract. This is a quite useful characterisation of meet-continuity. As an application, laid out in Section 3.4, it enables us to prove that any dcpo with a core-compact function space must be meet-continuous, which in turn implies that any cartesian closed full subcategory of quasicontinuous domains actually consists of continuous domains entirely.

Since quasicontinuous domains as well as continuous domains are core-compact and sober in the Scott topology, we wonder whether the above results can be generalised to the category of core-compact and sober dcpos. That is, does every cartesian closed full subcategory of core-compact sober dcpos consist of continuous domains as well? In Chapter 4, we attempt to attack this question. We are not able to give a full answer to it, but we find two subcategories, the category of stably compact

dcpos and that of L-dcpo, such that any cartesian closed full subcategory of pointed core-compact sober dcpo is entirely contained in one of them. This result is presented as Theorem 4.4.3, and we call it the *Dichotomy Theorem* for locally compact sober dcpo. In order to prove our dichotomy result, we first investigate *coherence* (Section 4.1) and *bicompleteness* (Section 4.2) in general. Coherence of a topological space means that the intersection of any two compact saturated subsets is again compact saturated. We prove that in well-filtered dcpo (in particular, in sober dcpo), to establish coherence it suffices to show compactness of the intersection of any two principal filters, [JJL16b]. This observation enables us to give a characterisation of Lawson-compactness of arbitrary dcpo, presented in Theorem 4.1.7. Bicompleteness is studied from an order-theoretical viewpoint. By considering typical non-bicomplete dcpo we conclude that the dcpo in our interest must be bicomplete. With this fact, sober L-dcpo can be understood easily via a forbidden structure. Finally, the equivalent descriptions of coherence and L-dcpo are sufficient ingredients for us to prove the dichotomy result.

One should note that there is a quite versatile setting proposed by Alex Simpson [Sim03] to model higher-order functional programming languages with probabilistic features: the category of *topological domains* and continuous functions. Indeed, this is a cartesian closed category. In [BS06], Ingo Battenfeld and A. Simpson defined two kinds of probabilistic powerdomain constructions on topological domains, the *free convex space construction* and the *observationally-induced powerdomains*, respectively. A remarkable result is that these two constructions are closed in the category of topological domains and both coincide with the classical powerdomain constructions on the category of ω -continuous pointed dcpo, while, in general, neither of them is Jones and Plotkin's classical construction. Generally, the category of topological domains is a nice category to work with and a lot of computational effects

can be modelled in it, see [Bat08] for an overview. One less satisfactory fact may be that topological domains are not closed under sobrification [GS06], thus one can not expect a localic description of them via Stone duality or a logical interpretation along the lines of *Abramsky's Domain Theory in Logical Form*.

Chapter 2

Basic concepts and preliminary results

In this chapter, we collect some basic definitions and well-known results which are of central interests for our discussions in Chapter 3 and Chapter 4. They mainly consist of the notions like *directed-complete partial ordered set* (*dcpo* for short), *Scott-continuous function*, *Scott topology*, *continuous domain* and *quasicontinuous domain*. We will focus on topological properties of both continuous domains and quasicontinuous domains with the Scott topology, and show that they both are *locally compact and sober* spaces in the Scott topology. Following these results, as generalisations of these order structures, we introduce *core-compact dcpo*s and *sober dcpo*s, over which most of our research will be conducted. Properties of core-compact dcpo are listed in this chapter. In particular, connections among core-compactness, sobriety, local compactness and *well-filteredness* will be illustrated. At the end of this chapter, the so-called Jung-Tix problem is introduced and explained, which is in fact at the heart of the research motivation to this thesis. Most of the contents in this chapter are

already in existing work. In Section 2.7, however, a novel example is given to explain the difference between well-filtered dcpos and sober ones.

2.1 Directed-Complete Partial Ordered Sets

Definition 2.1.1. A set L with a binary relation \leq on it is called a *poset* and denoted by (L, \leq) if for all $x, y, z \in L$:

1. $x \leq x$ (reflexivity);
2. $x \leq y, y \leq z$ imply $x \leq z$ (transitivity);
3. $x \leq y, y \leq x$ imply $x = y$ (antisymmetry).

The terminology “poset” is short for *partially ordered set* and the relation \leq is called a *partial order* on L . The relation \leq is called a *preorder* if it only satisfies conditions 1 and 2. Intuitively, $x \leq y$ can be read as x is smaller or less than y . For a partial order \leq , we use $x < y$ to mean that x is strictly smaller than y , that is, $x \leq y$ and $x \neq y$. If no ambiguity arises, we often omit the symbol \leq and simply use L instead of (L, \leq) .

We use “partial” here since the definition itself does not require every two elements to be related via \leq or \geq . Comparing to this, we have the following two extreme situations:

1. If any two elements x, y in a non-empty poset (L, \leq) are related via \leq , i.e., $x \leq y$ or $y \leq x$ for all $x, y \in L$, then L is called a *totally ordered poset*, or a *chain*.

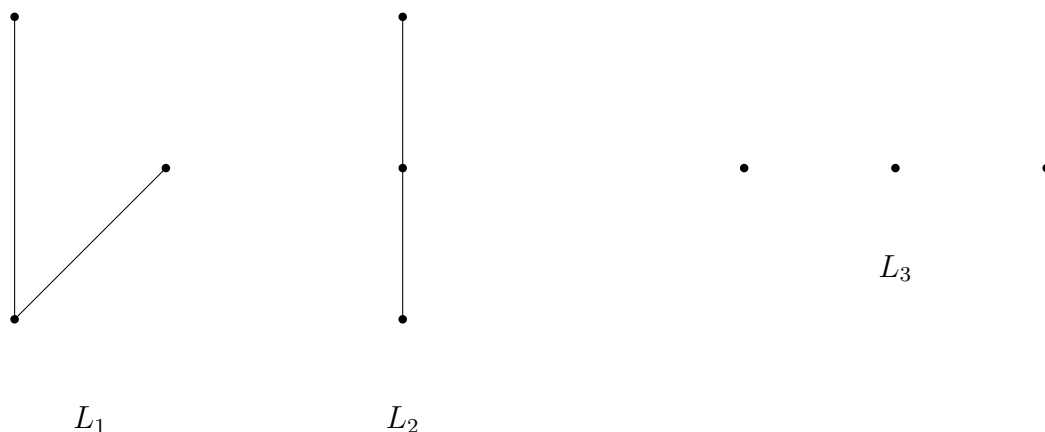


FIGURE 2.1: Examples of posets.

2. If no two different elements x, y in a non-empty poset (L, \leq) are related via \leq , then L is called an *anti-chain*. The order on an anti-chain is often called the *discrete order*.

For the sake of intuition, any finite poset can be represented as a line diagram by using dots to represent elements in the poset and interconnecting lines to indicate the order relation. A detailed explanation can be found in [DP02, Chapter 1]. An ordinary poset L_1 , a chain L_2 and an anti-chain L_3 are illustrated in Figure 2.1, respectively. In this thesis, infinite posets will also be suggested diagrammatically by showing their finite parts which indicate the building principle.

There are many ways through which we can construct new posets out of the given ones. The following definition gives four general constructions.

- Definition 2.1.2.**
1. Given a poset (L, \leq) and a subset A of L , we can equip A with the *induced order* \leq_A , defined as $a \leq_A b$ for $a, b \in A$ if and only if $a \leq b$ in L . Obviously, (A, \leq_A) is also a poset.
 2. Given a poset (L, \leq) , we set $L^{op} = L$, and define a binary relation \leq^{op} on L^{op} by $x \leq^{op} y$ if and only if $y \leq x$ in L . One easily verifies that \leq^{op} is a partial order on L^{op} . The poset (L^{op}, \leq^{op}) is called the *dual poset* of (L, \leq) .

3. Given a family of posets $(L_i, \leq_i), i \in I$, we define a binary relation \leq on the cartesian product $\prod_{i \in I} L_i$ by $(x_i)_{i \in I} \leq (y_i)_{i \in I}$ if and only if $x_i \leq_i y_i$ in L_i for all $i \in I$. Again, one can verify that \leq is a partial order on $\prod_{i \in I} L_i$. We call $(\prod_{i \in I} L_i, \leq)$ the *product* of $(L_i, \leq_i), i \in I$. For two posets L and M , we use $L \times M$ to denote the product.
4. Given a family of posets $(L_i, \leq_i), i \in I$, we define a binary relation \leq on the disjoint union $\coprod_{i \in I} L_i$ by $(i, x) \leq (j, y)$ if and only if $i = j$ and $x \leq_i y$ in L_i . Then \leq is a partial order on $\coprod_{i \in I} L_i$, and $(\coprod_{i \in I} L_i, \leq)$ is called the *disjoint sum* of $(L_i, \leq_i), i \in I$. For two posets L and M , we use $L + M$ to denote the disjoint sum.

Functions between posets considered in this thesis are always preserving the order.

Definition 2.1.3. Let L and M be posets.

1. A function $f: L \rightarrow M$ is called *monotone* if $f(x) \leq f(y)$ in M whenever $x \leq y$ in L , for any $x, y \in L$.
2. The identity function on L is denoted by id_L . L and M are said to be *isomorphic* (in symbols, $L \cong M$) if there exist monotone functions $f: L \rightarrow M$ and $g: M \rightarrow L$ such that $g \circ f = \text{id}_L$ and $f \circ g = \text{id}_M$.

We now define order structures of our interest, the so-called directed-complete partial ordered sets (dcpo for short). Before this, let us first fix some terminology.

Given a poset L and any element $x \in L$, we set $\uparrow x = \{y \in L \mid x \leq y\}$, $\downarrow x = \{y \in L \mid y \leq x\}$ and call them *principal filters* and *principal ideals*, respectively. For a subset $A \subseteq L$, define $\uparrow A = \bigcup_{x \in A} \uparrow x$ and $\downarrow A = \bigcup_{x \in A} \downarrow x$. A is called an *upper set* (resp., a *lower set*) if $A = \uparrow A$ (resp., $A = \downarrow A$). An element $a \in A$ is called the *least element*

(resp., *greatest element*) of A if $A \subseteq \uparrow a$ (resp., $A \subseteq \downarrow a$). A poset L is called *pointed* if L has a least element; such an element is often called *bottom* and denoted by \perp . The greatest element of L is also called the *top element* and denoted by \top . From the antisymmetry of partial orders, the least and greatest elements of a subset (if they exist) are unique. An element $x \in A$ is said to be *maximal* (resp., *minimal*) in A if for all $y \in A$ with $x \leq y$ (resp., $y \leq x$), one has $x = y$. Unlike the greatest or least elements, maximal or minimal elements are not necessarily unique. For instance, all elements of L_3 in Figure 2.1 are both maximal and minimal. A subset $A \subseteq L$ is called a *finitely generated upper set* if there exists a finite subset $F \subseteq A$ such that $A = \uparrow F$. For a poset L , if there exists a set M_L of minimal elements of L such that $L = \uparrow M_L$, then we say L is *grounded*, and we say L is *finitely grounded* if M_L is finite. Obviously, a poset L is finitely grounded if and only if it is, as an upper set, finitely generated.

For a subset $A \subseteq L$, an element $x \in L$ is called an *upper bound* (resp., *lower bound*) of A if $A \subseteq \downarrow x$ (resp., $A \subseteq \uparrow x$). We use A^u (resp., A^l) to denote the set of all upper (resp., lower) bounds of A . A is called *bounded* if A^u is non-empty. If there exists a least (resp., a greatest) element a in A^u (resp., A^l), then a is called the *supremum* (resp., *infimum*) of A . We use $\bigvee A$ or $\sup A$ (resp., $\bigwedge A$ or $\inf A$) to denote the supremum (resp., infimum) of A . For any two elements $a, b \in L$, $a \vee b$ (resp., $a \wedge b$) is used to denote the supremum (resp., infimum) of a, b .

A *lattice* L is a poset in which $\bigvee A$ and $\bigwedge A$ exist for any non-empty finite subset A . If for arbitrary subsets A of L , both $\bigvee A$ and $\bigwedge A$ exist, we call L a *complete lattice*. If $\bigvee A$ exists for non-empty A , we call L is *join-complete*. Dually, L is called *meet-complete* if $\bigwedge A$ exists for non-empty $A \subseteq L$. L is called *bounded-complete* if $\bigvee A$ exists for any bounded subset $A \subseteq L$. Note that a poset is meet-complete if and only if it is bounded-complete.

Definition 2.1.4. Let L be a poset.

1. A non-empty subset D of L is called *directed* (resp., *filtered*) if for any two elements $a, b \in D$, there exists an element $c \in D$, such that $a, b \leq c$ (resp., $c \leq a, b$). A lower directed subset of L is called an *ideal*, and an upper filtered one is called a *filter*.
2. L is called a *directed-complete partially ordered set* (*dcpo* for short) if for any directed subset $D \subseteq L$, the supremum $\bigvee D$ exists.

In general, for a dcpo L , the dual poset L^{op} may not be a dcpo. If, in addition, L^{op} is a dcpo, we call L a *bicomplete* dcpo. Every finite poset is a bicomplete dcpo. From the definition, a dcpo L is bicomplete if and only if $\bigwedge A$ exists for any filtered subset A of L .

Proposition 2.1.5. *Given any two dcpos L and M , both the product $L \times M$ and the disjoint sum $L + M$ are again dcpos.*

Proof. Let $\{(a_i, b_i)\}_{i \in I}$ be a directed subset of $L \times M$. Then $\{a_i\}_{i \in I}$ (resp., $\{b_i\}_{i \in I}$) is directed in L (resp., in M). So $\bigvee_{i \in I} a_i$ and $\bigvee_{i \in I} b_i$ exist. It is easy to see that $(\bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i)$ is the supremum of $\{(a_i, b_i)\}_{i \in I}$ in $L \times M$.

$L + M$ is directed-complete since any directed subset of $L + M$ is entirely contained in L or M . □

Remark 2.1.6. The above proposition also holds for products and disjoint sums of infinitely many dcpos, and the argument is similar.

To check that some poset L is a dcpo we need to compute the supremum of every directed subset D of L , actually sometimes it is more convenient to compute the supremum of a *cofinal* subset of D .

Definition 2.1.7. A subset E of a directed set D is called *cofinal* in D if for any $d \in D$ there exists some $e \in E$ such that $d \leq e$.

The following fact about cofinal subsets is trivial.

Proposition 2.1.8. *Let L be a poset and D a directed subset of L . If E is a cofinal subset of D , then*

1. E is directed;
2. $\sup E$ exists if and only if $\sup D$ exists, and they are equal. □

The following lemma also supplies an easier way of verifying the directed-complete structure. Since chains are directed, if a poset is directed-complete, then every chain in it has a supremum. Surprisingly, the reverse statement also holds.

Lemma 2.1.9. *A poset L is a dcpo if and only if every chain in L has a supremum.* □

This is a well-known result in order theory. The proof, which uses the Axiom of Choice, goes back to a lemma of Iwamura [Iwa44], and can be found in [Mar76]. A different proof from P.M. Cohn is also available in [Coh65].

2.2 Scott-continuous functions, topologies and the specialisation order

2.2.1 Scott-continuous functions

Morphisms between dcpos considered in this thesis are supposed to preserve the dcpo structure, i.e., the directed suprema. With this in mind, the following definition is canonical.

Definition 2.2.1. Let L and M be dcpos. A function $f: L \rightarrow M$ is called *Scott-continuous* if for every directed subset D of L , $f(\bigvee D) = \bigvee f(D)$.

Since every two comparable elements form a directed subset, every Scott-continuous function is automatically monotone. The category of all dcpos with Scott-continuous functions between them is denoted by **DCPO**.

Remark 2.2.2. 1. In this thesis, we always define categories with their morphisms being Scott-continuous functions until stated otherwise.

2. For a subcategory **C** of dcpos, we use \mathbf{C}_\perp to denote the full subcategory of **C**, with its objects being pointed ones from **C**. For example, \mathbf{DCPO}_\perp denotes the category of all pointed dcpos and Scott-continuous functions between them.

Unlike continuous functions between topological spaces, the following property of Scott-continuous functions between dcpos comes for free.

Proposition 2.2.3. [GHK⁺03, Lemma II-2.8] *Let L, M and N be dcpos and f a function from $L \times M$ to N . Then f is Scott-continuous if and only if it is Scott-continuous in each variable separately; that is,*

- *for all $a \in L$, the function $x \mapsto f(a, x) : M \rightarrow N$ is Scott-continuous,*
- *for all $b \in M$, the function $x \mapsto f(x, b) : L \rightarrow N$ is Scott-continuous.*

Proof. We prove the non-trivial direction. Suppose that f is Scott-continuous in each variable separately, then we know that f is monotone in each variable as well. Now given $(a, b) \leq (c, d)$ in $L \times M$, we have that $f(a, b) \leq f(a, d) \leq f(c, d)$, so f is monotone from $L \times M$ to N .

Let D be a directed subset of $L \times M$. Since f is monotone, it is obvious that $\bigvee f(D) \leq f(\bigvee D)$.

For the reverse inequality, we write $D = \{(a_i, b_i)\}_{i \in I}$. Then we know that $\bigvee D = (\bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i)$. So we have:

$$\begin{aligned} f(\bigvee D) &= f((\bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i)) \\ &= \bigvee_{i \in I} f((a_i, \bigvee_{i \in I} b_i)) \\ &= \bigvee_{i \in I} \bigvee_{j \in I} f((a_i, b_j)) \end{aligned}$$

and

$$\bigvee f(D) = \bigvee_{i \in I} f((a_i, b_i)).$$

For any $(a_i, b_j), i, j \in I$, we can find some $k \in I$ such that $(a_i, b_i), (a_j, b_j) \leq (a_k, b_k)$ since D is directed. Hence $(a_i, b_j) \leq (a_k, b_k)$, and this together with the fact that f is monotone imply that $f((a_i, b_j)) \leq f((a_k, b_k)) \leq \bigvee f(D)$. So finally, we have $f(\bigvee D) \leq \bigvee f(D)$ and hence $f(\bigvee D) = \bigvee f(D)$. \square

Remark 2.2.4. The above proposition can be read as: a function between dcpos is jointly Scott-continuous if and only if it is separately Scott-continuous.

Given any two dcpos L and M , we can use Scott-continuous functions between them to form a new dcpo $[L \rightarrow M]$, called the *function space* from L to M .

Definition 2.2.5. For any two dcpos L and M , $[L \rightarrow M]$ is defined to be the set of all Scott-continuous functions from L to M with the *pointwise* order, that is, for $f, g \in [L \rightarrow M]$, $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in L$.

Proposition 2.2.6. *For any dcpos L and M , the function space $[L \rightarrow M]$ is a dcpo in the pointwise order, with suprema computed pointwise.*

Proof. Let $\{f_\alpha\}_{\alpha \in A}$ be a directed family of Scott-continuous functions from L to M . We define a function $f: L \rightarrow M$ that maps $x \in L$ to the directed supremum

$\bigvee_{\alpha \in A} f_\alpha(x)$, and claim that f is the supremum of $f_\alpha, \alpha \in A$ in $[L \rightarrow M]$. We first prove that f is Scott-continuous. Indeed, for any directed subset $D \subseteq L$,

$$\begin{aligned} f(\bigvee D) &= \bigvee_{\alpha \in A} f_\alpha(\bigvee D) \\ &= \bigvee_{\alpha \in A} \bigvee_{d \in D} f_\alpha(d) \\ &= \bigvee_{d \in D} \bigvee_{\alpha \in A} f_\alpha(d) \\ &= \bigvee f(D). \end{aligned}$$

That f is the least upper bound of $\{f_\alpha\}_{\alpha \in A}$ is trivial. □

Proposition 2.2.7. *Let L and M be dcpos. Then the evaluation mapping*

$$\text{eval} : (f, x) \mapsto f(x) : [L \rightarrow M] \times L \rightarrow M$$

is Scott-continuous.

Proof. From the previous proposition $[L \rightarrow M]$ is a dcpo with directed suprema calculated pointwise. By Proposition 2.2.3, we prove the Scott-continuity of eval by checking that it is Scott-continuous in each variable.

First, we fix an element $x \in L$ and let $\{f_\alpha\}_{\alpha \in A}$ be a directed subset of $[L \rightarrow M]$.

Then

$$\begin{aligned} \text{eval}\left(\bigvee_{\alpha \in A} f_\alpha, x\right) &= \left(\bigvee_{\alpha \in A} f_\alpha\right)(x) \\ &= \bigvee_{\alpha \in A} f_\alpha(x) \\ &= \bigvee_{\alpha \in A} \text{eval}(f_\alpha, x). \end{aligned}$$

Now we fix $f \in [L \rightarrow M]$ and let D be a directed subset of L . Then

$$\begin{aligned} \text{eval}(f, \bigvee D) &= f(\bigvee D) \\ &= \bigvee_{d \in D} f(d) \\ &= \bigvee_{d \in D} \text{eval}(f, d). \end{aligned}$$

So eval is indeed Scott-continuous by Proposition 2.2.3. \square

The function space construction of dcpos is of great importance. Actually, function spaces are the exponential objects in the category **DCPO**.

Definition 2.2.8. Let \mathbf{C} be a category with finite products. An object Y in \mathbf{C} is called *exponentiable*, if the functor $-\times Y : \mathbf{C} \rightarrow \mathbf{C}$ has a right adjoint $-^Y$, that is, if for all objects X and Z there is a natural bijection $E : \mathbf{C}(X \times Y, Z) \rightarrow \mathbf{C}(X, Z^Y)$. In this case, Z^Y is called an *exponential object* in \mathbf{C} . If in addition, for all objects Y, Z in \mathbf{C} , the exponential object Z^Y exists, then we say that the category \mathbf{C} is *cartesian closed*.

The following lemma is essentially due to M. Smyth [Smy83a, Lemma 5]. We collect here a more general version from A. Jung.

Lemma 2.2.9. [Jun89, Lemma 1.21] *Let \mathbf{C} be a cartesian closed full subcategory of **DCPO**. Then the following holds for any two objects $A, B \in \mathbf{C}$.*

1. *The terminal object T (the empty product) of \mathbf{C} is isomorphic to the one-point dcpo.*
2. *The categorical product of A and B is isomorphic to the cartesian product $A \times B$ as defined in Definition 2.1.2.*

3. The exponential object A^B is isomorphic to $[B \rightarrow A]$.

As explained in the Introduction, cartesian closedness is required when modelling a higher-type programming language, [Str06]. This lemma states that in any full subcategory of **DCPO**, exponential objects are precisely the function spaces between dcpos; therefore, one only needs to focus on the function spaces when working with cartesian closedness in **DCPO**. We take this observation as one of the justifications of working within this category.

Corollary 2.2.10. *The category **DCPO** is cartesian closed.*

Proof. Straightforward from Proposition 2.1.5, Proposition 2.2.6 and Lemma 2.2.9. □

2.2.2 Topologies on posets

In this section, several topologies are defined on posets, among which the Scott topology is the most fundamental one. We will show in the sequel that those topologies are useful in characterising order-theoretic properties of dcpos. Before we proceed, let us fix some notions and notations for topological spaces first, adopting the conventions of [GHK⁺03] and [GL13].

Definition 2.2.11. Let (X, τ) be a topological space.

1. The open sets of X form a complete lattice which we denote by $\mathcal{O}(X)$. For any subset $A \subseteq X$, we use \overline{A} and A° to denote the topological closure and interior of A , respectively.
2. A subset A of X is called *saturated* if it is an intersection of open subsets of X . For any subset $B \subseteq X$, the set $\bigcap\{U \in \mathcal{O}(X) \mid B \subseteq U\}$ is the smallest

saturated set containing B , which is called the *saturation* of B and denoted by $\text{sat}(B)$.

3. X is called *coherent* if the intersection of any two compact saturated subsets is again compact.
4. X is said to be *locally compact* if for every open set U and $x \in U$, there exist a compact set K and an open set V such that $x \in V \subseteq K \subseteq U$.
5. X is called *well-filtered* if for every filtered family $\{K_i \mid i \in I\}$ of compact saturated sets with the intersection $\bigcap_{i \in I} K_i$ being a subset of an open set U , U contains K_i for some $i \in I$ already.
6. A non-empty subset $S \subseteq X$ is called *irreducible* if $S \subseteq A \cup B$ for closed subsets A and B implies $S \subseteq A$ or $S \subseteq B$.
7. X is called a *sober* space if X is a T_0 space and every irreducible closed subset of X is the closure of some singleton.
8. X is called *stably compact* if it is compact, locally compact, coherent and sober.
9. The *patch topology* on X arises by taking all closed sets together with all compact saturated sets as a subbasis for the closed sets.

Definition 2.2.12. Let L be a dcpo. The *Scott-open sets* on L consist of those subsets U that satisfy:

1. U is an upper set, that is, $U = \uparrow U$;
2. for any directed subset $D \subseteq L$, $\sup D \in U$ implies $D \cap U \neq \emptyset$.

All Scott-open sets on L form a topology on L called the *Scott topology*.

Remark 2.2.13. The Scott topology can be also defined on arbitrary posets. In this case, one only replaces the “directed subset” in the above definition by “directed subset whose supremum exists”.

The space $(L, \sigma(L))$ is denoted by ΣL . While the set of all Scott-open subsets of L is denoted by $\sigma(L)$, that of all Scott-closed subsets of L is denoted by $\Gamma(L)$. In this thesis, when we discuss dcpos as topological spaces, we always mean them equipped with the Scott topology unless stated otherwise. For example, we call a dcpo L a sober dcpo if and only if L with the Scott topology is a sober topological space, and for a subset $A \subseteq L$, \bar{A} and A° denote the closure and interior of A with respect to the Scott topology, respectively. Sometimes $\text{cl}_\sigma A$ and $\text{int}_\sigma A$ are also used for \bar{A} and A° , respectively, if we need to emphasise the Scott topology.

The following proposition is straightforward from the definition of the Scott topology.

Proposition 2.2.14. *A subset A of a dcpo L is Scott-closed if and only if*

1. *A is a lower set, that is, $A = \downarrow A$;*
2. *A is closed under the formation of directed sups, that is, for any directed subset $D \subseteq A$, $\sup D \in A$.* □

From this observation, Scott-open sets are also said to be inaccessible by the suprema of directed subsets.

Proposition 2.2.15. *Let L be a dcpo.*

1. *The principal ideal $\downarrow x$ is Scott-closed for every $x \in L$, and $\downarrow x = \overline{\{x\}}$.*
2. *In general, if D is a directed subset of L , $\bar{D} = \downarrow \sup D$.*

Proof. (1) That $\downarrow x$ is Scott-closed for every $x \in L$ is straightforward. Since $\downarrow x$ is a Scott-closed subset containing x , so $\overline{\{x\}} \subseteq \downarrow x$. That $\downarrow x \subseteq \overline{\{x\}}$ is true since $\overline{\{x\}}$ is Scott-closed hence a lower set and $x \in \overline{\{x\}}$.

(2) Since $D \subseteq \downarrow \sup D$ and $\downarrow \sup D$ is Scott-closed, $\overline{D} \subseteq \downarrow \sup D$. For the converse containment, \overline{D} is a Scott-closed subset containing D , so it follows from Proposition 2.2.14 that $\sup D \in \overline{D}$. So $\downarrow \sup D \subseteq \overline{D}$ since \overline{D} is a lower set. \square

Proposition 2.2.16. *Let L be a dcpo.*

1. ΣL is a T_0 topological space.
2. ΣL is T_1 if and only if L is an anti-chain.

Proof. (1) Let x, y be two different elements in L . Without loss of generality, we can assume that $x \not\leq y$. Then $L \setminus \downarrow y$ is a Scott-open subset containing x but missing y .

(2) Recall that a topological space is T_1 if and only if every singleton set is closed. In our case, this means $\{x\} = \overline{\{x\}} = \downarrow x$ for every $x \in L$, which is equivalent to saying that L is an anti-chain. \square

The following propositions give us a general impression of Scott-closed subsets and compact subsets in dcpos, respectively. To prove them, we need the Hausdorff Maximality Principle.

Lemma 2.2.17 (Hausdorff Maximality Principle). *In a poset, every chain is contained in a maximal chain. Here maximality of chains is considered in the poset of all chains ordered by set inclusion.* \square

Proposition 2.2.18. *Let A be a Scott-closed subset of a dcpo L . Then every element $x \in A$ is below some maximal element of A .*

Proof. We find a maximal chain C containing x in A by applying Lemma 2.2.17 to (A, \leq_A) . Since A is Scott-closed, $\sup C \in A$. One easily sees that $x \leq \sup C$. The fact that $\sup C$ is a maximal element of A comes from the maximality of C . \square

Proposition 2.2.19. *Let K be a compact subset of a dcpo L . Then every element $x \in K$ is above some minimal element of K .*

Proof. Assume that K is compact in L . For every $x \in L$, by the Hausdorff Maximality Principle, there exists a maximal chain $C \subseteq K$ containing x . Since $\downarrow c$ is Scott-closed for $c \in C$ and K is compact, the filtered intersection $\bigcap \{\downarrow c \mid c \in C\} \cap K$ is non-empty. By the maximality of C , this non-empty intersection must be a singleton set that consists of a minimal element of K . Obviously, this minimal element is below x . \square

In Definition 2.2.1, we have defined Scott-continuous functions between dcpos. Meanwhile, it is also natural to consider topologically continuous functions between dcpos equipped with the Scott topology. The following proposition states that these two kinds of morphisms actually coincide, which justifies the terminology of Scott-continuity.

Proposition 2.2.20. *Let f be a function from dcpo L to M . Then f is Scott-continuous if and only if f is continuous from ΣL to ΣM .*

Proof. Assume that f is Scott-continuous and U a Scott-open subset of M . Since f is Scott-continuous, it is monotone. Thus $f^{-1}(U)$ is an upper set in L . Let D be any directed subset of L with its supremum $\sup D$ in $f^{-1}(U)$. We have $f(\sup D) \in U$. It follows from the Scott-continuity of f that $f(D)$ is directed and $\sup f(D) \in U$. Then $f(d) \in U$ for some $d \in D$. Hence $d \in f^{-1}(U)$, and it implies that $f^{-1}(U)$ is Scott-open.

For the converse, let f be a continuous function from ΣL to ΣM and D a directed subset of L . We first claim that f is monotone. Indeed, for every $x, y \in L$, $x \leq y$ implies that $x \in \downarrow y = \overline{\{y\}}$. Since f is continuous, we have that $f(x) \in f(\overline{\{y\}}) \subseteq \overline{\{f(y)\}} = \downarrow f(y)$, that is $f(x) \leq f(y)$. So indeed f is monotone. Now from the monotonicity of f , $f(D)$ is directed, $\sup f(D)$ exists in M and $\sup f(D) \leq f(\sup D)$. We proceed to proving $f(\sup D) \leq \sup f(D)$. Assume this is not true, that is, $f(\sup D) \in M \setminus \downarrow \sup f(D)$, i.e., $\sup D \in f^{-1}(M \setminus \downarrow \sup f(D))$. Since f is continuous and $M \setminus \downarrow \sup f(D)$ is Scott-open, $f^{-1}(M \setminus \downarrow \sup f(D))$ is Scott-open. Thus we have some $d \in D$ such that d is also in $f^{-1}(M \setminus \downarrow \sup f(D))$. So $f(d) \in M \setminus \downarrow \sup f(D)$, and obviously this is impossible. So we have proved that $f(\sup D) \leq \sup f(D)$ and indeed f is Scott-continuous. \square

From Proposition 2.2.15, the ideal $\downarrow x$ is always Scott-closed for every $x \in L$. One wonders whether all Scott-closed subsets can be generated by those principal ideals, that is, whether $\downarrow x, x \in L$ can form a subbasis of closed sets of the Scott topology. However, this is not true. Consider the dcpo of natural numbers with the discrete order: Every subset of it is Scott-closed, while $\downarrow x, x \in L$, only generate finite subsets of natural numbers as closed sets. This observation leads us to the following definition.

Definition 2.2.21. The *upper topology* on a poset L , denoted by $\nu(L)$, is the topology generated by $L \setminus \downarrow x$ for all $x \in L$ as a subbasis.

Since $L \setminus \downarrow x$ is always Scott-open for every $x \in L$, the upper topology is coarser than the Scott topology.

Dually, we can define the lower topology.

Definition 2.2.22. The *lower topology* on a poset L , denoted by $\omega(L)$, is the topology generated by $L \setminus \uparrow x$ for all $x \in L$ as a subbasis.

A same argument as in Proposition 2.2.16 shows that neither upper topology or lower topology on a poset L is T_1 unless L is discrete. However, the following topology, as a refinement of the Scott topology and the lower topology, is always T_1 on any dcpo.

Definition 2.2.23. The *Lawson topology* on a dcpo L , denoted by $\lambda(L)$, is defined to be the common refinement of the Scott topology and the lower topology. The space $(L, \lambda(L))$ is denoted by $\Lambda(L)$.

A dcpo is called *Lawson-compact* if it is compact in the Lawson topology.

Proposition 2.2.24. *The Lawson topology $\lambda(L)$ on a dcpo L is T_1 .*

Proof. Note that for any $x \in L$, $\downarrow x, \uparrow x$ are closed in the Scott topology and lower topology, respectively. Hence $\{x\} = \downarrow x \cap \uparrow x$ is closed in the Lawson topology, which implies that the Lawson topology is T_1 . \square

2.2.3 The specialisation order

We have seen that many topologies can be defined on posets; conversely, for any topological space X , there is a natural way to define a preorder \leq_s on X . We say $x \leq_s y$ for $x, y \in X$ if and only if for any open set U , $y \in U$ whenever $x \in U$. One easily verifies that \leq_s is a preorder on X . The pair (X, \leq_s) is denoted by $\Omega(X)$ and we call \leq_s the *specialisation preorder* on X . For any $x \in X$, we set $\uparrow_s x = \{y \in X \mid x \leq_s y\}$ and $\downarrow_s x = \{y \in X \mid y \leq_s x\}$. For $A \subseteq X$, set $\uparrow_s A = \bigcup_{x \in A} \uparrow_s x$ and $\downarrow_s A = \bigcup_{x \in A} \downarrow_s x$. As in the partial order case, we also call A an upper set (resp., lower set) if $A = \uparrow_s A$ (resp., $A = \downarrow_s A$).

Proposition 2.2.25. *Let X be a topological space. Then*

1. *for all $x, y \in X$, $y \leq_s x$ if and only if $y \in \overline{\{x\}}$, that is, $\downarrow_s x = \overline{\{x\}}$ for any $x \in X$;*

2. the space X is T_0 if and only if \leq_s is a partial order;
3. open sets (resp., closed sets) in X are always upper sets (resp., lower sets) in $\Omega(X)$;
4. for any subset A of X , A is saturated in X if and only if A is an upper set in $\Omega(X)$. □

Proof. The first three statements are obvious and we only prove the 4th one. Since open sets are upper sets in $\Omega(X)$, every saturated set, as an intersection of open sets, is also an upper set.

Conversely, assume A is an upper set in $\Omega(X)$. We claim that $A = \bigcap \{U \mid A \subseteq U \text{ \& } U \in \mathcal{O}(X)\}$. Indeed, for any $x \notin A$, since A is an upper set in $\Omega(X)$, $A \subseteq X \setminus \downarrow_s x = X \setminus \overline{\{x\}} \in \mathcal{O}(X)$. So $x \notin \bigcap \{U \mid A \subseteq U \text{ \& } U \in \mathcal{O}(X)\}$. Hence $\bigcap \{U \mid A \subseteq U \text{ \& } U \in \mathcal{O}(X)\} \subseteq A$. The reverse containment is obvious. □

For a dcpo (L, \leq) , we can equip L with the Scott topology $\sigma(L)$ and obtain a T_0 topological space ΣL . Now the specialisation preorder \leq_s can be defined on ΣL . From the previous proposition, for any $x \in L$, we have $\downarrow_s x = \text{cl}_\sigma\{x\} = \downarrow x$, so \leq_s is equal to the original order \leq on L , and hence $(L, \leq) = \Omega(\Sigma L)$.

Corollary 2.2.26. *Let L be a dcpo and $A \subseteq L$.*

1. *The subset A is saturated in ΣL if and only if it is an upper set in L .*
2. *If A is compact and saturated, then A is of the form $\uparrow M$, where M is the set of minimal elements in A .*

Proof. (1) From Proposition 2.2.25 we know that A is saturated in ΣL if and only if it is an upper set in $\Omega(\Sigma L)$, and we have seen that $\Omega(\Sigma L) = (L, \leq)$.

(2) This is straightforward from the first statement and Proposition 2.2.19. □

For a T_0 topological space, the poset $\Omega(X)$ is always a poset, but it need not be a dcpo in general. We now define an interesting class of topological spaces whose specialisation order is directed-complete:

Definition 2.2.27. A topological space X is called a *monotone convergence space* if $\Omega(X)$ is a dcpo, and every directed subset converges to its supremum. In other words, open sets in X are Scott-open in $\Omega(X)$, i.e., $\mathcal{O}(X) \subseteq \sigma(\Omega(X))$.

Proposition 2.2.28. *Every sober space is a monotone convergence space.*

Proof. Let X be a sober space. Since every sober space is T_0 , $\Omega(X)$ is a poset. We prove that $\Omega(X)$ is a dcpo. First, we claim that every directed subset $D \subseteq \Omega(X)$ is an irreducible set in X . To this end let A, B be closed sets such that $D \subseteq A \cup B$. If $D \not\subseteq A$ and $D \not\subseteq B$, then we could find some $a, b \in D$, such that $a \notin A$ and $b \notin B$. Since D is directed in $\Omega(X)$, we find an upper bound $c \in D$ of a, b . This c cannot be in A or B either; otherwise we would have that $a \in A$ or $b \in B$. Hence we conclude that $D \not\subseteq A \cup B$. This contradiction implies that D is an irreducible subset, and hence the closure \overline{D} of D is also irreducible. Since X is sober, there exists a unique $s \in X$ such that $\overline{D} = \overline{\{s\}}$. By Proposition 2.2.25, s is an upper bound of D since $D \subseteq \overline{D} = \overline{\{s\}}$. Actually, s is the least upper bound of D . Indeed, for any s' such that $D \subseteq \overline{\{s'\}}$, we have $s \in \overline{\{s\}} = \overline{D} \subseteq \overline{\{s'\}}$, which implies that $s \leq_s s'$. So s is the supremum of D , and hence $\Omega(X)$ is a dcpo.

Now we prove that $\mathcal{O}(X) \subseteq \sigma(\Omega(X))$. Let $U \in \mathcal{O}(X)$ and D be a directed subset of $\Omega(X)$ with $\sup D \in U$. Then from above we know that $\overline{D} \cap U = \overline{\sup D} \cap U \neq \emptyset$, and this implies that $D \cap U \neq \emptyset$. Hence U is Scott-open in $\Omega(X)$. \square

The converse of the previous proposition does not hold, that is, not every monotone convergence space is sober. Actually, for any dcpo L , ΣL is a monotone convergence space: The specialisation order is directed-complete because it agrees with the given

order. That directed sets converge to their supremum is the defining property of Scott-open sets. However, we will see in Chapter 2.5 that ΣL need not be sober in general.

2.3 Continuous domains

In a theory of approximation, ideal elements are supposed to be completely determined by the behaviour of their approximants. In the language of dcpos, this is interpreted as saying that a given element can be written as a supremum of a directed subset, and elements in the directed subset are regarded as approximants of the given element. Some approximants are found to be necessarily needed for the approximation in the sense that no approximation can be made until the approximating sequence has passed those elements. This intuition can be encoded by the so-called *way-below relation* on dcpos.

- Definition 2.3.1.**
1. Let L be a dcpo and $a, b \in L$. We say a is *way-below* b (in symbols $a \ll b$) if and only if for all directed subsets $D \subseteq L$, $b \leq \sup D$ implies $a \leq d$ for some $d \in D$.
 2. An element $x \in L$ is said to be *compact* if $x \ll x$. The subset of all compact elements in L is denoted by $K(L)$.
 3. For $a \in L$, we set $\downarrow a = \{x \in L \mid x \ll a\}$ and $\uparrow a = \{x \in L \mid a \ll x\}$. For a subset $A \subseteq L$, $\downarrow A$ is defined as $\bigcup \{\downarrow x \mid x \in A\}$, and $\uparrow A$ is defined dually, i.e., $\bigcup \{\uparrow x \mid x \in A\}$.

The following are some basic properties of the way-below relation and proofs are omitted.

Proposition 2.3.2. *In a dcpo L the following statements hold for all $a, b, c, d \in L$.*

1. $a \ll b$ implies $a \leq b$;
2. $a \leq b \ll c \leq d$ implies $a \ll d$;
3. $a \ll c$ and $b \ll c$ imply $a \vee b \ll c$ whenever $a \vee b$ exists in L ;
4. $\perp \ll a$ whenever L has a least element \perp ;
5. An element a is compact if and only if $\uparrow a$ is Scott-open. □

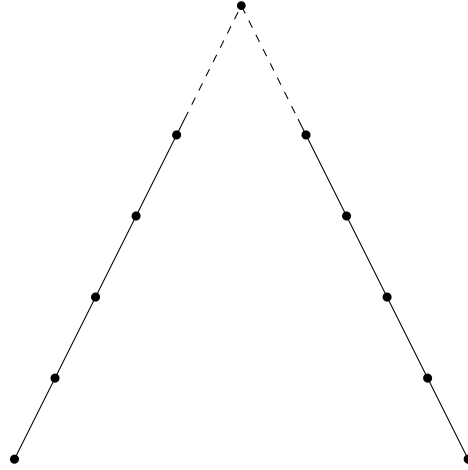
Continuous domains are those dcpos that are well-behaved regarding approximations.

Definition 2.3.3 (continuous domain, algebraic domain). 1. A dcpo L is called *continuous* if for every $x \in L$, $\downarrow x$ is directed and $\sup \downarrow x = x$. Continuous dcpos are also called *continuous domains* or simply *domains*. A *continuous lattice* is a continuous domain that is simultaneously a complete lattice. The category of all continuous domains is denoted by **CONT**.

2. A dcpo L is called *algebraic* if for every $x \in L$, $\downarrow x \cap K(L)$ is directed and $\sup(\downarrow x \cap K(L)) = x$. Algebraic dcpos are also called *algebraic domains*. An algebraic domain L is called *countably based* if $K(L)$ is a countable set. An *algebraic lattice* is an algebraic domain that is simultaneously a complete lattice. The category of all algebraic domains is denoted by **ALG**.

Example 2.3.4. 1. The unit interval $[0, 1]$ is a continuous lattice in the usual order, where $a \ll b$ in $[0, 1]$ iff $a = b = 0$ or $a < b$.

2. The set \mathbb{Z}^- of negative integers with the natural ordering is an algebraic domain. Every element in it is compact.

FIGURE 2.2: A non-continuous dcpo A .

3. Every finite poset is an algebraic domain, with every element in it being compact.
4. [GHK⁺03, Proposition I-1.4] For a locally compact topological space X , the lattice $\mathcal{O}(X)$ of open sets of X , under the containment partial order, is a continuous lattice. For any two open subsets U and V , $U \ll V$ in $\mathcal{O}(X)$ if and only if there exists a compact saturated subset K such that $U \subseteq K \subseteq V$.
5. The dcpo A in Figure 2.2 is not continuous, since $\downarrow x = \emptyset$ for all $x \in A$.

The following proposition simplifies the verification of continuity.

Proposition 2.3.5. *A dcpo L is continuous iff for every $x \in L$ there exists a directed subset $A \subseteq \downarrow x$ such that $\sup A = x$.*

Proof. We show the non-trivial direction. Fix $x \in L$ and the corresponding directed subset $A \subseteq \downarrow x$ with $\sup A = x$; we prove that $\downarrow x$ is directed. To this end, let $a, b \ll x$. Since A is directed and $\sup A = x$, we have some $c, d \in A$ such that $a \leq c$ and $b \leq d$. Again by the directedness of A , we can find some $e \in A$ greater than both c and d . Then we have $a, b \leq e \in A \subseteq \downarrow x$. That is, e is an upper bound of a and b in $\downarrow x$. Hence $\downarrow x$ is directed. \square

The following proposition shows an important property that every continuous domain enjoys, which enables us to examine the Scott topology on continuous domains more clearly.

Proposition 2.3.6 (Interpolation Property). *Let L be a continuous domain. If $a \ll b$ in L , then there exists some element $c \in L$ such that $a \ll c \ll b$.*

Proof. We consider the set $B := \{x \mid \exists y : x \ll y \ll b\}$, and claim that B is a directed subset of L and $\sup B = b$. Obviously B is not empty. For $x_1, x_2 \in B$, there exist y_1, y_2 such that $x_1 \ll y_1 \ll b$ and $x_2 \ll y_2 \ll b$. Since L is continuous, $\downarrow b$ is directed. We have some $y_3 \in \downarrow b$ such that $y_1, y_2 \leq y_3$. By Proposition 2.3.2, we have $x_1, x_2 \in \downarrow y_3$. Again from the directedness of $\downarrow y_3$, there is some $x_3 \in \downarrow y_3$ above x_1 and x_2 . So we have $x_3 \ll y_3 \ll b$, which implies $x_3 \in B$ and hence B is directed. Now we prove that $\sup B = b$. First, $\sup B$ is larger than every element y in $\downarrow b$ since $\downarrow y \subseteq B$ and that $y = \sup \downarrow y$. So $b = \sup \downarrow b \leq \sup B$. Conversely, $\sup B$ is obviously less than b since $B \subseteq \downarrow b \subseteq \downarrow b$. Hence we have $\sup B = b$.

Since $a \ll b$ and $\sup B = b$, we have some $x \in B$ such that $a \leq x$. Since $x \in B$, there exists some c such that $x \ll c \ll b$. So we have $a \leq x \ll c \ll b$, which in light of Proposition 2.3.2 implies that $a \ll c \ll b$. \square

As promised, with the Interpolation Property at hand, the Scott topology on continuous domains becomes more transparent.

Proposition 2.3.7. *Let L be a dcpo. Then $\text{int}_\sigma \uparrow x \subseteq \uparrow x$ for all $x \in L$. If in addition, L is continuous, then $\uparrow x$ is Scott-open and $\text{int}_\sigma \uparrow x = \uparrow x$.*

Proof. Take $a \in \text{int}_\sigma \uparrow x$ and assume that $a \leq \sup E$ for some directed subset $E \subseteq L$. It follows from the Scott-openness of $\text{int}_\sigma \uparrow x$ that $\sup E \in \text{int}_\sigma \uparrow x$ and $E \cap \text{int}_\sigma \uparrow x \neq \emptyset$. This means $e \in \text{int}_\sigma \uparrow x \subseteq \uparrow x$ for some $e \in E$, which implies $x \ll a$. So $\text{int}_\sigma \uparrow x \subseteq \uparrow x$.

From Proposition 2.3.2 $\uparrow x$ is obviously an upper set. For the Scott-openness of $\uparrow x$, let D be a directed subset with $\sup D \in \uparrow x$, i.e., $x \ll \sup D$. If L is continuous, we employ the Interpolation Property to find some $y \in L$ such that $x \ll y \ll \sup D$. This implies that $y \leq d$ for some $d \in D$ and $x \ll d$ for this d , i.e., $D \cap \uparrow x \neq \emptyset$. So $\uparrow x$ is Scott-open. Now the equality $\text{int}_\sigma \uparrow x = \uparrow x$ follows from the fact that $\uparrow x$ is Scott-open and $\uparrow x \subseteq \uparrow x$. \square

Proposition 2.3.8. *Every continuous domain L is locally compact in the Scott topology. More precisely, for any Scott-open set $U \subseteq L$ and $x \in U$, there exists some $y \in L$ such that $x \in \uparrow y \subseteq \uparrow y \subseteq U$. In particular, the sets $\uparrow x, x \in L$ form a basis of the Scott topology $\sigma(L)$.*

Proof. Let U be a Scott-open subset of L and $x \in U$. Since L is continuous, we have that $\downarrow x$ is directed and $\sup \downarrow x = x \in U$. From the openness of U , some $y \in \downarrow x$ is already in U . So we have $x \in \uparrow y \subseteq \uparrow y \subseteq U$. Obviously, $\uparrow y$ is compact in the Scott topology. From the previous proposition, we have that $\uparrow y$ is a compact neighbourhood of x inside U . \square

Alternatively, for continuous domains we can find another basis of the Scott topology, namely, the set of Scott-open filters, where a *Scott-open filter* is a Scott-open set which is also a filter in the sense of Definition 2.1.4.

Proposition 2.3.9. *Let L be a continuous domain and U a Scott-open subset of L . For every $x \in U$, there exists a Scott-open filter F such that $x \in F \subseteq U$.*

Proof. Given any Scott-open set U and $x \in U$. From Proposition 2.3.8 there exists some $y \in U$ such that $x \in \uparrow y \subseteq \uparrow y \subseteq U$. By the Interpolation Property, we can find some y_1 such that $y \ll y_1 \ll x$. Doing this inductively, we end up with a sequence $y_i, i \in \mathbb{N}$ such that $y \ll \dots \ll y_n \ll \dots \ll y_1 \ll x$. Let $F = \bigcup_{i \in \mathbb{N}} \uparrow y_i$. Obviously, F is

Scott-open and $x \in F \subseteq U$. We verify that F is a filter. To this end, let $a, b \in F$. Then there exist $i, j \in \mathbb{N}$ and $a \in \uparrow y_i, b \in \uparrow y_j$. Without loss of generality we assume $i \leq j$, then $a, b \in \uparrow y_j \subseteq \uparrow y_j \subseteq \uparrow y_{j+1} \subseteq F$. So we find inside F a lower bound y_j of both a and b . Thus F is a filter. \square

Proposition 2.3.10. *Every continuous domain with the Scott topology is a sober space.*

Proof. Let L be a continuous domain and A an irreducible closed subset of ΣL . We consider the set $\downarrow A$ and show that it is directed. For any $a, b \in \downarrow A$, we have that $\uparrow a$ and $\uparrow b$ are two Scott-open subsets intersecting A . Since A is irreducible, $\uparrow a \cap \uparrow b \cap A$ is not empty, say $c \in \uparrow a \cap \uparrow b \cap A$. Since L is continuous, $\downarrow c$ is directed; therefore we can find some $d \in \downarrow c$ such that $a, b \leq d$. Thus $d \in \uparrow a \cap \uparrow b \cap \downarrow c \subseteq \uparrow a \cap \uparrow b \cap \downarrow A$, and hence $\downarrow A$ is a directed subset of L and $\sup \downarrow A$ exists. Again since L is continuous, every element in A can be approximated by a directed subset of $\downarrow A$, So $\sup A = \sup \downarrow A$. Moreover, we have $\sup A \in A$ since A is Scott-closed and $\downarrow A$ is a directed subset of it. Finally, we can conclude that $A = \downarrow \sup A$. So ΣL is sober. \square

Notice that from Proposition 2.3.8 we can find for every $x \in U$ a compact neighbourhood $\uparrow y$ inside U . The set $\uparrow y$ is not only a compact saturated set, it is also a principal filter and enjoys the fact that any Scott-open cover of $\uparrow y$ has a member covering $\uparrow y$ already. This phenomenon can be abstracted in general.

Definition 2.3.11. Let X be a topological space. A subset A of X is called *supercompact* if for any open cover \mathcal{U} of A , i.e., $A \subseteq \bigcup \mathcal{U}$, there exists a member $U \in \mathcal{U}$ such that $A \subseteq U$. Equivalently, a subset A is supercompact if and only if for arbitrary family $\{C_i\}_{i \in I}$ of closed sets, $A \cap \bigcap_{i \in I} C_i \neq \emptyset$ whenever $K \cap C_i \neq \emptyset$ for all $i \in I$. X is called a *locally supercompact space* if for any open set U and $x \in U$, there exists a supercompact neighbourhood K of x inside U , that is, $x \in K^\circ \subseteq K \subseteq U$.

Every supercompact set is compact, and every locally supercompact space is locally compact. Combining Propositions 2.3.8 and 2.3.10 gives us the following refined result.

Theorem 2.3.12. *Every continuous domain with the Scott topology is a locally supercompact sober space.* \square

Conversely, every locally supercompact and sober topological space actually arises as a continuous domain equipped with the Scott topology. To prove this, let us first rewrite supercompact subsets in the language of the specialisation preorder.

Proposition 2.3.13. *Let X be a topological space. Then the supercompact saturated subsets of X are exactly the sets of the form $\uparrow_s x$ with $x \in X$.*

Proof. From Proposition 2.2.25 $\uparrow_s x$ is saturated for all $x \in X$. Moreover, any open cover of $\uparrow_s x$ contains a member U with $x \in U$, then $\uparrow_s x \subseteq U$ since open sets are upper sets. Hence $\uparrow_s x$ is supercompact.

Reversely, assume that K is a supercompact saturated subset of X . Then $K \cap \bigcap_{c \in K} \overline{\{c\}} = K \cap \bigcap_{c \in K} \downarrow_s c$ is not empty. Pick x in the intersection and we claim that $\uparrow_s x = K$. Obviously, $\uparrow_s x \subseteq K$ since $x \in K$ and K is saturated. Conversely, for any $c \in K$, $x \leq_s c$ since $x \in \bigcap_{c \in K} \downarrow_s c$. So $K \subseteq \uparrow_s x$. \square

Theorem 2.3.14. *Let X be a topological space. Then the following statements are equivalent:*

1. X is a locally supercompact sober space;
2. X is a locally supercompact monotone convergence space;
3. $\Omega(X)$ is a continuous domain, and $\mathcal{O}(X) = \sigma(\Omega(X))$, i.e., $X = \Sigma\Omega(X)$.

Proof. (1 \Rightarrow 2) This is straightforward from Proposition 2.2.28.

(2 \Rightarrow 3) Since X is a monotone convergence space, $\Omega(X)$ is a dcpo. We proceed to show that $\Omega(X)$ is continuous. For any $x \in X$, consider the set $A_x = \{y \in X \mid x \in (\uparrow_s y)^\circ\}$. Since X is locally supercompact, A_x is a directed subset. Moreover, $\sup A_x = x$. In fact if $\sup A_x < x$, we know that $x \in X \setminus \downarrow_s \sup A_x$ which is open. Thus we use local supercompactness to find some z such that $x \in (\uparrow_s z)^\circ \subseteq \uparrow_s z \subseteq X \setminus \downarrow_s \sup A_x$. Hence we obtain that $z \in A_x$ and $z \not\leq \sup A_x$ at the same time, which is a contradiction. Finally, since X is a monotone convergence space, the sets $(\uparrow_s y)^\circ, y \in A_x$ are Scott-open and therefore we have $y \ll x$ for all $y \in A_x$. Hence $\Omega(X)$ is a continuous domain by Proposition 2.3.5. To prove $\mathcal{O}(X) = \sigma(\Omega(X))$, we only need to prove $\sigma(\Omega(X)) \subseteq \mathcal{O}(X)$ since the reverse containment is always true for monotone convergence spaces. To this end let U be a Scott-open set in $\Omega(X)$. For any $x \in U$, we know that $x = \sup A_x$, so $A_x \cap U \neq \emptyset$. Pick $y \in A_x \cap U$. Then we have that $x \in (\uparrow_s y)^\circ \subseteq \uparrow_s y \subseteq U$. This implies that $\sigma(\Omega(X)) \subseteq \mathcal{O}(X)$.

(3 \Rightarrow 1) This is straightforward from Theorem 2.3.12. □

Corollary 2.3.15. *L is a continuous domain if and only if ΣL is a locally supercompact sober space.*

Proof. Straightforward from Theorems 2.3.12, 2.3.14 and the fact that $L = \Omega(\Sigma L)$. □

If we denote the category of all locally supercompact sober spaces and continuous functions by **LSS**. Then from the discussion above and Proposition 2.2.20 we have the following categorical isomorphism.

Corollary 2.3.16. *The category **CONT** is isomorphic to **LSS**.* □

2.4 Quasicontinuous domains

Quasicontinuity is a generalisation of the notion of continuity of dcpos, introduced in the early 80s by Gierz, Lawson and Stralka [GLS83].

Instead of focusing on the way-below relation between points, in the “quasi” setting we talk about the way-below relation between finite sets and points, that is to say, we use finite sets to approximate a given point. In general, one says that a subset G of a dcpo L is *way-below* a subset H if for every directed set D , $\sup D \in \uparrow H$ implies $d \in \uparrow G$ for some $d \in D$. This generalises the usual way-below relation between elements which justifies writing $G \ll H$ for it. If H consists of a single element x then one writes $G \ll x$ instead of $G \ll \{x\}$. For any subset F of L , set $\uparrow\uparrow F = \{x \in L \mid F \ll x\}$ (Be aware that $\uparrow\uparrow F$ is different from $\uparrow F$ which was defined in Definition 2.3.1).

Consistent with this we define a preorder (called the *Smyth preorder*) between subsets G, H by $G \leq H \Leftrightarrow \uparrow H \subseteq \uparrow G$. This implies that a family \mathcal{F} of subsets is *directed* if the corresponding family $\{\uparrow G \mid G \in \mathcal{F}\}$ is filtered in the inclusion order. Note that the Smyth preorder becomes a partial order on any family consisting of upper sets and is equal to reverse containment.

The following properties of the way-below relation between subsets are trivial.

Proposition 2.4.1. *In a dcpo L the following statements hold for all subsets A, B, C, D of L .*

1. $A \ll B$ implies $A \leq B$, i.e., $B \subseteq \uparrow A$;
2. $A \leq B \ll C \leq D$ implies $A \ll D$;
3. $A \ll B$ and $C \ll D$ imply $A \cup C \ll B \cup D$. □

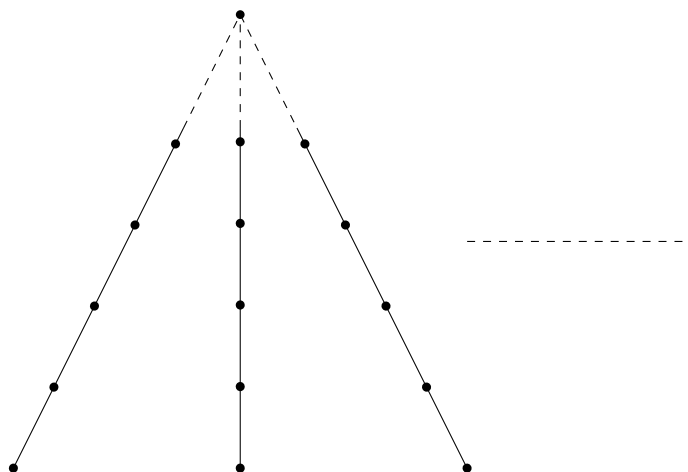


FIGURE 2.3: A non-quasicontinuous dcpo.

Definition 2.4.2 (quasicontinuous domain). A dcpo L is called *quasicontinuous* (or a *quasicontinuous domain*) if for each $x \in L$ the family

$$\text{fin}(x) = \{F \mid F \text{ is finite, } F \ll x\}$$

is a directed family, and whenever $x \not\leq y$, then there exists $F \in \text{fin}(x)$ with $y \notin \uparrow F$, i.e., $\uparrow x = \bigcap \{\uparrow F \mid F \in \text{fin}(x)\}$.

The category of quasicontinuous domains is denoted by **qCONT**.

Proposition 2.4.3. *Every continuous domain is quasicontinuous.*

Proof. Straightforward. □

Example 2.4.4. Quasicontinuous domains strictly generalise continuous ones. The dcpo A in Figure 2.2 is a quasicontinuous domain which is not continuous. Meanwhile, not every dcpo is quasicontinuous, a counterexample is the dcpo in Figure 2.3. Concretely, this dcpo is obtained by taking infinitely many disjoint copies of the natural numbers (with the usual order) and gluing them together at ∞ . The verification is easy and omitted here.

Quasicontinuous domains share many properties with continuous domains. Especially we will see that every quasicontinuous domain is locally compact and sober in the Scott topology, a property that every continuous domain enjoys (see Theorem 2.3.12). Before we proceed, let us first recall Rudin’s lemma, which relies on Axiom of Choice and is important for us to work with quasicontinuity.

Lemma 2.4.5 (Rudin’s Lemma). [GHK⁺03, Lemma III-3.3] *Let \mathcal{F} be a directed family of non-empty finite subsets of a partially ordered set L . Then there exists a directed set $D \subseteq \bigcup_{F \in \mathcal{F}} F$ such that $D \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.* \square

The following key fact is a consequence of Rudin’s Lemma:

Proposition 2.4.6. (cf. [HK13, Lemma 4.1]) *Let L be a dcpo and $G, H \subseteq L$. Then $G \ll H$ if and only if for any directed family \mathcal{F} of non-empty finite sets, that $\bigcap \{\uparrow F \mid F \in \mathcal{F}\} \subseteq \uparrow H$ implies that $F \subseteq \uparrow G$ for some $F \in \mathcal{F}$.*

Proof. The “if” direction is obvious. To prove the “only if” part, we assume that $F \not\subseteq \uparrow G$ for all $F \in \mathcal{F}$. Then $\{F \setminus \uparrow G \mid F \in \mathcal{F}\}$ is also a directed family of non-empty finite sets. By Rudin’s lemma, there exists a directed set $D \subseteq (\bigcup_{F \in \mathcal{F}} F) \setminus \uparrow G$ such that $D \cap (F \setminus \uparrow G) \neq \emptyset$ for all $F \in \mathcal{F}$. So $\sup D \in \uparrow F$ for all F , which implies $\sup D \in \bigcap \{\uparrow F \mid F \in \mathcal{F}\} \subseteq \uparrow H$. One then obtains that $\uparrow G \cap D \neq \emptyset$ since $G \ll H$, which is a contradiction to the fact that $D \subseteq (\bigcup_{F \in \mathcal{F}} F) \setminus \uparrow G$. \square

Remark 2.4.7. Roughly speaking, the previous proposition indicates that one could define alternatively the way-below relation between finitely generated subsets by considering the usual way-below relation between them, as defined in Definition 2.3.1, in the poset of finitely generated upper sets with the reverse inclusion order. One then can define quasicontinuity of a dcpo by stipulating that its finitely generated subsets form a continuous poset. For a detailed investigation of this idea, one may refer to [HK13, Section 4].

We now use the previous proposition to prove a convenient criterion for quasicontinuity, which can be regarded as a quasi-version of Proposition 2.3.5.

Proposition 2.4.8. *A dcpo L is quasicontinuous if for every $x \in L$ the family $\text{fin}(x)$ contains a directed subfamily \mathcal{G} such that $\uparrow x = \bigcap \{\uparrow G \mid G \in \mathcal{G}\}$.*

Proof. We only need to prove that the family $\text{fin}(x)$ is directed. For $F, H \in \text{fin}(x)$, since $F, H \ll x$ and $\bigcap \{\uparrow G \mid G \in \mathcal{G}\} = \uparrow x$, by Proposition 2.4.6, there exist $G_1, G_2 \in \mathcal{G}$ such that $G_1 \subseteq \uparrow F$ and $G_2 \subseteq \uparrow H$. Then some $G \in \mathcal{G}$ is included in $\uparrow F \cap \uparrow H$ since \mathcal{G} is directed. \square

Similar to Proposition 2.3.6, we also have a quasi-version of the Interpolation Property for quasicontinuous domains.

Proposition 2.4.9. [GHK⁺03, Theorem III-3.5] *Let L be a quasicontinuous domain and $H \subseteq L$. If $H \ll x$, then there exists a finite set F such that $H \ll F \ll x$.*

Proof. For the given x we first consider the collection

$$\mathcal{G} = \{G : G \in \text{fin}(x) \mid \exists F \in \text{fin}(x), G \ll F \ll x\},$$

and prove that \mathcal{G} is a directed family and $\bigcap_{G \in \mathcal{G}} \uparrow G = \uparrow x$. Obviously, we have $\uparrow x \subseteq \bigcap_{G \in \mathcal{G}} \uparrow G$. For the reverse, if $z \notin \uparrow x$, i.e., $x \not\leq z$, from the quasicontinuity there exists a finite $F \subseteq L$ such that $F \ll x$ and $z \notin \uparrow F$. For each $a \in F$, we have $a \not\leq z$, then similarly a finite subset $F_a \subseteq L$ can be found satisfying that $F_a \ll a$ and $z \notin \uparrow F_a$. Set $K = \bigcup_{a \in F} F_a$. By Proposition 2.4.1 one easily sees that $K \ll F \ll x$, hence $K \in \mathcal{G}$. Moreover we have $z \notin \uparrow K$, which implies that $z \notin \bigcap_{G \in \mathcal{G}} \uparrow G$ and hence $\bigcap_{G \in \mathcal{G}} \uparrow G \subseteq \uparrow x$. For directedness of \mathcal{G} , suppose $G_i \in \mathcal{G}$ and $G_i \ll F_i \ll x$ for F_i finite and $i = 1, 2$. Since L is quasicontinuous, there exists $F \in \text{fin}(x)$ such that

$F \subseteq \uparrow F_1 \cap \uparrow F_2$. Then $G_i \ll F_i \leq F$ implies $G_i \ll F$ for $i = 1, 2$. It then follows that $G_i \ll b$ for all $b \in F$ and $i = 1, 2$. Again since L is quasicontinuous, $\text{fin}(b)$ is directed for every $b \in F$; therefore, there exists a finite set E_b for every $b \in F$ such that $E_b \ll b$ and $E_b \subseteq \uparrow G_1 \cap \uparrow G_2$. Set $E = \bigcup_{b \in F} E_b$. Then E is finite and $E \subseteq \uparrow G_1 \cap \uparrow G_2$, and again by Proposition 2.4.1 we have $E \ll F \ll x$. This means that $E \in \mathcal{G}$ and hence \mathcal{G} is directed.

Since $H \ll x$ and \mathcal{G} is a directed family with $\bigcap_{G \in \mathcal{G}} \uparrow G = \uparrow x$, from Proposition 2.4.6 we have that $G \subseteq \uparrow H$ for some $G \in \mathcal{G}$. Since $G \ll F \ll x$ for some finite F , and finally we conclude that $H \ll F \ll x$. \square

Proposition 2.4.10. [GHK⁺03, Proposition III-3.6] *Let L be a quasicontinuous domain.*

1. *A subset U of L is Scott-open iff for each $x \in U$ there exists a finite $F \ll x$ such that $\uparrow F \subseteq U$.*
2. *The sets $\uparrow F = \{x \mid F \ll x\}$ for all finite F of L are Scott-open and they are equal to $\text{int}_\sigma(\uparrow F)$, hence from 1, the family $\{\uparrow F \mid F \subseteq_{\text{fin}} L\}$ forms a basis for the Scott topology.*

Proof. (1) Let U be a Scott-open subset and $x \in U$. From the definition of Scott-open sets we know that $U \ll x$. We employ Proposition 2.4.9 to find a finite subset F such that $U \ll F \ll x$. Since U is an upper set, we have $\uparrow F \subseteq U$. Conversely, let U be any subset satisfying the assumption. Then U must be an upper set, since for any $x \in U$, there exists a finite F with $F \ll x$ and $\uparrow F \subseteq U$, this implies that $\uparrow x \subseteq \uparrow F \subseteq U$. Now assume that D is a directed subset with $\text{sup } D \in U$. Again from the assumption, there exists some finite G such that $G \ll \text{sup } D$ and $\uparrow G \subseteq U$. So some $d \in D$ must be in $\uparrow G$ hence in U from the definition of the way-below relation. So U is Scott-open.

(2) The Scott-openness of $\uparrow F$ follows from (1) since by Proposition 2.4.9 for each $x \in \uparrow F$ there exists a finite set H such that $F \ll H \ll x$, i.e., $H \ll x$ and $\uparrow H \subseteq \uparrow F$. Since $\uparrow F$ is Scott-open and $\uparrow F \subseteq \uparrow F$, we have $\uparrow F \subseteq \text{int}_\sigma(\uparrow F)$. The reverse containment is obvious. Hence we have $\uparrow F = \text{int}_\sigma(\uparrow F)$. Finally, that the family $\{\uparrow F \mid F \subseteq_{\text{fin}} L\}$ forms a basis for the Scott topology is a direct consequence of the first statement. \square

We arrive at one of the main results of this section, which says that quasicontinuous domains generalise continuous ones in the realm of locally compact sober deposes.

Theorem 2.4.11. [GHK⁺03, Proposition III-3.7] *Every quasicontinuous domain with the Scott topology is a locally compact sober space.*

Proof. Let L be a quasicontinuous domain and U a Scott-open set in L and $x \in U$. By Proposition 2.4.10, there exists a finite set F such that $F \ll x$ and $\uparrow F \subseteq U$. Thus we have $x \in \uparrow F \subseteq \uparrow F \subseteq U$. Since $\uparrow F$ is Scott-open from Proposition 2.4.10, and $\uparrow F$ is obviously compact, it then follows that $\uparrow F$ is a compact neighbourhood of x inside U . Hence L is locally compact.

For sobriety, let A be a closed irreducible subset of L . Consider the collection $\mathcal{F} = \{F : F \text{ is finite} \mid \uparrow F \cap A \neq \emptyset\}$. We claim that \mathcal{F} is a directed family of finite sets. Obviously, \mathcal{F} is not empty because of quasicontinuity. Given $F_1, F_2 \in \mathcal{F}$, we have that $\uparrow F_1 \cap A \neq \emptyset$ and $\uparrow F_2 \cap A \neq \emptyset$. From the previous proposition $\uparrow F_1, \uparrow F_2$ are Scott-open, it then follows that $\uparrow F_1 \cap \uparrow F_2 \cap A \neq \emptyset$ since A is irreducible. Pick some a in the intersection; by the quasicontinuity of L , there exists some finite $G \in \text{fin}(a)$ such that $G \subseteq \uparrow F_1 \cap \uparrow F_2$. This implies that $G \in \mathcal{F}$ and \mathcal{F} is directed. Since A is a closed set, the collection $\{F \cap A \mid F \in \mathcal{F}\}$ is also a directed family of finite non-empty sets. We apply Rudin's lemma to this collection and find a directed set D such that $D \subseteq \bigcup_{F \in \mathcal{F}} (F \cap A) \subseteq A$ and $D \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. We proceed by showing that

$\sup D$ is the largest element of A . Since $D \subseteq A$ and A is Scott-closed, $\sup D \in A$. Assume that there exists $t \in A$ such that $t \not\leq \sup D$. Since L is quasicontinuous we have some finite set $E \in \text{fin}(t)$ such that $\sup D \notin \uparrow E$. However, since $t \in \uparrow E \cap A$, E is in \mathcal{F} by definition and hence $\sup D \in \uparrow E$. This contradiction implies that $\sup D$ is the largest element in A . Hence $A = \downarrow \sup D$ and L is sober in the Scott topology. \square

Quasicontinuous domains are special locally compact sober dcpos. As can be seen, the compact neighbourhoods found in the above proof are of a special form: they are finitely generated upper sets. We make this property into the following definition.

Definition 2.4.12. A topological space X is called *locally finitary compact* if for any open set U and $x \in U$, there exists a finite set $F \subseteq U$ such that $x \in (\text{sat}(F))^\circ \subseteq \text{sat}(F) \subseteq U$.

Since $\text{sat}(F)$ is always compact saturated for any finite set F , every locally finitary compact space is always locally compact.

The following statement is obvious from the proof of Theorem 2.4.11.

Proposition 2.4.13. *Every quasicontinuous domain is locally finitary compact and sober.* \square

We know from Theorem 2.3.14 that locally supercompact sober spaces arise as continuous domains equipped with the Scott topology; locally finitary compact sober spaces, meanwhile, are actually quasicontinuous domains equipped with the Scott topology. To prove this, we need the following result first.

Proposition 2.4.14. *Let L be a dcpo and $\{F_\alpha\}_{\alpha \in A}$ a directed family of non-empty finite subsets of L . If the intersection $\bigcap_{\alpha \in A} \uparrow F_\alpha$ is included in a Scott-open set U ,*

then some F_α is already contained in U . Moreover, the intersection $\bigcap_{\alpha \in A} \uparrow F_\alpha$ is compact.

Proof. Assume the statement is not true, that is, $F_\alpha \setminus U \neq \emptyset$ for all $\alpha \in A$. Then $\{F_\alpha \setminus U \mid \alpha \in A\}$ is a directed family of finite sets. By Rudin's Lemma, there exists a directed subset $D \subseteq \bigcup_{\alpha \in A} (F_\alpha \setminus U)$ such that $D \cap (F_\alpha \setminus U) \neq \emptyset$ for all $\alpha \in A$. The latter inequality implies that $\sup D \in \bigcap_{\alpha \in A} \uparrow F_\alpha \subseteq U$. Hence some $d \in D$ is in U since U is Scott-open. However, this is impossible since $d \in D \subseteq \bigcup_{\alpha \in A} (F_\alpha \setminus U) = (\bigcup_{\alpha \in A} F_\alpha) \setminus U$. Any Scott-open cover of the intersection $\bigcap_{\alpha \in A} \uparrow F_\alpha$ actually covers some $\uparrow F_\alpha$, and consequently finitely many members suffice to cover $\uparrow F_\alpha$ since F_α is finite. Hence the same finite members cover $\bigcap_{\alpha \in A} \uparrow F_\alpha$ already. \square

Theorem 2.4.15 (cf. [Law85, Theorem 2]). *Let X be a topological space. Then the following statements are equivalent:*

1. X is a locally finitary compact sober space;
2. X is a locally finitary compact monotone convergence space;
3. $\Omega(X)$ is a quasicontinuous domain, and $\mathcal{O}(X) = \sigma(\Omega(X))$, i.e., $X = \Sigma\Omega(X)$.

Proof. (1 \Rightarrow 2) This is from Proposition 2.2.28.

(2 \Rightarrow 3) Since X is a monotone convergence space, $\Omega(X)$ is a dcpo. We show that $\Omega(X)$ is quasicontinuous. For any $x \in \Omega(X)$, consider the family

$$A_x = \{F \mid F \subseteq_{\text{fin}} X \ \& \ x \in (\uparrow_s F)^\circ\}.$$

We know that $A_x \subseteq \text{fin}(x)$ since X is a monotone convergence space. Also, since X is locally finitary compact, A_x is directed and $\bigcap_{F \in A_x} \uparrow_s F = \uparrow_s x$. It then follows from Proposition 2.4.8 that $\Omega(X)$ is quasicontinuous. To prove that $\mathcal{O}(X) = \sigma(\Omega(X))$,

let $U \in \sigma(\Omega(X))$ and $x \in U$. Then A_x defined above is a directed family of finite sets and $\bigcap_{F \in A_x} \uparrow_s F = \uparrow_s x \subseteq U$, hence by Proposition 2.4.14 there exists some $F \in A_x$ such that $F \subseteq U$. This implies that $x \in (\uparrow_s F)^\circ \subseteq \uparrow_s F \subseteq U$. Thus we have $U \in \mathcal{O}(X)$ and $\sigma(\Omega(X)) \subseteq \mathcal{O}(X)$. The reverse containment holds since X is a monotone convergence space.

(3 \Rightarrow 1) This is straightforward from Proposition 2.4.13. \square

If we denote the category of all locally finitary compact sober spaces and continuous functions by **LFS**. Then analogously to Corollary 2.3.16 we have the following.

Corollary 2.4.16. *The category **qCONT** is isomorphic to **LFS**.* \square

2.5 Core-compactness and sobriety

In Sections 2.3 and 2.4 we have seen that both continuous domains and quasicontinuous domains are locally compact sober dcpos. In this section, we take a closer look at those two properties, i.e., local compactness and sobriety, and the interplay between them. For a greater generality, we start from the following definition.

Definition 2.5.1 (core-compact space). A topological space X is called *core-compact* if its open sets form a continuous lattice in the inclusion order. In particular, a dcpo L is called a *core-compact dcpo* if $\sigma(L)$ is a continuous lattice.

Remark 2.5.2. From Example 2.3.4 every locally compact topological space is core-compact. In particular, every quasicontinuous domain is locally compact by Theorem 2.4.11 and hence core-compact.

The following observation is a direct consequence of the definition of core-compact spaces.

Proposition 2.5.3. *A topological space X is core-compact if and only if for any open set $U \in \mathcal{O}(X)$ and $x \in U$, there exists an open set V such that $x \in V \ll U$. \square*

Proposition 2.5.4. (cf. [GL13, Exercise 5.2.7]) *A topological space X is core-compact if and only if the set $(\ni) := \{(U, x) \in \mathcal{O}(X) \times X \mid x \in U\}$ is open in $\Sigma\mathcal{O}(X) \times X$, the topological product of $\Sigma\mathcal{O}(X)$ and X .*

Proof. Assume that X is core-compact, that is, $\mathcal{O}(X)$ is a continuous lattice in the inclusion order. For any $(U, x) \in (\ni)$, we know $x \in U$. Since $\mathcal{O}(X)$ is continuous, we can find some $V \in \mathcal{O}(X)$ such that $x \in V \ll U$. Again since $\mathcal{O}(X)$ is continuous, $\uparrow V$ is Scott-open. So $\uparrow V \times V$ is an open set in $\Sigma\mathcal{O}(X) \times X$ containing (U, x) . Note that $\uparrow V \times V$ is contained in (\ni) , so the set (\ni) is open in $\Sigma\mathcal{O}(X) \times X$.

Conversely, let U be any open subset of X and $x \in U$. Then $(U, x) \in (\ni)$. By assumption (\ni) is open in $\Sigma\mathcal{O}(X) \times X$, we find a Scott-open subset \mathcal{H} of $\mathcal{O}(X)$ and an open set $V \in \mathcal{O}(X)$ such that $(U, x) \in (\mathcal{H}, V) \subseteq (\ni)$. Note that the fact $(\mathcal{H}, V) \subseteq (\ni)$ implies that $V \subseteq \bigcap \mathcal{H}$, so we have $x \in V \subseteq \bigcap \mathcal{H} \subseteq U$. The Scott-openness of \mathcal{H} tells us that V is actually way-below U . \square

Core-compactness is an essential topological property in finding cartesian closed full subcategories of topological spaces. The core-compact spaces are precisely the exponentiable objects in the category of topological spaces and continuous functions. For a core-compact topological space Y and an arbitrary space Z , the exponential object Z^Y is the topological space with the underlying set

$$C[Y, Z] = \{f : Y \rightarrow Z \mid f \text{ is continuous}\}$$

equipped with the *Isbell topology* which is generated, as a subbasis, by subsets of the form:

$$N(\mathcal{H} \leftarrow U) = \{f \in C[Y, Z] \mid f^{-1}(U) \in \mathcal{H}\},$$

where \mathcal{H} is a Scott-open set in $\mathcal{O}(Y)$ and U an open set in Z .

Theorem 2.5.5. *For a topological space X , the following statements are equivalent:*

1. X is core-compact;
2. X is exponentiable in the category of topological spaces and continuous functions. □

A full development of this result can be found in [GHK⁺03, Section 2.4] and in [GL13, Section 5.4]. An elementary treatment is accessible at [EH02], and the above theorem appears as [EH02, Theorem 4.7].

For topological spaces C and Z , another common topology defined on $C[Y, Z]$ is the so-called *compact-open* topology, where opens in the subbasis of the topology are of the form:

$$N(K \rightarrow U) = \{f \in C[Y, Z] \mid f(K) \subseteq U\},$$

where K is a compact set in Y and U an open set in Z .

For dcpos L and M , the Isbell topology and the compact-open topology are defined on the function space $[L \rightarrow M]$ by viewing it as $C[\Sigma L, \Sigma M]$ in light of Proposition 2.2.20.

Proposition 2.5.6. *1. Let Y, Z be topological spaces. Then the compact-open topology is coarser than the Isbell topology on $C[Y, Z]$. If Y is locally compact, then the two topologies coincide.*

2. *Let L, M be dcpos. Then the Isbell topology is coarser than the Scott topology on the function space $[L \rightarrow M]$.*

Proof. (1) The compact-open topology is always coarser than the Isbell topology. Indeed, for any compact subset K of Y , let $\mathcal{K} = \{V \in \mathcal{O}(Y) \mid K \subseteq V\}$. One easily

verifies that \mathcal{K} is a Scott-open set in $\mathcal{O}(Y)$ and $N(\mathcal{K} \leftarrow U) = N(K \rightarrow U)$ for any open set $U \in \mathcal{O}(Z)$.

Now we assume that Y is locally compact. Let \mathcal{H} be a Scott-open set in $\mathcal{O}(Y)$ and U an open set in Z . For any $f \in N(\mathcal{H} \leftarrow U)$, $f^{-1}(U) \in \mathcal{H}$. Since Y is locally compact,

$$f^{-1}(U) = \bigcup \{K^\circ \mid K \subseteq f^{-1}(U) \text{ \& } K \text{ is compact}\}.$$

This is a directed union, so there exists a compact set $K \subseteq f^{-1}(U)$ such that $K^\circ \in \mathcal{H}$. Then one has $f(K) \subseteq U$, i.e., $f \in N(K \rightarrow U)$. Moreover, for any $g \in N(K \rightarrow U)$, $K^\circ \subseteq K \subseteq g^{-1}(U)$, so $g^{-1}(U) \in \mathcal{H}$, i.e., $g \in N(\mathcal{H} \leftarrow U)$. To sum up, one has that $f \in N(K \rightarrow U) \subseteq N(\mathcal{H} \leftarrow U)$, and this implies that the Isbell topology is coarser than the compact-open topology.

(2) Let \mathcal{H} be a Scott-open set in $\sigma(L)$ and U be a Scott-open set in M . We first prove that $N(\mathcal{H} \leftarrow U)$ is an upper set. To this end let $f \leq g$ in $[L \rightarrow M]$ and $f \in N(\mathcal{H} \leftarrow U)$. For any $x \in f^{-1}(U)$, $f(x) \in U$ hence $g(x) \in U$ since $f \leq g$. So we have $f^{-1}(U) \subseteq g^{-1}(U)$. Since $f^{-1}(U) \in \mathcal{H}$ by assumption, we have that $g^{-1}(U)$ is in the Scott-open set \mathcal{H} , which is equivalent to saying that $g \in N(\mathcal{H} \leftarrow U)$; therefore $N(\mathcal{H} \leftarrow U)$ is an upper set. Now take any directed family $\{f_i\}_{i \in I}$ of Scott-continuous functions and assume $\sup_{i \in I} f_i \in N(\mathcal{H} \leftarrow U)$. This means that $(\sup_{i \in I} f_i)^{-1}(U) \in \mathcal{H}$. Since $f_i, i \in I$, are Scott-continuous and U is Scott-open, with a simple computation we know that $(\sup_{i \in I} f_i)^{-1}(U) = \bigcup_{i \in I} f_i^{-1}(U) \in \mathcal{H}$. Note that $\{f_i^{-1}(U)\}_{i \in I}$ is a directed family of open sets, then we have some $i \in I$ such that $f_i^{-1}(U) \in \mathcal{H}$, and hence $f_i \in N(\mathcal{H} \leftarrow U)$. So $N(\mathcal{H} \leftarrow U)$ is Scott-open, and hence the Isbell topology is coarser than the Scott topology. \square

We have seen that by Theorem 2.5.5 core-compact spaces can be characterised as exponentiable objects in the category of topological spaces; core-compact dcpos,

meanwhile, can be further described via products of the Scott topology.

Theorem 2.5.7. (cf. [GHK⁺03, Theorem II-4.13]) *Let L be a dcpo. Then the following statements are equivalent:*

1. L is core-compact, i.e., $\sigma(L)$ is a continuous lattice;
2. for every dcpo S one has $\Sigma(S \times L) = \Sigma S \times \Sigma L^1$;
3. for every complete lattice S one has $\Sigma(S \times L) = \Sigma S \times \Sigma L$;
4. $\Sigma(\sigma(L) \times L) = \Sigma(\sigma(L)) \times \Sigma L$. □

This result originally appears as [GHK⁺80, Theorem II-4.11], we present a more direct proof here.

Proof. (1 \Rightarrow 2) We prove that the topologies on both $\Sigma(S \times L)$ and $\Sigma S \times \Sigma L$ coincide. It is easy to see that every open set in $\Sigma S \times \Sigma L$ is Scott-open in $S \times L$, we prove the inverse provided L is core-compact. To this end let O be a Scott-open set in $S \times L$ and $(a, b) \in O$. We consider the set $B = \{y \in L \mid (a, y) \in O\}$. Obviously B is a Scott-open set in L containing b . From core-compactness of L there exists some Scott-open set $V \in \sigma(L)$ such that $b \in V \ll B$. Moreover, since $\sigma(L)$ is continuous, we employ the Interpolation Property to find a sequence $\{V_i\}_{i \in \mathbb{N}}$ of Scott-open sets in L such that $b \in V \ll \dots \ll V_n \ll \dots \ll V_1 \ll B$. For each $i \in \mathbb{N}$, we define $U_i = \{x \in S \mid \{x\} \times V_i \subseteq O\}$, and let $U = \bigcup_{i \in \mathbb{N}} U_i$. Since for each $i \in \mathbb{N}$, $\{a\} \times V_i \subseteq \{a\} \times B \subseteq O$, we have that $a \in U_i$ for each $i \in \mathbb{N}$, and hence $a \in U$. Moreover, for any $c \in U$, there exists some $j \in \mathbb{N}$ such that $c \in U_j$, hence $\{c\} \times V_j \subseteq O$. Note that $\{c\} \times V \subseteq \{c\} \times V_j \subseteq O$, this implies that $U \times V \subseteq O$.

¹Note that these two products are taken in the category of dcpos and that of topological spaces, respectively.

To sum up, we have $(a, b) \in U \times V \subseteq O$. Since V is Scott-open in L , we finish our proof by showing that U is Scott-open in S . Indeed, the set U is obviously an upper set since each $U_i, i \in \mathbb{N}$ is. Now let D be a directed subset of S and $\sup D \in U$. For each $d \in D$, set $W_d = \{y \in L \mid (d, y) \in O\}$. It is easy to see that $\{W_d\}_{d \in D}$ is a directed family of Scott-open sets in L . Since $\sup D \in U$, there exists some $n \in \mathbb{N}$ such that $\sup D \in U_n$, which means that $\{\sup D\} \times V_n \subseteq O$. Then for each $y \in V_n$, $(\sup D, y) \in O$; therefore, there exists some $e \in D$ such that $(e, y) \in O$, i.e., $y \in W_e$. This implies that $V_n \subseteq \bigcup_{d \in D} W_d$. Remember that $V_{n+1} \ll V_n$, it follows that $V_{n+1} \subseteq W_d$ for some $d \in D$. Then for this d , by definition of W_d we know that $\{d\} \times V_{n+1} \subseteq O$, and this means that $d \in U_{n+1} \subseteq U$. So U is indeed Scott-open in S .

(2 \Rightarrow 3) Obvious.

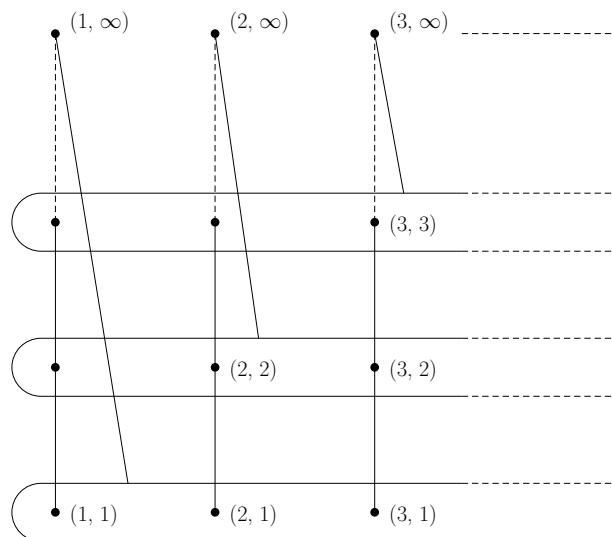
(3 \Rightarrow 4) Obvious.

(4 \Rightarrow 1) By Proposition 2.5.4, we prove that L is core-compact by showing that the set $(\ni) = \{(U, x) \mid x \in U \in \sigma(L)\}$ is open in $\Sigma(\sigma(L)) \times \Sigma L$. From the assumption, we only need to prove that (\ni) is Scott-open in $\sigma(L) \times L$. This is obvious and we omit it here. \square

The above theorem enables us to prove that certain dcpos are sober in the Scott topology.

Theorem 2.5.8. *Let L be a join-complete dcpo, that is, $\sup A$ exists for all non-empty $A \subseteq L$. If L is core-compact, then L is a sober dcpo.*

Proof. We first note that the binary join operation $\vee : (a, b) \mapsto a \vee b : L \times L \rightarrow L$ is Scott-continuous, which is straightforward from Proposition 2.2.3, and this is equivalent to saying that \vee is continuous from $\Sigma(L \times L)$ to ΣL . Since L is core-compact, from Theorem 2.5.7 $\Sigma(L \times L) = \Sigma L \times \Sigma L$. We have that \vee is also continuous from $\Sigma L \times \Sigma L$ to ΣL .

FIGURE 2.4: Johnstone's non-sober dcpo \mathcal{J} .

Now we assume that L is not sober in the Scott topology, that is, there exists an irreducible closed set $A \subseteq L$ such that A does not have a greatest element. Since A is Scott-closed, we then have at least two different maximal elements in A , say a and b . It is easy to see that $a \vee b \in L \setminus A$. Since \vee is continuous from $\Sigma L \times \Sigma L$ to ΣL , we can find Scott-open sets U and V in L such that $a \in U$, $b \in V$ and $U \vee V = \{u \vee v \mid u \in U \ \& \ v \in V\} \subseteq L \setminus A$. Since U and V are upper sets, $U \vee V = U \cap V$. This implies that $U \cap V \subseteq L \setminus A$, which is equivalent to $A \subseteq (L \setminus U) \cup (L \setminus V)$. Remembering that A is an irreducible closed set, we have either $A \subseteq L \setminus U$ or $A \subseteq L \setminus V$. However, neither case is possible since $a \in A \cap U$ and $b \in A \cap V$. This contradiction implies that L is actually sober. \square

Not every dcpo is sober in the Scott topology. In fact, this question was not clear until Johnstone [Joh81] gave a non-sober dcpo in 1981. His example is depicted in Figure 2.4.

Example 2.5.9 (Johnstone's non-sober dcpo \mathcal{J}). Let \mathbb{N} be the set of natural numbers and $\mathcal{J} = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ with the partial order defined by $(m, n) \leq (m', n')$ iff either $m = m'$ and $n \leq n' \leq \infty$ or $n' = \infty$ and $n \leq m'$.

It is easy to see from the figure that \mathcal{J} is a dcpo. We now claim that \mathcal{J} itself is a Scott-closed irreducible set. In fact, for any two non-empty Scott-open subsets U and V , we have some $(i, \infty) \in U$ and $(j, \infty) \in V$. Since U, V are Scott-open and $(i, \infty) = \bigvee_{k \in \mathbb{N}} (i, k)$, $(j, \infty) = \bigvee_{k \in \mathbb{N}} (j, k)$, we have some k_i and k_j such that $(i, k_i) \in U$ and $(j, k_j) \in V$. Without loss of generality we assume that $k_i \leq k_j$. Then from the order, we know that $(i, k_i), (j, k_j) \leq (k_j, \infty)$. This implies that $(k_j, \infty) \in U \cap V$. So \mathcal{J} is indeed an irreducible set. However, since (n, ∞) is maximal in \mathcal{J} for all $n \in \mathbb{N}$, \mathcal{J} cannot be written as a Scott closure of some singleton. We conclude that \mathcal{J} is not sober.

In general, to verify that a space is sober is quite complicated. The following theorem, called the Hofmann-Mislove Theorem, supplies an important characterisation of sober spaces via a correspondence between compact saturated subsets and Scott-open filters of opens.

Before we proceed, let us fix some notations first. For a topological spaces X , we set $\mathcal{Q}(X)$ to be the set of all compact saturated subsets (including the empty set) of X . For a dcpo L , we use $\text{OFilt}(L)$ to denote the set of all Scott-open filters in L . We equip $\mathcal{Q}(X)$ with the reverse inclusion order and $\text{OFilt}(L)$ with the inclusion order. The Hofmann-Mislove Theorem states that $\mathcal{Q}(X)$ and $\text{OFilt}(\mathcal{O}(X))$ are isomorphic for any sober space X .

Theorem 2.5.10 (Hofmann-Mislove Theorem). *Let X be a sober space. Then the mapping*

$$\Phi : \mathcal{Q}(X) \rightarrow \text{OFilt}(\mathcal{O}(X)), \Phi(K) = \{U \in \mathcal{O}(X) \mid K \subseteq U\}$$

which assigns to a compact saturated subset K of X the open filter of all open sets containing K is an order isomorphism between $\mathcal{Q}(X)$ (ordered by reverse inclusion)

and $\text{OFilt}(\mathcal{O}(X))$. The inverse sends an open filter of open sets to its intersection:

$$\Psi : \text{OFilt}(\mathcal{O}(X)) \rightarrow \mathcal{Q}(X), \Psi(\mathcal{F}) = \bigcap \mathcal{F} = \bigcap \{U \mid U \in \mathcal{F}\}.$$

Moreover, for any T_0 space X , if either the function Φ or Ψ defined above is an isomorphism, then X is sober. \square

This theorem originates from [HM81], where the proof was given by K. Hofmann and M. Mislove via spectral theory. A direct and more accessible proof was formalised by K. Keimel and J. Paseka in [KP94]. The second part of this theorem is a straightforward consequence of [GHK⁺03, Theorem II-1.21].

Proposition 2.5.11. *Every sober space is well-filtered.*

Proof. Let X be a sober space, and assume that $\{K_\alpha\}_{\alpha \in A}$ is a filtered family of compact saturated subsets with its intersection $\bigcap_{\alpha \in A} K_\alpha$ contained in an open subset U of X . We consider the family $\mathcal{F} = \{V \in \mathcal{O}(X) \mid K_\alpha \subseteq V \text{ for some } \alpha \in A\}$ of open sets. One can easily check that this family is a Scott-open filter in $\mathcal{O}(X)$ and $\bigcap \mathcal{F} = \bigcap_{\alpha \in A} K_\alpha$. By the Hofmann-Mislove Theorem, any open set containing $\bigcap \mathcal{F}$ is actually in \mathcal{F} . In particular, we have that U is in \mathcal{F} , and by the definition we know that some K_α is contained in U . \square

Proposition 2.5.12. *Let X be a well-filtered space and $\{K_\alpha\}_{\alpha \in A}$ a filtered family of compact saturated subsets of X . Then $\bigcap_{\alpha \in A} K_\alpha$ is compact.*

Proof. Let \mathcal{U} be a family of open sets covering $\bigcap_{\alpha \in A} K_\alpha$, that is, $\bigcap_{\alpha \in A} K_\alpha \subseteq \bigcup \mathcal{U}$. Since X is well-filtered, we have some $\alpha \in A$ such that $K_\alpha \subseteq \bigcup \mathcal{U}$. Since K_α is compact, finitely many open sets in \mathcal{U} cover K_α , and these finite open sets cover the intersection $\bigcap_{\alpha \in A} K_\alpha$ as well. \square

Example 2.5.13. Johnstone's dcpo \mathcal{J} offers a non-sober dcpo; \mathcal{J} is also a non-well-filtered dcpo. In fact, from Example 2.5.9 we easily see that the induced topology on the set of maximal elements of \mathcal{J} is equal to the co-finite topology. This implies that any subset of maximal elements of \mathcal{J} is actually compact and saturated. If we denote the set $\{(i, \infty) \in \mathcal{J} \mid i \geq n\}$ by M_n , then $\{M_n \mid n \in \mathbb{N}\}$ is a filtered family of compact saturated subsets of \mathcal{J} with an empty intersection. However, none of such $M_n, n \in \mathbb{N}$ is empty. So \mathcal{J} is not well-filtered.

One sees from the definition of well-filteredness, as a contrapositive statement, that a space X is well-filtered if and only if, for every closed subset $C \subseteq X$, and for every filtered family $\{K_\alpha\}_{\alpha \in A}$ of compact saturated sets such that K_α intersects C for all $\alpha \in A$, the intersection $\bigcap_{\alpha \in A} K_\alpha$ also intersects C . Moreover, we have the following stronger statement.

Proposition 2.5.14. *A space X is well-filtered if and only if, for every filtered family $\{C_\beta\}_{\beta \in B}$ of closed subsets, and for every filtered family $\{K_\alpha\}_{\alpha \in A}$ of compact saturated sets such that K_α intersects C_β for each $\alpha \in A, \beta \in B$, the intersection $\bigcap_{\alpha \in A} K_\alpha \cap \bigcap_{\beta \in B} C_\beta$ is not empty.*

Proof. For any fixed $\alpha \in A$, we know that $K_\alpha \cap C_\beta \neq \emptyset$ for all $\beta \in B$. Since K_α is compact and $\{C_\beta\}_{\beta \in B}$ is filtered, we have that $\bigcap_{\beta \in B} C_\beta \cap K_\alpha \neq \emptyset$. Note this holds for all $\alpha \in A$, so from the contrapositive statement above, $\bigcap_{\alpha \in A} K_\alpha \cap \bigcap_{\beta \in B} C_\beta \neq \emptyset$. \square

Sobriety and well-filteredness are two different notions for T_0 topological spaces. One can easily find well-filtered spaces which are not sober. For instance, the space of real numbers with the co-countable topology, i.e., open sets consist of the empty set and those subsets with countable complements. In this space, compact subsets are precisely the finite subsets, hence the space is well-filtered, while it is not sober since the whole space itself is irreducible.

However, whether sobriety and well-filteredness coincide on dcpos was unknown for a long time. In fact, R. Heckmann asked this question in [Hec92] as an open problem. After 9 years, H. Kou [Kou01] answered it in the negative by giving a non-sober well-filtered dcpo. Moreover, he proved that these two notions coincide over locally compact dcpos [Kou01, Theorem 2.2]. In general, this also holds for T_0 topological spaces.

Proposition 2.5.15. [Kou01, Theorem 2.3] *Let X be a locally compact T_0 space. Then X is sober if and only if it is well-filtered.*

Proof. From Proposition 2.5.11 we know sober spaces are well-filtered. We prove that well-filteredness and local compactness imply sobriety. To this end, let A be a closed irreducible subset of X . Consider the family $\mathcal{K} = \{K \in \mathcal{Q}(X) \mid A \cap K^\circ \neq \emptyset\}$ of compact saturated subsets K with their interior K° intersecting A . Since A is irreducible and X is locally compact, \mathcal{K} is a filtered family of compact saturated subsets. By Proposition 2.5.14, $\bigcap \mathcal{K}$ intersects A . Pick $a \in \bigcap \mathcal{K} \cap A$. The closure of $\{a\}$ is obviously contained in A . Conversely, for any $x \in A$ and open set U containing x , since X is locally compact, we find some compact saturated neighbourhood Q of x inside U . Then we have that $Q \in \mathcal{K}$ and this implies that $a \in Q \subseteq U$; therefore, x is in the closure of $\{a\}$. Hence $A \subseteq \overline{\{a\}}$. To sum up, the closed irreducible subset A can be written as the closure of the point a . Since X is T_0 , we conclude that X is sober. \square

In light of the Hofmann-Mislove Theorem, the proof of the following result, which is borrowed from [GL13, Theorem 8.3.10], is more topological in nature than the original one appearing in [HL78].

Theorem 2.5.16. *Let X be a sober space. Then X is core-compact if and only if X is locally compact.*

Proof. We prove the non-trivial direction. Assume that X is core-compact and $U \subseteq X$ is an open set containing x . Since X is core-compact, $\mathcal{O}(X)$ is continuous. We find an open set V such that $x \in V \ll U$. Since $\mathcal{O}(X)$ is continuous, $\uparrow V$ is Scott-open in $\mathcal{O}(X)$ by Proposition 2.3.7 and $U \in \uparrow V$. By Proposition 2.3.9, we find a Scott-open filter $\mathcal{F} \subseteq \mathcal{O}(X)$ such that $U \in \mathcal{F} \subseteq \uparrow V$. Now the Hofmann-Mislove Theorem tells us that $\bigcap \mathcal{F}$, the intersection of all open sets in \mathcal{F} , is actually compact. Moreover, since $U \in \mathcal{F} \subseteq \uparrow V$, we know $x \in V \subseteq \bigcap \mathcal{F} \subseteq U$. Hence $\bigcap \mathcal{F}$ is a compact neighbourhood of x inside U . So X is locally compact. \square

The following result is now a corollary to Theorem 2.5.8 and Theorem 2.5.16.

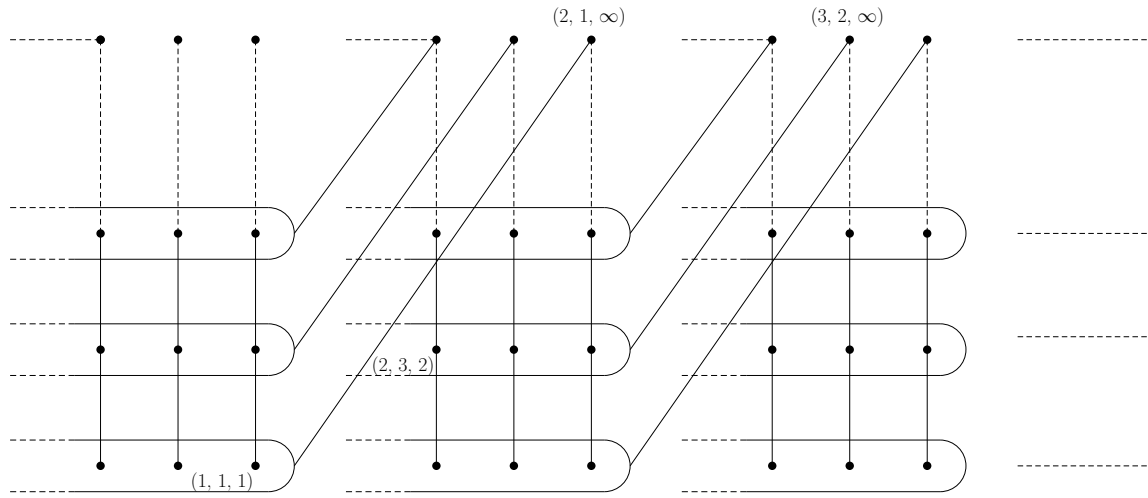
Corollary 2.5.17. *For a join-complete poset L , L is a locally compact dcpo if and only if it is core-compact.* \square

We end this section with the following observation which comes from a combination of Proposition 2.5.15 and Theorem 2.5.16.

Theorem 2.5.18. *For a T_0 space X , the following statements are equivalent:*

1. X is locally compact and sober;
2. X is locally compact and well-filtered;
3. X is core-compact and sober. \square

Question 2.5.19. What can we say about core-compact and well-filtered T_0 spaces? In particular, are these spaces sober?

FIGURE 2.5: A non-sober well-filtered dcpo \mathcal{L} .

2.6 A simple non-sober well-filtered dcpo

We have mentioned in the last section that Kou [Kou01] gave a non-sober well-filtered dcpo for answering a question posed by Heckmann [Hec92]. In this section, we give another well-filtered dcpo which is not sober in the Scott topology. This dcpo seems to us simpler than Kou's.

Example 2.6.1. Let $\mathcal{L} = \mathbb{N} \times \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$, where \mathbb{N} is the set of natural numbers.

We define an order \leq on \mathcal{L} as follows:

$(i_1, j_1, k_1) \leq (i_2, j_2, k_2)$ if and only if:

- $i_1 = i_2, j_1 = j_2, k_1 \leq k_2 \leq \infty$;
- or $i_2 = i_1 + 1, k_1 \leq j_2, k_2 = \infty$.

\mathcal{L} can be easily depicted as in Figure 2.5. It is obvious that \mathcal{L} is a dcpo, since any infinite directed subset of \mathcal{L} is contained in $\downarrow(i, j, \infty)$ for some $i, j \in \mathbb{N}$ with its supremum being (i, j, ∞) .

Claim 2.6.2. \mathcal{L} is not sober in the Scott topology.

Proof. Since (i, j, ∞) is maximal in \mathcal{L} for any $i, j \in \mathbb{N}$, we prove that \mathcal{L} is not sober by showing that \mathcal{L} itself is an irreducible set. To this end, take any two non-empty Scott-open sets U and V . Then we know $(i_1, j_1, \infty) \in U$ and $(i_2, j_2, \infty) \in V$ for some $i_1, j_1, i_2, j_2 \in \mathbb{N}$. Without loss of generality, we assume $i_1 \leq i_2$. Since $\bigvee_{k \in \mathbb{N}} (i_1, j_1, k) = (i_1, j_1, \infty) \in U$, we have some $k_1 \in \mathbb{N}$ such that $(i_1, j_1, k_1) \in U$. So $(i_1 + 1, k_1, \infty) \in U$ since $(i_1, j_1, k_1) \leq (i_1 + 1, k_1, \infty)$ and U is an upper set. By Scott-openness of U , we have $(i_1 + 1, k_1, k_2) \in U$ for some $k_2 \in \mathbb{N}$. Now by induction we can reach some n such that $(i_2, k_n, k_{n+1}) \in U$. Note that $(i_2, j_2, \infty) \in V$, so $(i_2, j_2, m) \in V$ for large enough m . Take $n = \max\{k_{n+1}, m\}$. Then $(i_2 + 1, n, \infty)$ is greater than both (i_2, k_n, k_{n+1}) and (i_2, j_2, m) . Thus $(i_2 + 1, n, \infty) \in U \cap V$, which implies that any two non-empty Scott-open sets in \mathcal{L} intersect. So \mathcal{L} is an irreducible set. \square

To prove that \mathcal{L} is not well-filtered, we first investigate what compact saturated subsets look like in \mathcal{L} . For any compact saturated subset $K \subseteq \mathcal{L}$, from Corollary 2.2.26 we could write K as $\uparrow M_K$, where M_K is the set of minimal elements of K . So we locate K by looking at the set M_K .

We use \mathcal{L}^∞ to denote the set of all maximal elements of \mathcal{L} , i.e., $\mathcal{L}^\infty = \{(i, j, \infty) \mid i, j \in \mathbb{N}\}$, and we denote the set $\mathcal{L} \setminus \mathcal{L}^\infty$ by $\mathcal{L}^{<\infty}$. For any $(i_0, j_0, \infty) \in \mathcal{L}^\infty$, we define $l_{(i_0, j_0, \infty)} = \{(i_0, j, \infty) \mid j_0 \leq j\} \cup \{(i, j, \infty) \mid i < i_0\}$. As can be seen from Figure 2.5, $l_{(i_0, j_0, \infty)}$ actually denotes the set of all maximal elements in \mathcal{L} which are on the left side of (i_0, j_0, ∞) , containing (i_0, j_0, ∞) itself. Moreover, one sees that for any $(i_0, j_0, \infty) \in \mathcal{L}^\infty$, $\downarrow l_{(i_0, j_0, \infty)}$ is Scott-closed.

Claim 2.6.3. Given a non-empty saturated subset $K \subseteq \mathcal{L}$, then K is compact in the Scott topology if and only if:

1. $M_K \cap \mathcal{L}^{<\infty}$ is finite;
2. there exist $i_0, j_0 \in \mathbb{N}$, such that $l_{(i_0, j_0, \infty)} \cap K = \{(i_0, j_0, \infty)\}$, and
3. for all $(i, j, \infty) \in K \cap \mathcal{L}^\infty$, i ranges over some finite subset of \mathbb{N} .

Proof. Let K be any saturated subset satisfying those three conditions and \mathcal{U} be a Scott-open cover of K . We prove that finitely many Scott-open sets in \mathcal{U} suffice to cover M_K , hence K itself. Obviously, $M_K \cap \mathcal{L}^{<\infty}$ will be covered by finitely many members of \mathcal{U} since it is finite from Condition 1. For $K \cap \mathcal{L}^\infty$, from Condition 2, there exist $i_0, j_0 \in \mathbb{N}$, such that $l_{(i_0, j_0, \infty)} \cap K = \{(i_0, j_0, \infty)\}$, hence some open set $U \in \mathcal{U}$ covers (i_0, j_0, ∞) . Note that $(i_0, j_0, \infty) = \bigvee_{k \in \mathbb{N}} (i_0, j_0, k)$. So there exists some $k_0 \in \mathbb{N}$ such that $(i_0, j_0, k_0) \in U$. Hence $(i_0 + 1, j, \infty) \in U$ for all $j \geq k_0$. In other words, there are only finitely many j 's with $(i_0 + 1, j, \infty)$ not in U . Inductively, for any $i \geq i_0$ there are only finitely many j 's such that (i, j, ∞) are not in U . Hence from 3 we know that only finitely many elements in $K \cap \mathcal{L}^\infty$ are not in U . So $K \cap \mathcal{L}^\infty$ must be covered by finitely many members from \mathcal{U} . Finally note that $M_K \subseteq (M_K \cap \mathcal{L}^{<\infty}) \cup (K \cap \mathcal{L}^\infty)$, this implies that finitely many members of \mathcal{U} would cover M_K hence K .

Conversely, for any non-empty compact saturated subset K , the set $M_K \cap \mathcal{L}^{<\infty}$ cannot be infinite. Otherwise the family $\{\mathcal{L} \setminus \downarrow(M_K \cap \mathcal{L}^{<\infty} \setminus \{x\}) \mid x \in M_K \cap \mathcal{L}^{<\infty}\}$ is a Scott-open cover of K , but any finitely many members of it fail to cover K . The second condition is true since $\{\downarrow l_{(i, j, \infty)} \mid (i, j, \infty) \in K\}$ is a filtered family of Scott-closed subsets, with all of its members intersecting with K . By the compactness of K , the intersection $\bigcap \{\downarrow l_{(i, j, \infty)} \mid (i, j, \infty) \in K\} \cap K$ is not empty. If some (i_0, j_0, ∞) is in this intersection, then $l_{(i_0, j_0, \infty)} \cap K = \{(i_0, j_0, \infty)\}$. If we find some (i, j, k) in this intersection, then obviously, $l_{(i, j, \infty)} \cap K = \{(i, j, \infty)\}$. Finally, for all $(i, j, \infty) \in K \cap \mathcal{L}^\infty$, if i varies within an infinite subset G of \mathbb{N} , then we pick one j_i for every

$i \in G$ such that $(i, j_i, \infty) \in K \cap \mathcal{L}^\infty$. Then $\{\downarrow\{(i, j_i, \infty) \mid i \in G \setminus F\} \mid F \subseteq_{\text{fin}} G\}$ is a filtered family of Scott-closed subsets, with each member of it intersecting K . However, the intersection of this family is empty, which contradicts the compactness of K . \square

Now we have enough ingredients to prove that \mathcal{L} is well-filtered.

Claim 2.6.4. \mathcal{L} is well-filtered in the Scott topology.

Proof. Assume that $\{K_\alpha\}_{\alpha \in A}$ is a filtered family of non-empty compact saturated subsets in \mathcal{L} , with its intersection $\bigcap_{\alpha \in A} K_\alpha$ contained in some Scott-open set U . Then $\{\uparrow(M_{K_\alpha} \cap \mathcal{L}^{<\infty}) \mid \alpha \in A\}$ is also a filtered family and $\bigcap_{\alpha \in A} \uparrow(M_{K_\alpha} \cap \mathcal{L}^{<\infty}) \subseteq U$. From the previous claim, $M_{K_\alpha} \cap \mathcal{L}^{<\infty}$ is finite for all $\alpha \in A$. So by Proposition 2.4.14, we have some α_0 such that $\uparrow(M_{K_{\alpha_0}} \cap \mathcal{L}^{<\infty}) \subseteq U$.

Now we consider $K_\alpha \cap \mathcal{L}^\infty, \alpha \in A$. They also form a filtered family in the inclusion order. We apply Claim 2.6.3 to find for each K_α the element $(i_\alpha, j_\alpha, \infty)$ such that $l_{(i_\alpha, j_\alpha, \infty)} \cap K_\alpha = \{(i_\alpha, j_\alpha, \infty)\}$. Fix some $\alpha_1 \in A$, since $\{K_\alpha\}_{\alpha \in A}$ is filtered, again from Claim 2.6.3, i_α ranges over some finite set $F \subseteq \mathbb{N}$ for all $\alpha \geq \alpha_1$. Let $i = \max\{i_\alpha \mid \alpha \geq \alpha_1\}$, then the set $\{j_\alpha \mid i_\alpha = i \ \& \ \alpha \geq \alpha_1\}$ is actually finite since when $i_\alpha = i_\beta, K_\alpha \subseteq K_\beta$ implies $j_\alpha \leq j_\beta$. This implies that there exists some point (i, j, ∞) being in all K_α , hence this point (i, j, ∞) is also in U . From the openness of U , for those α with $i_\alpha = i$ at most finitely many elements of $K_\alpha \cap \mathcal{L}^\infty$ are not in U . Since $\bigcap_{\alpha \in A} K_\alpha$ is contained in U , for some big enough α_2 , we have $K_{\alpha_2} \cap \mathcal{L}^\infty \subseteq U$. Now we can conclude that $K_\alpha \subseteq U$ for $\alpha > \alpha_0, \alpha_2$, and hence \mathcal{L} is well-filtered. \square

Recently, X. Xi and J. Lawson [XL17] proved that every complete lattice is well-filtered in the Scott topology. Hence Isbell's non-sober complete lattice [Isb82] serves as another dcpo which is not sober but well-filtered. We also want to mention that

X. Xi and D. Zhao [XZ15] gave a uniform construction of dcpo models of T_1 topological spaces, and proved this construction preserves sobriety and well-filteredness. Thus, one can obtain non-sober well-filtered dcpos by constructing the dcpo model of non-sober well-filtered topological spaces. The construction, however, is out of the scope of this thesis and can be found in [XZ15]. We would like to end this section with advertising the simplicity of our example \mathcal{L} of which the order structure, as shown in Figure 2.5, is quite transparent.

2.7 Scott-continuous retractions

We have introduced Scott-continuous functions between dcpos and seen that they work well with the Scott topology (see for example Proposition 2.2.20). However, this is not the case for continuity or quasicontinuity of dcpos. More precisely, neither continuity nor quasicontinuity of dcpos are preserved by Scott-continuous functions. Consider the following example:

Example 2.7.1. Let L be any non-continuous dcpo, and M be the dcpo consisting of the same elements of L but with the discrete order. Then the function that sends $x \in M$ to the same element $x \in L$ is Scott-continuous. However, M as a dcpo is continuous since every element of it is compact.

This leads us to a new class of functions between dcpos which do preserve continuity.

Definition 2.7.2. In general, for topological spaces X and Y , a continuous function $f: X \rightarrow Y$ is called a *retraction* if there exists a continuous map $g: Y \rightarrow X$ such that $f \circ g = \text{id}_Y$. The function g is called a *section* of f , and Y is called a *retract* of X .

For dcpos L and M , L is called a *retract* of M if ΣL is a retract of ΣM . From Proposition 2.2.20, retractions and sections between dcpos are always Scott-continuous.

Example 2.7.3. Let L be a dcpo. For each element $a \in L$, the principal ideal $\downarrow a$ in the induced order is a retract of L under the retraction r_a from L to $\downarrow a$, defined as:

$$r_a(x) = \begin{cases} x, & x \in \downarrow a; \\ a & \text{otherwise.} \end{cases}$$

The corresponding section is the inclusion map.

Example 2.7.4. Let L be a dcpo. For each *compact element* $a \in L$, the principal filter $\uparrow a$ in the induced order is a retract of L under the retraction f_a from L to $\uparrow a$, defined as:

$$f_a(x) = \begin{cases} x, & x \in \uparrow a; \\ a & \text{otherwise.} \end{cases}$$

The corresponding section is the inclusion map.

The images of continuous domains under retractions are indeed continuous again.

Proposition 2.7.5. *Let f be a retraction from a continuous domain L to a dcpo M . Then M is continuous.*

Proof. Let $g : M \rightarrow L$ be a Scott-continuous function such that $f \circ g = \text{id}_M$. For given $a \in M$, we claim $f(x) \ll a$ for every $x \ll g(a)$ in L . To this end, let $D \subseteq M$ be a directed subset and $a \leq \sup D$. It follows that $g(a) \leq g(\sup D) = \sup g(D)$ since g is Scott-continuous. So $x \leq g(d)$ for some $d \in D$, which implies $f(x) \leq f(g(d)) = d$. Thus $f(x) \ll a$ for every $x \ll g(a)$, that is $\{f(x) \mid x \ll g(a)\} \subseteq \downarrow a$. We now prove that $\sup \downarrow a = a$ and $\downarrow a$ is directed.

Since L is continuous, $\downarrow g(a)$ is directed. We have the following equations from the Scott-continuity of f .

$$\sup\{f(x) \mid x \ll g(a)\} = \sup f(\downarrow g(a)) = f(\sup \downarrow g(a)) = f(g(a)) = a.$$

Since $\{f(x) \mid x \ll g(a)\} \subseteq \downarrow a$, this implies $\sup \downarrow a = a$. The directedness of $\downarrow a$ follows from Proposition 2.3.5 by noticing that $\{f(x) \mid x \ll g(a)\}$ is directed. \square

Proposition 2.7.6. *Every continuous domain L arises as the retract of an algebraic domain.*

Proof. We give a sketch of the proof and details can be found at [GHK⁺03, Theorem I-4.17].

Take all the ideals of L and order them by the set inclusion. One can verify that this is an algebraic domain in which principal ideals are compact elements. The function that sends each ideal I of L to its supremum $\sup I$ is the wanted retraction, and the corresponding section sends each $x \in L$ to the ideal $\downarrow x$. \square

Corollary 2.7.7. *Let L be a core-compact dcpo and U be a Scott-open subset of L . Then U is also a core-compact dcpo in the induced order.*

Proof. Obviously, U with the induced order is a dcpo and the Scott topology on U is the induced Scott-topology from L . Now the restriction map that sends any Scott-open set $V \in \sigma(L)$ to $U \cap V$ is a retraction from $\sigma(L)$ to $\sigma(U)$. It then follows from Proposition 2.7.5 that $\sigma(U)$ is continuous. \square

Similar to continuity, quasicontinuity is also preserved by retractions:

Proposition 2.7.8. *Let f be a retraction from a quasicontinuous domain L to a dcpo M . Then M is quasicontinuous.*

Proof. Since f is a retraction, then by definition there exists a Scott-continuous function g from M to L such that $f \circ g = \text{id}_M$. For every $x \in M$ and finite set $F \ll g(x)$, we claim that $f(F) \ll x$. Indeed, let D be a directed set of M with $x \leq \sup D$; then $g(x) \leq g(\sup D) = \sup g(D)$ and we obtain an element $d \in D$ such that $g(d) \in \uparrow F$ because $F \ll g(x)$. So we get $d = f(g(d)) \in f(\uparrow F) \subseteq \uparrow f(F)$, and the claim is true.

Given $x, y \in M$ with $x \not\leq y$, then $x = f(g(x)) \in M \setminus \downarrow y$ and we get $g(x) \in f^{-1}(M \setminus \downarrow y)$. Since L is quasicontinuous and $f^{-1}(M \setminus \downarrow y)$ is Scott-open, we get from Proposition 2.4.10 that there exists $G \in \text{fin}(g(x))$ such that $G \subseteq f^{-1}(M \setminus \downarrow y)$. This means that $f(G) \subseteq M \setminus \downarrow y$ or equivalently $y \notin \uparrow f(G)$. By the claim above we know that $f(G) \ll x$, that is, $f(G) \in \text{fin}(x)$. So for every $x \in M$, we have $\uparrow x = \bigcap \{\uparrow f(G) \mid G \in \text{fin}(g(x))\}$.

Finally, we note that the family $\{f(G) \mid G \in \text{fin}(g(x))\}$ is directed because $\text{fin}(g(x))$ is and f preserves the order. Proposition 2.4.8 now allows us to conclude that M is quasicontinuous. \square

A function that preserves quasicontinuity need not be a retraction. Actually, J. Goubault-Larrecq [GL12] proposed the so-called *quasi-retractions* between topological spaces as generalisations of retractions, and proved that quasi-retractions also preserve quasicontinuity of dcpos equipped with the Scott topology. Moreover, these maps preserve nearly all topological properties that we have encountered so far. In this thesis, however, we only need retractions between dcpos, and they are already powerful enough for our purpose.

Proposition 2.7.9. *Let L, M be dcpos and $f : L \rightarrow M$ a retraction. Then M is compact, locally compact, core-compact, coherent, well-filtered, sober, respectively, whenever L is, respectively. In particular, M is stably compact if L is stably compact.*

Proof. We prove the proposition for sobriety and core-compactness. One may consult in [Jun04, Proposition 2.17] for detailed proofs.

Let g be a section of f , and assume that L is sober. For A an arbitrary closed irreducible subset of M , one has $g(A)$ is an irreducible subset of L . Indeed, if $g(A) \subseteq B \cup C$ for closed sets B and C , then $A \subseteq g^{-1}(B) \cup g^{-1}(C)$. Hence $A \subseteq g^{-1}(B)$ or $A \subseteq g^{-1}(C)$ since A is irreducible, which implies that $g(A) \subseteq B$ or $g(A) \subseteq C$. One easily verifies that the Scott closure $\overline{g(A)}$ of $g(A)$ is also irreducible and from the sobriety of L , $\overline{g(A)} = \downarrow a$ for some $a \in L$. Now we do the following calculations:

$$A = \overline{A} = f(g(\overline{A})) \subseteq f(\overline{g(A)}) = f(\downarrow a) \subseteq \downarrow f(a), \quad f(a) \in f(\overline{g(A)}) \subseteq \overline{f(g(A))} = A.$$

It follows that $A = \downarrow f(a)$. Hence M is sober.

Now we assume that L is core-compact, i.e., $\sigma(L)$ is continuous. Since f, g are Scott-continuous, both $f^{-1} : \sigma(M) \rightarrow \sigma(L)$ and $g^{-1} : \sigma(L) \rightarrow \sigma(M)$ are well-defined and Scott-continuous. Since $f \circ g = \text{id}_M$, for any open set $U \in \sigma(M)$, one has $g^{-1} \circ f^{-1}(U) = (f \circ g)^{-1}(U) = U$. This implies that g^{-1} and f^{-1} form a pair of retraction and section between $\sigma(L)$ and $\sigma(M)$. Since $\sigma(L)$ is continuous, and continuity is preserved by retractions from Proposition 2.7.5, $\sigma(M)$ is also a continuous lattice. We conclude that M is core-compact. \square

Retractions between dcpos are also reconciled with the function space construction.

Proposition 2.7.10. *Let L, L' be dcpos and M, M' be retracts of them, respectively. Then $[M \rightarrow M']$ is a retract of $[L \rightarrow L']$.*

Proof. Let $f \in [L \rightarrow M], f' \in [L' \rightarrow M']$ be the retractions with their corresponding sections g and g' . We define a function from $[L \rightarrow L']$ to $[M \rightarrow M']$ by sending each Scott-continuous function $h \in [L \rightarrow L']$ to $f' \circ h \circ g \in [M \rightarrow M']$. One can easily

see this function is well-defined and Scott-continuous, and moreover, it has a section which maps $l \in [M \rightarrow M']$ to $g' \circ l \circ f \in [L \rightarrow L']$. \square

Corollary 2.7.11. *For any non-empty dcpo L , L is a retract of $[L \rightarrow L]$.*

Proof. Fix any element a in L . Then the constant map c_a that maps L onto $\{a\}$ is a retraction from L to $\{a\}$. Hence from Proposition 2.7.10 we know that $[\{a\} \rightarrow L]$ is a retract of $[L \rightarrow L]$. The statement holds since L is isomorphic to $[\{a\} \rightarrow L]$. \square

2.8 The Jung-Tix Problem

In this section, we introduce the Jung-Tix problem² which actually motivates the research in this thesis, and sum up the work that are meant to attack it.

Basically, in domain theory the Jung-Tix Problem relates to modelling higher-order functional programming languages with probabilistic features, and technically boils down to finding a cartesian closed full subcategory of continuous domains that is simultaneously closed under Jones and Plotkin's probabilistic powerdomain construction.

2.8.1 Jones and Plotkin's probabilistic powerdomain construction

In domain theory, a common method for modelling probabilistic features of functional programming languages is Jones and Plotkin's probabilistic powerdomain monad over dcpos [JP89, Jon90].

²The problem was named by J. Goubault-Larrecq in [GL12].

Jones and Plotkin's idea dates back to a general framework for semantics proposed by M. Smyth [Smy83b, Smy92]. In Smyth's dictionary, one compares computational concepts with topological spaces. A datatype τ is modelled by some topological space X_τ , and the set of semi-decidable properties (or observable properties in the sense of S. Abramsky [Abr91]), which is closed under unions and finite intersections, is simulated by a topology on X_τ . Moreover, a program of type $\tau \rightarrow \tau'$ is translated into a continuous function from X_τ to $X_{\tau'}$. In this framework, a random computation is modelled as something which has a probability of being in each open set. Mathematically, this can be achieved by giving, for each open set, a value between 0 and 1, which indicates the probability that the computation result lies in that open set. This idea is formalised by the so-called *valuations* on open sets.

Definition 2.8.1 (valuation). For a topological space X , a *probability valuation* on $\mathcal{O}(X)$, the set of open sets of X , is a function $\mu : \mathcal{O}(X) \rightarrow [0, 1]$ that is:

1. strict: $\mu(\emptyset) = 0$;
2. monotone: $V \subseteq U$ implies $\mu(V) \leq \mu(U)$;
3. modular: $\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V)$.

Definition 2.8.2. For a topological space X , a probability valuation $\mu : \mathcal{O}(X) \rightarrow [0, 1]$ is *continuous* if for any directed family \mathcal{D} of open sets with union $U = \bigcup \mathcal{D}$, we have $\mu(U) = \sup\{\mu(V) : V \in \mathcal{D}\}$.

Definition 2.8.3. (cf. [GHK⁺03, Definition IV-9.7]) For a topological space X , the *probabilistic powerdomain* $\mathcal{P}(X)$ of X is the set of all continuous probability valuations on $\mathcal{O}(X)$ with the pointwise order, sometimes called the *stochastic order*: $\mu \leq \nu$ iff $\mu(U) \leq \nu(U)$ for all open sets U . ($\mathcal{P}(X)$ is called the *subprobabilistic powerdomain* of X by some authors, e.g., [GL12].)

With the above definitions, a program $p: \iota \rightarrow \tau$ with probabilistic features can be modelled by a function $f_p: \llbracket \iota \rrbracket \rightarrow \mathcal{P}(\llbracket \tau \rrbracket)$, where $\llbracket \iota \rrbracket$ and $\llbracket \tau \rrbracket$ are topological spaces which are modelling the input type ι and output type τ , respectively. Given an input x of type ι , we find the denotation $\llbracket x \rrbracket$ in $\llbracket \iota \rrbracket$, then the behaviour of $p(x)$ is modelled by $f_p(\llbracket x \rrbracket)$, a probability valuation on $\mathcal{O}(\llbracket \tau \rrbracket)$. For any open set $U \in \mathcal{O}(\llbracket \tau \rrbracket)$ this probability valuation gives us the probability $f_p(\llbracket x \rrbracket)(U)$ that the denotation of the output lies in U .

It is straightforward to verify that for a directed family of continuous probability valuations, the pointwise supremum is another such, so the probabilistic powerdomain is a depo on any topological space. Moreover, Jones [Jon90] showed the highly non-trivial result that the probabilistic powerdomain of a continuous domain is again continuous.

Theorem 2.8.4 (cf. [GHK⁺03, Corollary IV-9.17]). *For a continuous domain L the probabilistic powerdomain $\mathcal{P}(L)$ on the topological space ΣL is again a continuous domain.* □

Theorem 2.8.4 states that the category **CONT** of continuous domains is closed under the probabilistic powerdomain construction. So one can interpret probabilistic computation in **CONT**. However, we will see in the next subsection that **CONT** is not cartesian closed. Thus **CONT** is not appropriate for modelling probabilistic computation rooted in a higher-order programming language. Due to this fact, one wants to find cartesian closed full subcategories of **CONT**, and G. Plotkin [Plo76], M. Smyth [Smy83a], and A. Jung [Jun89, Jun90b] have made essential contributions to this part.

2.8.2 Cartesian closed full subcategories of **CONT**

As hinted in the last subsection, the category **CONT** is not cartesian closed. In light of Lemma 2.2.9, this can be checked by investigating function spaces of certain continuous domains. For example, take the set \mathbb{Z}^- of negative integers with their natural ordering, then the function space $[\mathbb{Z}^- \rightarrow \mathbb{Z}^-]$ is not continuous. Indeed, for any Scott-continuous functions $f, g \in [\mathbb{Z}^- \rightarrow \mathbb{Z}^-]$ with $g \leq f$, we define a function f_n for each $n \in \mathbb{N}$ by:

$$f_n(x) = \begin{cases} f(x), & x \geq -n, \\ g(x) - 1, & \text{otherwise.} \end{cases}$$

Then the $f_n, n \in \mathbb{N}$, form a sequence of Scott-continuous functions with supremum f but none of them dominates g , so g is not way-below f . Thus any two Scott-continuous functions are not in the way-below relation (This example is taken from [Jun89, Page 30]). Since \mathbb{Z}^- is algebraic, by this example we also assert that the category **ALG** of algebraic domains is not cartesian closed either.

In general, we have the following useful criterion to exclude dcpos with a non-continuous function space.

Lemma 2.8.5. [Jun89, Theorem 1.37] *A dcpo with continuous function space is bicomplete.* □

Historically, in order to accommodate non-deterministic computation, Plotkin [Plö76] used the idea of bilimits and proposed the notion of SFP-domains (where SFP stands for sequence of finite posets). He proved that the category **SFP** of SFP-domains is a cartesian closed full subcategory of countably based algebraic domains and closed under the so-called *Plotkin powerdomain construction*. One possible definition of SFP-domains is as follows:

Definition 2.8.6 (SFP-domain). A dcpo L is an *SFP-domain* if it carries an ascending sequence of Scott-continuous endofunctions $f_n : L \rightarrow L$ such that for all $n \in \mathbb{N}$,

1. $\sup_{n \in \mathbb{N}} f_n = \text{id}_L$;
2. the image of f_n is finite;
3. $f_n \circ f_n = f_n$.

Theorem 2.8.7. [Plo76] **SFP** is cartesian closed. □

It was conjectured by Plotkin that the category **SFP** $_{\perp}$ is actually the largest cartesian closed full subcategory of the category $\omega\mathbf{ALG}_{\perp}$ of countably based pointed algebraic dcpos. In 1983, Smyth proved this conjecture [Smy83a] by showing the following result.

Theorem 2.8.8. [Smy83a, Theorem 1] *If L and $[L \rightarrow L]$ are countably algebraic, then L is strongly algebraic (i.e. SFP-domain).* □

As a consequence of Lemma 2.2.9, Theorem 2.8.7 and Theorem 2.8.8, we have the following.

Theorem 2.8.9. [Smy83a] **SFP** $_{\perp}$ is the largest cartesian closed full subcategory of $\omega\mathbf{ALG}_{\perp}$. □

The classification of general algebraic domains (not necessarily countably based) was given by Jung [Jun90a], making use of the notions of L-domains and bifinite domains which generalise Plotkin's SFP-domains.

Definition 2.8.10 (bifinite domain). A dcpo L is a *bifinite domain* if it carries a directed set of Scott-continuous endofunctions $f_a : L \rightarrow L$ such that for all $a \in A$,

1. $\sup_{a \in A} f_a = \text{id}_L$;
2. the image of f_a is finite;
3. $f_a \circ f_a = f_a$.

The category of bifinite domains is denoted by **BF**.

Definition 2.8.11 (L-dcpo). A dcpo in which every principal ideal $\downarrow x$ is a complete lattice (in its induced order) is called an *L-dcpo*. An L-domain (algebraic L-domain) is an L-dcpo which is also continuous (algebraic). The category of L-dcps is denoted by **L**, the category of L-domains is denoted by **cL** and that of algebraic L-domains is denoted by **aL**.

Theorem 2.8.12. [Jun90a, Corollary 3.8] *The category **ALG**_⊥ of pointed algebraic domains contains exactly two maximal full subcategories which are cartesian closed: **aL**_⊥ and **BF**_⊥.* □

Since the categories **BF**_⊥ and **aL**_⊥ are exactly the two maximal cartesian closed categories in the pointed algebraic case, searching in light of Proposition 2.7.6, people were conjecturing that the category **RB**_⊥ of retracts of pointed bifinite domains and the category **cL**_⊥ would be the two maximal cartesian closed full subcategories in the pointed continuous setting.

An equivalent description of retracts of bifinite domains is the so-called RB-domain.

Definition 2.8.13 (RB-domain). A dcpo L is an *RB-domain* if it carries a directed set of Scott-continuous endofunctions $f_a : L \rightarrow L$ such that for all $a \in A$,

1. $\sup_{a \in A} f_a = \text{id}_L$;
2. the image of f_a is finite.

And indeed we have:

Theorem 2.8.14. [Jun89, Theorem 2.11, Theorem 2.13]

1. The category \mathbf{RB}_\perp of pointed RB-domains is cartesian closed.
2. The category \mathbf{cL}_\perp of pointed L-domains is cartesian closed. □

However, to prove that \mathbf{RB}_\perp and \mathbf{cL}_\perp are maximal among cartesian closed full subcategories of \mathbf{CONT}_\perp turned out to be hopeless. In 1990, by introducing a new notion called FS-domain, Jung was able to fully classify the maximal cartesian closed full subcategories of \mathbf{CONT}_\perp [Jun89, Jun90b].

Definition 2.8.15 (FS-domain). A Scott-continuous function $\delta : L \rightarrow L$ on a dcpo L is *finitely separating* if there exists a finite set F_δ such that for each $x \in L$, there exists $y \in F_\delta$ such that $\delta(x) \leq y \leq x$. A dcpo L is an *FS-domain* if it carries a directed set of Scott-continuous endofunctions $f_a : L \rightarrow L, a \in A$, such that

1. $\sup_{a \in A} f_a = \text{id}_L$;
2. f_a is finitely separating for all $a \in A$.

The category of all FS-domains and Scott-continuous functions between them is denoted by \mathbf{FS} .

Theorem 2.8.16. [Jun90b, Theorem 3] *The category \mathbf{FS}_\perp is cartesian closed.* □

Jung's classification theorem relies on the following lemma.

Lemma 2.8.17. [Jun89, Lemma 4.23] *Let D and E be pointed bicomplete continuous dcpos. If E is not an L-domain and if D is not Lawson-compact then $[D \rightarrow E]$ is not continuous.* □

Since \mathbf{cL}_\perp is cartesian closed, it follows from the previous lemma that \mathbf{cL}_\perp is one of the maximal cartesian closed full subcategories of continuous domains. The category \mathbf{FS} is located via the following theorem which deals with continuous domains with a Lawson-compact function space.

Theorem 2.8.18. [Jun90b, Theorem 8] *If L and $[L \rightarrow L]$ are pointed continuous domains and Lawson-compact, then L is an FS-domain.* \square

Now the following classification theorem is just a corollary to Theorem 2.8.14, Theorem 2.8.16, Lemma 2.8.17 and Theorem 2.8.18.

Theorem 2.8.19. [Jun90b, Corollary 10] *Every cartesian closed full subcategory of \mathbf{CONT}_\perp is contained in \mathbf{cL}_\perp or in \mathbf{FS}_\perp .* \square

We end this subsection with the following remarks.

Remark 2.8.20. While for the purpose of modelling computation, a least element of dcpos is needed to model *non-termination*, it is possible to extend the previous results to the case of dcpos without a least element. One gets four maximal cartesian closed full subcategories of \mathbf{ALG} . The same technique also works for \mathbf{CONT} . See [Jun89, Chapter 3] for details.

Remark 2.8.21. One easily sees that from the definition every RB-domain is an FS-domain. However, the reverse becomes one of the oldest and best-known open problems in domain theory, more precisely:

Open: *Is every FS-domain an RB-domain?*

2.8.3 The conflict between cartesian closedness and \mathcal{P}

Now consider modelling a probabilistic higher-order functional programming language. We have cartesian closed categories $\mathbf{SFP}_\perp, \mathbf{BF}_\perp, \mathbf{AL}_\perp, \mathbf{RB}_\perp, \mathbf{FS}_\perp$ and \mathbf{cL}_\perp for potential semantic domains since the cartesian closedness could accommodate higher-order function types. Moreover, for modelling the probabilistic computation, the probabilistic powerdomain construction needs to be restricted to the wanted semantic category. This requirement rules out categories $\mathbf{SFP}_\perp, \mathbf{BF}_\perp, \mathbf{AL}_\perp$, since for a dcpo L the powerdomain $\mathcal{P}(L)$ would never be algebraic. This is because probability valuations are defined as functions into real numbers. So categories $\mathbf{RB}_\perp, \mathbf{FS}_\perp$ and \mathbf{cL}_\perp remain in our sight. In Jones' thesis [Jon90], she showed that the probabilistic powerdomain of L-domains need not be L-domains again, and in general any lattice structure would be destroyed by this powerdomain construction. Hence \mathbf{cL}_\perp was ruled out from the candidates. By Theorem 2.8.19, this means any potential candidate category will be entirely contained in \mathbf{FS}_\perp . So naturally one asks the following question:

Open: *Is the category \mathbf{FS}_\perp closed under Jones and Plotkin's probabilistic powerdomain construction?*

This question, however, turns out to be extremely difficult, and is the so-called *Jung-Tix problem*. In the following, we list what we know so far about the probabilistic powerdomain construction \mathcal{P} .

We know that \mathbf{CONT}_\perp is closed under \mathcal{P} . In the remarkable paper [JT98], Jung and Tix showed that \mathcal{P} can be restricted to the category \mathbf{LawC}_\perp of pointed Lawson-compact continuous domains.

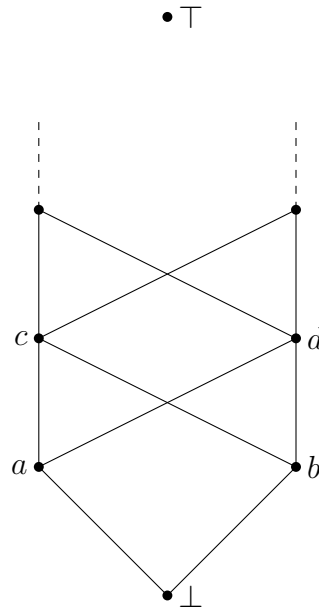


FIGURE 2.6: Plotkin's ladder.

Theorem 2.8.22. [JT98, Theorem 4.2] *Let D be a Lawson-compact, continuous domain with bottom element. Then the probabilistic powerdomain is also Lawson-compact.* □

Naturally, we wonder whether the category \mathbf{LawC}_\perp is cartesian closed. The answer, disappointingly, is no.

Example 2.8.23. Consider the dcpo L in Figure 2.6, which is called Plotkin's ladder. L can be easily verified as a Lawson-compact algebraic domain. The function space of L , however, is not continuous. Suppose that $[L \rightarrow L]$ is continuous. In particular, there exists a directed family $\{f_\alpha \mid \alpha \in A\}$ such that $f_\alpha \ll \text{id}_L$ for every $\alpha \in A$ and $\bigvee_{\alpha \in A} f_\alpha = \text{id}_L$. So we have $\bigvee_{\alpha \in A} f_\alpha(a) = a$. Since a is a compact element in L , we have some $\alpha \in A$ such that $f_\alpha(a) = a$. Similarly, we have some $\beta \in A$ with $f_\beta(b) = b$. We then find an upper bound f of both f_α and f_β in the directed family $\{f_\alpha \mid \alpha \in A\}$. Since $f \leq \text{id}_L$ and f is monotone, this f will fix a and b , hence every element in L . This means that $f = \text{id}_L$ and hence $\text{id}_L \ll \text{id}_L$. However, this is impossible, since

$\{r_x \mid x \in L\}$ (see Example 2.7.3) is a directed family of Scott-continuous functions with its supremum bigger than id_L , while none of them is above id_L . So $[L \rightarrow L]$ is not continuous.

Lawson-compactness can be characterised via the Scott topology. The following theorem which works for quasicontinuous domains will be generalised to arbitrary dcpos in Chapter 4 (see Theorem 4.1.7); we list it here for an ad hoc purpose.

Theorem 2.8.24. *Let L be a quasicontinuous dcpo. Then the following statements are equivalent:*

1. L is compact in the Lawson topology;
2. L is coherent and compact in the Scott topology;
3. L is stably compact in the Scott topology.

Proof. The equivalence between (1) and (2) can be found at [GHK⁺03, Theorem III-5.8], and that (2) is equivalent to (3) is obvious since every quasicontinuous domain is locally compact and sober in the Scott topology. \square

Note that by the previous theorem Lawson-compact continuous domains are precisely stably compact continuous domains, so Theorem 2.8.22 can also be read as:

Theorem 2.8.25. *Let D be a stably compact continuous domain with bottom element. Then the probabilistic powerdomain of D is also stably compact in the Scott topology.*

\square

Skipping out of the scope of dcpos,

Jung was able to generalise the above theorem to a space-like version. However, the topology on the powerdomain may not be the Scott topology again.

Theorem 2.8.26. [Jun04, Theorem 3.2, Theorem 3.3] $\mathcal{P}(X)$ on a stably compact space X is stably compact when equipped with the so-called weak topology. \square

Stably compact spaces (see Definition 2.2.11), which are T_0 -analogues of compact Hausdorff spaces, have a very nice duality with *compact pospaces*. The work dates back to [Nac65], and also can be found in [GHK⁺03, Chapter VI] or in [Jun04]. Another essential benefit of stably compact spaces is that they admit a logical counterpart, the so-called *strong proximity lattices*, along the lines of *Abramsky's Domain Theory in Logical Form* [Abr91]. For this work, see e.g., [JS96]. It was proved in [JKM01] that the category of stably compact spaces with *closed relations* is monoidal closed, however, neither the category of stably compact spaces with closed relations nor that of stably compact spaces with continuous functions is cartesian closed, hence the stably compact framework does not present an intrinsic function type construction.

In 2012, Jean Goubault-Larrecq proposed to study the Jung-Tix problem in the quasicontinuous setting. Analogously to RB-domains, Goubault-Larrecq introduced QRB-domains in a flavour of quasicontinuity, and proved that the powerdomain construction \mathcal{P} can be restricted to the category **QRB** of QRB-domains [GL12, GLJ14].

Definition 2.8.27 (QRB-domain). For a poset L we use $\text{Fin}(L)$ to denote the set $\{\uparrow F \mid F \subseteq_{\text{fin}} L\}$ and equip it with the Smyth order, i.e., the reverse inclusion.

A *quasi-deflation* on a poset L is a continuous map $\varphi : \Sigma L \rightarrow \Sigma \text{Fin}(L)$ such that $x \in \varphi(x)$ for every $x \in L$, and $\{\varphi(x) \mid x \in L\}$, the image of φ , is finite.

A dcpo L is called a *QRB-domain* if it carries a directed family of quasi-deflations $\{\varphi_i \mid i \in I\}$ such that $\uparrow x = \bigcap_{i \in I} \varphi_i(x)$ for each $x \in L$.

The category of all QRB-domains is denoted by **QRB**.

Theorem 2.8.28. [GLJ14, Theorem 6.2] *For every QRB-domain L , $\mathcal{P}(L)$ is again a QRB-domain.* \square

By imitating FS-domains, Li and Xu [LX13] introduced the QFS-domains.

Definition 2.8.29 (QFS-domain). For a poset L , a continuous map $\varphi : \Sigma L \rightarrow \Sigma \text{Fin}(L)$ is called *quasi-finitely separated* on L if there exists a finite set $M \subseteq_{\text{fin}} L$ such that for every $x \in L$ there is $m \in M$ such that $x \in \uparrow m \subseteq \varphi(x)$.

A dcpo L is called a *QFS-domain* if it carries a directed family of quasi-finitely separated functions $\{\varphi_i \mid i \in I\}$ such that $\uparrow x = \bigcap_{i \in I} \varphi_i(x)$ for each $x \in L$.

Surprisingly, it was independently proved by J. Goubault-Larrecq, A. Jung [GLJ14] and J. Lawson, X. Xi [LX14] that QFS-domains and QRB-domains are the same and are equivalent to Lawson-compact quasicontinuous domains. While, as we see from Remark 2.8.21, in the continuous setting whether RB-domains and FS-domains coincide has been open for decades.

Theorem 2.8.30. [GLJ14, Theorem 5.7],[LX14, Theorem 4.8] *For a dcpo L , the following are equivalent.*

1. L is a QRB-domain;
2. L is a QFS-domain;
3. L is a Lawson-compact quasicontinuous domain;
4. L is a stably compact quasicontinuous domain. \square

With the above result, Theorem 2.8.28 can be rewritten as:

Theorem 2.8.31. *Let L be a stably compact quasicontinuous domain. Then the probabilistic powerdomain $\mathcal{P}(L)$ is again stably compact in the Scott topology.* \square

	sober	cartesian closed	closed under \mathcal{P}
SFP _⊥	✓	✓	✗
BF _⊥	✓	✓	✗
AL _⊥	✓	✓	✗
cL _⊥	✓	✓	✗
RB _⊥	✓	✓	??
FS _⊥	✓	✓	<i>The Jung-Tix problem</i>
LawC _⊥	✓	✗	✓
CONT _⊥	✓	✗	✓
QRB _⊥	✓	✗	✓
SCS ^a	✓	✗	✓
DCPO _⊥	✗	✓	✓

^athe category of stably compact spaces and continuous functions

TABLE 2.1: The conflict between cartesian closedness and \mathcal{P} .

Unfortunately, although it is closed under \mathcal{P} , the category **QRB** is not cartesian closed. It is shown in [GL12] that the function space of Plotkin's ladder (L in Figure 2.6) is not a QRB-domain. Since **QRB** is not cartesian closed, a natural question is: what can one say about cartesian closed full subcategories of **QRB**? Or in general, can we find a cartesian closed full subcategory of quasicontinuous domains that is simultaneously closed under \mathcal{P} ? In the next chapter, we give an answer to these questions.

We end this chapter with summarising the results so far in Table 2.1.

Chapter 3

Meet-continuous dcpos

In the last chapter we have seen that the category \mathbf{QRB}_\perp is closed under the probabilistic powerdomain \mathcal{P} . Since \mathbf{QRB}_\perp is not cartesian closed, a natural question arises as: are there any new cartesian closed full subcategories in \mathbf{QRB}_\perp , or in general, in the category of quasicontinuous domains? We give a negative answer to this question in this chapter. This will imply that in the quasicontinuous setting any candidate category for the Jung-Tix Problem will actually lie in the category \mathbf{FS} . The crucial notion that we make use of in answering this question is the so-called *meet-continuity* over dcpos, which was initially introduced by H. Kou, Y. Liu and M. Luo in [KLL03] as a bridge between quasicontinuity and continuity of dcpos.

We organise this chapter as follows. First, we give the definition of meet-continuity of dcpos in Section 3.1, and clarify the interplays among meet-continuity, quasicontinuity and continuity of dcpos. In Section 3.2, we investigate topological properties of meet-continuous dcpos and give their topological characterisation. Section 3.3 displays the central work of this chapter. In this section, we give an order-theoretic

characterisation of meet-continuous dcpos via forbidden substructures. This characterisation enables us to give a necessary condition for dcpos having a core-compact function space. Precisely, we prove that a dcpo with a core-compact function space must be meet-continuous. This work is laid out in Section 3.4. Finally, as an application of this result, we conclude this chapter with the result that any cartesian closed full subcategory of quasicontinuous domains consists of continuous domains entirely. That is to say, similar to the case of the continuous setting, FS-domains and L-domains also form the only two maximal cartesian closed full subcategories of pointed quasicontinuous domains.

3.1 Connecting quasicontinuity to continuity

In this section, we introduce meet-continuous dcpos, and show that any quasicontinuous domain is continuous if and only if it is meet-continuous. For the sake of intuition, we first introduce meet-continuity on dcpos that have binary infima.

Definition 3.1.1. For a dcpo L which has binary infima, that is, $x \wedge y$ exists for all $x, y \in L$, L is said to be *meet-continuous* if for each $a \in L$, the function $a_\wedge : x \mapsto a \wedge x : L \rightarrow L$ is Scott-continuous, i.e., for any directed subset $D \subseteq L$, $a \wedge (\bigvee D) = \bigvee_{d \in D} (a \wedge d)$.

Roughly speaking, a dcpo that has binary infima is called meet-continuous if the meet operation is continuous with respect to the Scott topology. The following proposition provides an easier approach to checking meet-continuity.

Proposition 3.1.2. *For a dcpo L which has binary infima, L is meet-continuous if and only if for any $x \in L$ and directed subset $D \subseteq L$ with $x \leq \bigvee D$, $x \wedge (\bigvee D) = \bigvee_{d \in D} (x \wedge d)$.*

Proof. We prove the “if” direction. Let $a \in L$ and G an arbitrary directed subset of L . We compute $a \wedge (\bigvee G)$. Since $a \wedge (\bigvee G) \leq \bigvee G$, by the assumption, we have

$$\begin{aligned} a \wedge (\bigvee G) &= a \wedge (\bigvee G) \wedge (\bigvee G) \\ &= \bigvee_{g \in G} (a \wedge (\bigvee G) \wedge g) \\ &= \bigvee_{g \in G} (a \wedge g). \end{aligned}$$

Hence L is meet-continuous. □

We now try to define meet-continuity on arbitrary dcpos. Note that for a dcpo L which has binary infima, if we embed L into $\Gamma(L)$, the set of Scott-closed subsets of L , by the function $e : x \mapsto \downarrow x : L \rightarrow \Gamma(L)$, then $x \wedge y$ is mapped to $\downarrow(x \wedge y) = \downarrow x \cap \downarrow y$ for $x, y \in L$. One easily verifies that the embedding function e is Scott-continuous, and moreover it is a topological embedding from ΣL to $\Sigma(\Gamma(L))$. If L is meet-continuous, for any $a \in L$ the composite $e \circ a_\wedge : x \mapsto \downarrow(a \wedge x) = \downarrow a \cap \downarrow x : L \rightarrow \Gamma(L)$ is then Scott-continuous. Conversely, since e is a topological embedding, for any function $f : L \rightarrow L$, $e \circ f$ is Scott-continuous if and only if f is Scott-continuous. So we have that L is meet-continuous if and only if the function $e \circ a_\wedge : x \mapsto \downarrow a \cap \downarrow x : L \rightarrow \Gamma(L)$ is Scott-continuous for all $a \in L$. Moreover, similar to Proposition 3.1.2, we have that $e \circ a_\wedge$ is Scott-continuous for all $a \in L$ iff for all $a \in L$, $e \circ a_\wedge$ preserves the supremum of all directed subsets D with $a \leq \sup D$. Indeed, for any directed subset E ,

$$e \circ a_\wedge(\sup E) = \downarrow a \cap \downarrow \sup E$$

and

$$\begin{aligned}
\sup e \circ a_{\wedge}(E) &= \sup_{e \in E} (\downarrow a \cap \downarrow e) \\
&= \overline{\bigcup_{e \in E} (\downarrow a \cap \downarrow e)} \\
&= \downarrow a \cap \overline{\bigcup_{e \in E} \downarrow e} \\
&= \overline{\downarrow a \cap \downarrow E}.
\end{aligned}$$

One easily sees that $\sup e \circ a_{\wedge}(E) \subseteq e \circ a_{\wedge}(\sup E)$. Reversely, for any $x \in e \circ a_{\wedge}(\sup E)$, that is $x \in \downarrow a \cap \downarrow \sup E$, we have that $x \leq \sup E$. Hence from the assumption we know that $\sup e \circ x_{\wedge}(E) = e \circ x_{\wedge}(\sup E)$, and this means that $\downarrow x = \downarrow x \cap \downarrow \sup E = \overline{\downarrow x \cap \downarrow E}$. Remember that $x \in \downarrow a$, so finally we have

$$x \in \downarrow x = \overline{\downarrow x \cap \downarrow E} \subseteq \overline{\downarrow a \cap \downarrow E} = \sup e \circ a_{\wedge}(E).$$

So $e \circ a_{\wedge}$ preserves the supremum of any directed subset E .

Note that the definition of the function $e \circ a_{\wedge} : x \mapsto \downarrow a \cap \downarrow x : L \rightarrow \Gamma(L)$ no longer involves binary infima. Now we can define meet-continuity of an arbitrary dcpo L by stipulating the Scott-continuity of $e \circ a_{\wedge}$ for all $a \in L$, which, as argued above, is equal to saying that for all $a \in L$, $e \circ a_{\wedge} : x \mapsto \downarrow a \cap \downarrow x : L \rightarrow \Gamma(L)$ preserves the supremum of all directed subsets $D \subseteq L$ with $a \leq \sup D$.

Note that for $a \leq \sup D$:

$$e \circ a_{\wedge}(\sup D) = \downarrow a \cap \downarrow \sup D = \downarrow a,$$

and

$$\sup e \circ a_{\wedge}(D) = \overline{\downarrow a \cap \downarrow D}.$$

Obviously, the equation $e \circ a_{\wedge}(\sup D) = \sup e \circ a_{\wedge}(D)$ holds for directed D with $a \leq \sup D$ if and only if $a \in \overline{\downarrow a \cap \downarrow D}$. This computation enables us to define meet-continuity on arbitrary dcpos as follows.

Definition 3.1.3. [KLL03, Definition 2.2] A dcpo L is called a *meet-continuous dcpo* if for all $x \in L$ and directed subsets $D \subseteq L$ with $x \leq \sup D$, one has $x \in \overline{\downarrow x \cap \downarrow D}$.

Remark 3.1.4. As can be seen from above, the two definitions of meet-continuity in Definitions 3.1.1 and 3.1.3 coincide on dcpos that have binary infima.

The following proposition is straightforward from the above.

Proposition 3.1.5. *Let L be a dcpo. Then L is meet-continuous if and only if for any $a \in L$, the mapping $x \mapsto \downarrow a \cap \downarrow x$ is Scott-continuous from L to $\Gamma(L)$. \square*

Proposition 3.1.6. *Every continuous domain is meet-continuous.*

Proof. Let L be a continuous domain and $x \in L$ with $x \leq \sup D$ for D directed. For any element $a \in \downarrow x$, by definition there exists some $d \in D$ such that $a \leq d$, that is, $a \in \downarrow D$. So we have $a \in \downarrow x \cap \downarrow D$, and hence $\downarrow x \subseteq \downarrow x \cap \downarrow D \subseteq \overline{\downarrow x \cap \downarrow D}$. Since $\downarrow x$ is directed, its supremum x is in the Scott-closed set $\overline{\downarrow x \cap \downarrow D}$. \square

Meet-continuity is preserved by retractions between dcpos.

Proposition 3.1.7. *Let f be a retraction from a meet-continuous dcpo L to a dcpo M . Then M is meet-continuous.*

Proof. Let $g : M \rightarrow L$ be a Scott-continuous function such that $f \circ g = \text{id}_M$. Let D be a directed subset in M and $x \leq \sup D$. Then we have that $g(x) \leq g(\sup D) = \sup g(D)$. Since L is meet-continuous, $g(x) \in \overline{\downarrow g(x) \cap \downarrow g(D)}$. We then have

$$\begin{aligned}
 x &= f(g(x)) \\
 &\in f(\overline{\downarrow g(x) \cap \downarrow g(D)}) \\
 &\subseteq \overline{f(\downarrow g(x) \cap \downarrow g(D))} \\
 &\subseteq \overline{\downarrow f(g(x)) \cap \downarrow f(g(D))} \\
 &= \overline{\downarrow x \cap \downarrow D}.
 \end{aligned}$$

So M is meet-continuous. □

From Example 2.7.3 we have seen that for each element a in a dcpo L , $\downarrow a$ is a retract of L . Together with the previous proposition, it follows that each principal ideal is meet-continuous in a meet-continuous dcpo. Moreover, we see the converse holds as well.

Proposition 3.1.8. *Let L be a dcpo. Then L is meet-continuous if and only if $\downarrow a$ with the induced order is meet-continuous for all $a \in L$.*

Proof. As explained above, we only need to prove the “if” direction. Let $D \subseteq L$ be a directed subset of L , and $x \leq \sup D$ for $x \in L$. Set $\sup D = k$. From the assumption we know that $\downarrow k$ is meet-continuous. Thus, we have $x \in \overline{\downarrow x \cap \downarrow D}_{\downarrow k}$, where $\overline{\downarrow x \cap \downarrow D}_{\downarrow k}$ means that we take the closure of $\downarrow x \cap \downarrow D$ with respect to the Scott topology on $\downarrow k$. However, since $\downarrow k$ is Scott-closed in L , $\overline{\downarrow x \cap \downarrow D}_{\downarrow k} = \overline{\downarrow x \cap \downarrow D}$. So we have $x \in \overline{\downarrow x \cap \downarrow D}$. Hence L is meet-continuous. □

We now proceed to the main result of this section. Before that, we need the following lemma.

Lemma 3.1.9. [GHK⁺03, Lemma III-2.10] *If F is a finite set in a meet-continuous dcpo, then we have*

$$\text{int}_\sigma(\uparrow F) \subseteq \uparrow F.$$

Proof. We assume that $\text{int}_\sigma(\uparrow F) \not\subseteq \bigcup\{\uparrow x \mid x \in F\}$, that is, there exists an element $a_1 \in \text{int}_\sigma(\uparrow F)$ such that $x \not\ll a_1$ for any $x \in F$. We set the finite F as $\{x_1, \dots, x_n\}$. Since $x_1 \not\ll a_1$, by definition there exists a directed set D_1 such that $a_1 \leq \sup D_1$ but $x_1 \notin \downarrow D_1$. Remember that the dcpo is meet-continuous, so $a_1 \leq \sup D_1$ implies $a_1 \in \overline{\downarrow a_1 \cap \downarrow D_1}$. So we have $a_1 \in \text{int}_\sigma(\uparrow F) \cap \overline{\downarrow a_1 \cap \downarrow D_1}$. Since $\text{int}_\sigma(\uparrow F)$ is Scott-open, we know $\downarrow a_1 \cap \downarrow D_1 \cap \text{int}_\sigma(\uparrow F) \neq \emptyset$. Choose $a_2 \in \downarrow a_1 \cap \downarrow D_1 \cap \text{int}_\sigma(\uparrow F)$. Since $a_2 \leq a_1$ and $x_2 \not\ll a_1$, we have $x_2 \not\ll a_2$. Again, by definition we find a directed set D_2 with $a_2 \leq \sup D_2$ but $x_2 \notin \downarrow D_2$. Moreover, from meet-continuity, we have $a_2 \in \overline{\downarrow a_2 \cap \downarrow D_2} \subseteq \overline{\downarrow a_1 \cap \downarrow D_1 \cap \downarrow D_2}$. By induction, we can find directed sets D_i such that $x_i \notin \downarrow D_i$ for $i = 1, \dots, n$ and an element $a_n \in \text{int}_\sigma(\uparrow F)$ with $a_n \in \overline{\downarrow a_1 \cap \bigcap_{i=1}^n \downarrow D_i}$. So we have $a_n \in \text{int}_\sigma(\uparrow F) \cap \overline{\downarrow a_1 \cap \bigcap_{i=1}^n \downarrow D_i}$, and this implies that $\text{int}_\sigma(\uparrow F) \cap \downarrow a_1 \cap \bigcap_{i=1}^n \downarrow D_i \neq \emptyset$, hence $\uparrow F \cap \bigcap_{i=1}^n \downarrow D_i \neq \emptyset$. However, this is impossible, since $x_i \notin \downarrow D_i$ for $i = 1, \dots, n$, one has $x_i \notin \bigcap_{i=1}^n \downarrow D_i$ for any $i = 1, \dots, n$. This contradiction shows that our assumption must have been wrong. \square

Remark 3.1.10. Note that the reverse of the above lemma does not hold. Consider the dcpo in Figure 2.3. For any finite set F in this dcpo, one easily sees that $\text{int}_\sigma(\uparrow F) = \emptyset$. However, the dcpo itself is not meet-continuous.

Corollary 3.1.11. *Let L be a dcpo. If L is quasicontinuous and meet-continuous, then for any finite subset F , one has $\uparrow\uparrow F = \uparrow F$.*

Proof. Obviously, that $\uparrow F \subseteq \uparrow\uparrow F$ holds in arbitrary dcpos. We show the converse containment.

Since L is meet-continuous, from Lemma 3.1.9 we know that $\text{int}_\sigma(\uparrow F) \subseteq \uparrow F$. Remember L is also quasicontinuous, we have $\text{int}_\sigma(\uparrow F) = \uparrow\uparrow F$ from Proposition 2.4.10. Hence $\uparrow F \subseteq \uparrow\uparrow F$. \square

The following theorem, which initially appears as [KLL03, Theorem 2.5], is the central result of this section. It states that meet-continuity is exactly what we need to bridge the gap between quasicontinuity and continuity.

Theorem 3.1.12. *A dcpo L is continuous if and only if it is quasicontinuous and meet-continuous.*

Proof. Obviously, every continuous dcpo is quasicontinuous, and it is meet-continuous from Proposition 3.1.6. Conversely, assume that L is quasicontinuous and meet-continuous. For each $a \in L$, from the quasicontinuity, we know that $\text{fin}(a) = \{F \subseteq_{\text{fin}} L \mid F \ll a\}$ is a directed family of non-empty finite sets and $\bigcap_{F \in \text{fin}(a)} \uparrow F = \uparrow a$. Moreover, from the corollary above we have that $\uparrow\uparrow F = \uparrow F$. This implies that $\downarrow a \cap F \neq \emptyset$ for each $F \in \text{fin}(a)$. Hence the family $\{\downarrow a \cap F \mid F \in \text{fin}(a)\}$ is also a directed set of non-empty finite sets. We now apply Rudin's lemma to find a directed subset $D \subseteq \bigcup \{\downarrow a \cap F \mid F \in \text{fin}(a)\} \subseteq \downarrow a$ such that $D \cap \downarrow a \cap F$ is not empty for all $F \in \text{fin}(a)$. So we have $\sup D \in \bigcap_{F \in \text{fin}(a)} \uparrow F = \uparrow a$. Moreover, since $D \subseteq \downarrow a$, we conclude that $\sup D = a$. Now we easily see that L is continuous with the help of Proposition 2.3.5. \square

Remark 3.1.13. A different proof of Theorem 3.1.12 can be found in [HJ16], where Stone duality was employed as a central technique.

3.2 Topological characterisations of meet-continuous dcpos

In this section, we examine topological properties of meet-continuous dcpos with the Scott topology. To begin with, we have the following characterisation which is a refinement of [GHK⁺03, Proposition III-2.3].

Lemma 3.2.1. *Let L be a dcpo. Then the following statements are equivalent:*

1. L is meet-continuous;
2. for any $x \in L$, if a subset $A \subseteq \downarrow x$ is Scott-open in $\downarrow x$, then $\uparrow A$ is Scott-open in L ;
3. for any Scott-open set U and any $x \in L$, $\uparrow(U \cap \downarrow x)$ is Scott-open.

Proof. (1 \Rightarrow 2) Assume that D is a directed subset of L and $\sup D \in \uparrow A$. So there exists some $y \in A$ such that $y \leq \sup D$. Then it follows from the meet-continuity that $y \in \overline{\downarrow y \cap \downarrow D}$. Since $y \in A \subseteq \downarrow x$, $\overline{\downarrow y \cap \downarrow D}$ is also the Scott closure of $\downarrow y \cap \downarrow D$ in $\downarrow x$. Now the fact that A is a Scott-open neighbourhood of y in $\downarrow x$ implies that $A \cap \downarrow y \cap \downarrow D \neq \emptyset$. It then follows that $\uparrow A \cap D \neq \emptyset$, thus $\uparrow A$ is indeed Scott-open in L .

(2 \Rightarrow 3) This is obvious since for any Scott-open set U in L , the set $U \cap \downarrow x$ is Scott-open in $\downarrow x$.

(3 \Rightarrow 1) Let D be a directed set and $x \leq \sup D$. For any Scott-open set U with $x \in U$, we know from the assumption that $\uparrow(U \cap \downarrow x)$ is a Scott-open set, and $x \in \uparrow(U \cap \downarrow x) \subseteq U$. Since $x \leq \sup D$, we have $\sup D \in \uparrow(U \cap \downarrow x)$, and hence $D \cap \uparrow(U \cap \downarrow x) \neq \emptyset$. So there exist $d \in D, a \in U \cap \downarrow x$ such that $a \leq d$, and this implies that $a \in U \cap \downarrow x \cap \downarrow D$. So any Scott-open set U containing x actually intersects $\downarrow x \cap \downarrow D$, hence $x \in \overline{\downarrow x \cap \downarrow D}$. So L is meet-continuous. \square

Meet-continuity simplifies the verification of compact elements. In meet-continuous dcpos, compactness of an element can be checked by only looking at directed subsets below this element.

Corollary 3.2.2. *Let L be a meet-continuous dcpo and $x, y \in L$. Then*

1. *the element x is compact if and only if for any directed subset D with $x = \sup D$, we have that $x \in D$;*
2. *if, in addition, L has binary infima, $x \ll y$ if and only if $x \in \downarrow D$ for every directed subset D with $\sup D = y$.*

Proof. (1) The interesting part is the “if” direction. Assume $x \in D$ for any directed subset D with $x = \sup D$. This means that $\{x\}$ is Scott-open in $\downarrow x$. By Lemma 3.2.1 it follows that $\uparrow x$ is Scott-open in L , which is equivalent to saying that x is a compact element in L .

(2) We also check the “if” direction. Assume that E is an arbitrary directed subset of L and $y \leq \sup E$. Since L is meet-continuous, we know that $y = y \wedge \sup E = \sup_{e \in E} (y \wedge e)$. Since the set $\{y \wedge e \mid e \in E\}$ is directed, from the assumption we have that $x \in \downarrow \{y \wedge e \mid e \in E\} \subseteq \downarrow E$. So $x \ll y$ holds. \square

Question 3.2.3. We do not know whether the statement in Corollary 3.2.2(2) holds for arbitrary meet-continuous dcpos.

Corollary 3.2.4. *Every minimal element in a meet-continuous dcpo is compact.*

Proof. Let L be a meet-continuous dcpo and $x \in L$ a minimal element. From Corollary 3.2.2, we only need to prove that $x \in D$ for any directed subset D with $\sup D = x$. However, since x is minimal in L , $\{x\}$ is the only directed subset with x as its supremum. \square

Corollary 3.2.5. *Let L be a meet-continuous dcpo and A be a finite subset of L . If every element in A is compact, then every minimal upper bound of A is also compact.*

Proof. Assume that x is a minimal upper bound of A . Since A is a finite set of compact elements, $\bigcap_{a \in A} \uparrow a$ is Scott-open in L . Thus, $\downarrow x \cap \bigcap_{a \in A} \uparrow a = \{x\}$ is Scott-open in $\downarrow x$. Since L is meet-continuous, by Lemma 3.2.1 we have $\uparrow x$ is Scott-open in L , so x is compact. \square

In a meet-continuous dcpo L , for any Scott-open set U and $x \in U$, from Theorem 3.2.1, we know that $\uparrow(U \cap \downarrow x)$ is a Scott-open neighbourhood of x inside U . Moreover, the set $\uparrow(U \cap \downarrow x)$ has the property that it contains at least one lower bound of x and y , for any element $y \in \uparrow(U \cap \downarrow x)$. In general, we give the following definition to capture this property.

Definition 3.2.6.

1. Let X be a topological space. An open neighbourhood V of x is called a *compatible* neighbourhood of x if for any $y \in V$, there exists some $z \in V$, such that $z \leq_s x, y$, where \leq_s is the specialisation preorder.
2. X is called a *locally compatible space* if for every open set U and $x \in U$, there exists a compatible open neighbourhood V of x , such that $x \in V \subseteq U$.

Theorem 3.2.7. *Let L be a dcpo. Then the following statements are equivalent:*

1. L is meet-continuous;
2. ΣL is a locally compatible space.

Proof. (1 \Rightarrow 2) Let U be a Scott-open subset of L and $x \in U$. Then $\uparrow(U \cap \downarrow x)$ is Scott-open from Lemma 3.2.1 and $x \in \uparrow(U \cap \downarrow x) \subseteq U$. It is easy to check that $\uparrow(U \cap \downarrow x)$ is a compatible open neighbourhood of x .

(2 \Rightarrow 1) Let D be a directed subset and $x \leq \sup D$. We prove that $x \in \overline{\downarrow x \cap \downarrow D}$. To this end, let U be any Scott-open set and $x \in U$. Since L is locally compatible in the Scott topology, there exists a compatible open neighbourhood V of x such that $x \in V \subseteq U$. Since $x \leq \sup D$ and V is Scott-open, we have some $d \in D$ such that $d \in V$. Note that V is a compatible neighbourhood of x , we can find in V an element z such that $z \leq d, x$. So $z \in V \cap \downarrow x \cap \downarrow d \subseteq U \cap \downarrow x \cap \downarrow D$. Thus, every Scott-open set U containing x intersects $\downarrow x \cap \downarrow D$. Hence $x \in \overline{\downarrow x \cap \downarrow D}$. \square

Remark 3.2.8. In a dcpo L , every open filter U is a compatible open neighbourhood of x for every $x \in U$. If L is continuous, it follows from Proposition 2.3.9 that ΣL is a locally compatible space, hence L is meet-continuous from the above theorem. This offers a topological viewpoint for Proposition 3.1.6.

One sees that both Lemma 3.2.1 and Theorem 3.2.7 give local characterisations for meet-continuity. The following theorem, which is essentially from [KLL03, Theorem 2.4], in a flavour of Stone duality, gives an equivalent description of meet-continuity via a global viewpoint from the Scott-closed sets.

Theorem 3.2.9. *Let L be a dcpo. Then the following statements are equivalent:*

1. L is a meet-continuous dcpo;
2. The set $\Gamma(L)$ of Scott-closed sets of L , under the inclusion order, is a meet-continuous dcpo.

Proof. (1 \Rightarrow 2) Since $\Gamma(L)$ is a complete lattice under the inclusion order, we prove the equation $A \cap \overline{\bigcup_{i \in I} B_i} = \overline{\bigcup_{i \in I} A \cap B_i}$ holds for arbitrary closed set A and directed collection $\{B_i\}_{i \in I}$ of closed sets. To this end, let $x \in A \cap \overline{\bigcup_{i \in I} B_i}$ and U be a Scott-open set containing x . Since L is meet-continuous, from Lemma 3.2.1 $\uparrow(\downarrow x \cap U)$ is also a Scott-open set containing x . Since $x \in \overline{\bigcup_{i \in I} B_i}$, we know that $\uparrow(\downarrow x \cap U) \cap (\bigcup_{i \in I} B_i)$

is not empty. This means that there exist $b \in B_i$ for some $i \in I$ and $a \in \downarrow x \cap U$ such that $a \leq b$. Then we have $a \in \downarrow x \cap \downarrow b \cap U \subseteq A \cap B_i \cap U$, which implies that U intersects $\bigcup_{i \in I} A \cap B_i$. Hence $x \in \overline{\bigcup_{i \in I} A \cap B_i}$ and $A \cap \overline{\bigcup_{i \in I} B_i} \subseteq \overline{\bigcup_{i \in I} A \cap B_i}$. The converse containment is trivial.

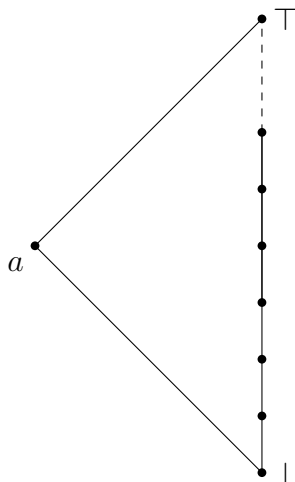
(2 \Rightarrow 1) Let D be a directed subset of L and $x \leq \sup D$. Then one has that $\downarrow x \subseteq \downarrow \sup D = \overline{\bigcup_{d \in D} \downarrow d}$. Since $\Gamma(L)$ is meet-continuous, we know that $x \in \downarrow x = \downarrow x \cap \overline{\bigcup_{d \in D} \downarrow d} = \overline{\bigcup_{d \in D} (\downarrow x \cap \downarrow d)} = \overline{\downarrow x \cap \downarrow D}$. So L is meet-continuous. \square

3.3 A simpler definition and forbidden substructures of meet-continuous dcpos

In the last section, we have given many characterisations for meet-continuous dcpos. In this section, we investigate the reason why a dcpo may fail to be meet-continuous. To start with, we consider a general construction of non-meet-continuous dcpos of a special form.

Recall that a chain C is said to be *well-ordered* if every non-empty subset $A \subseteq C$ has a least element. For any $c \in C$, we use $c + 1$ to denote the least element of $\{x \in C \mid c < x\}$, provided this set is not empty. We say a chain C is *downward well-ordered* if its order dual C^{op} is well-ordered.

Definition 3.3.1. For every well-ordered chain C without a top element, we define the poset $\mathcal{M}(C) = C \cup \{\top, a\}$, where a and \top are not in C and the order on $\mathcal{M}(C)$ is: $x \leq y$ iff $x = y = a$ or $y = \top$ or $x, y \in C$, $x \leq y$ in C . Define $\mathcal{M}(C)_\perp$ to be the lifting of $\mathcal{M}(C)$ by adding a least element \perp . Figure 3.1 shows $\mathcal{M}(\mathbb{N})_\perp$ (where \mathbb{N} is the ordered chain of natural numbers).

FIGURE 3.1: A non-meet-continuous dcpo $\mathcal{M}(\mathbb{N})_{\perp}$.

For every well-ordered chain C without a top element, neither $\mathcal{M}(C)$ nor $\mathcal{M}(C)_{\perp}$ is meet-continuous. We examine this for $\mathcal{M}(C)$. Indeed, from the definition we have $a \leq \top = \sup C$. However, by definition $\downarrow a \cap \downarrow C = \emptyset$, so a is not in the Scott closure of $\downarrow a \cap \downarrow C$.

Actually, we will now show that any non-meet-continuous dcpo contains $\mathcal{M}(C)$ or $\mathcal{M}(C)_{\perp}$ for some well-ordered chain C , as a retract inside it. As can be seen easily, any nontrivial directed subsets in $\mathcal{M}(C)$ and $\mathcal{M}(C)_{\perp}$ are actually chains. Meanwhile, the definition of meet-continuity involves arbitrary directed subsets. In order to eliminate this disharmony, we first try to replace “directed subset” in the definition of meet-continuity by “well-ordered chain”. To this end, we recall some facts about well-ordered chains first.

- Proposition 3.3.2.**
1. *Every well-ordered chain C with a top element \top is an algebraic lattice, and the set of compact elements in C is equal to $\{\perp\} \cup \{c+1 \mid c \in C \setminus \{\top\}\}$, where \perp is the least element in C .*
 2. *Every downward well-ordered chain E is an algebraic domain, with every element in it being compact.*

Proof. (1) Let C be a well-ordered chain with top element \top and least element \perp . By well-orderedness, C has infima for non-empty subsets and since we assume a top element, it follows that C is a complete lattice. Let $c \in C \setminus \{\top\}$. Then we know that $C \setminus \downarrow c$ is a non-empty Scott-open set and $C \setminus \downarrow c = \{x \in C \mid c < x\} = \uparrow(c+1)$. It then follows from Proposition 2.3.2 that $c+1$ is compact. For every non-compact element a one has $a = \sup\{x \in C \mid x < a\} \leq \sup\{x+1 \mid x \in C, x < a\} \leq a$. The first equality follows from the fact that $\{x \in C \mid x < a\} = C \setminus \uparrow a$ is not Scott-closed. So every element in C is the supremum of compact elements below it. Thus, C is an algebraic lattice. If $k \in C \setminus \{\perp\}$ is compact, then the set $\{x \in C \mid x < k\}$ is non-empty and its supremum s is strictly smaller than k . It is now easy to verify that $k = s+1$.

(2) Let D be a non-empty subset of E . Then D has a greatest element since it has a least element in E^{op} , so E is a dcpo. Let $x \in E$, we find $x+1$ in E^{op} . Then in E we easily see that $\uparrow x = E \setminus \downarrow(x+1)$. The element x is compact since $E \setminus \downarrow(x+1)$ is Scott-open. Since every element is compact, E is an algebraic domain. \square

We are interested in well-ordered chains which are subsets of dcpos and compatible with the dcpo structure.

Definition 3.3.3. We say a chain C is *limit embedded* in the dcpo L , if whenever $x = \sup D$ in C , for D a non-empty subset of C , then x is also the supremum of D considered as a subset of L . Equivalently, C is limit embedded in L if the embedding of C into L preserves existing directed suprema.

Proposition 3.3.4. 1. *The image of a well-ordered set under a monotone function is well-ordered.*

2. *Let C be a bounded-complete chain and $f: C \rightarrow L$ a Scott-continuous function into a dcpo L . Then the image of f is limit embedded in L .*

Proof. (1) Assume that f is a monotone function from a well-ordered set C to a poset Q and A is a non-empty subset of $f(C)$. Then $f^{-1}(A)$ is a non-empty subset of C and hence contains a least element a . It is now easy to see that $f(a)$ is the least element of A .

(2) Let D be a non-empty subset in the image of the Scott-continuous function $f: C \rightarrow L$. If $f^{-1}(D)$ is bounded in C then it has a supremum c there. Since $f^{-1}(D)$ is automatically directed we can use Scott-continuity of f to conclude that $f(c) = \sup D$ which shows that the supremum of D lies in the image of f .

If $f^{-1}(D)$ is unbounded in C then because C is a chain this means that $f^{-1}(D)$ is cofinal in C . This implies that D is cofinal in $f(C)$ and therefore if it has a supremum in $f(C)$ then that is the largest element of $f(C)$ and clearly also the supremum of D in L . \square

Definition 3.3.5. A dcpo L is *meet*-continuous* if for any $x \in L$ and any well-ordered chain C limit embedded in L , $x \leq \sup C$ implies that x is in the Scott closure of $\downarrow x \cap \downarrow C$.

Although seemingly weaker than meet-continuity, we now show that meet*-continuity is in fact sufficient to establish the former. To this end we recall Iwamura's decomposition of directed sets, [Iwa44], as presented by Markowsky:

Theorem 3.3.6. [Mar76, Theorem 1] *If D is an infinite directed set, then there exists a transfinite sequence D_α , $\alpha < |D|$, of directed subsets of D having the following properties:*

1. *for each α , if α is finite, so is D_α , while if α is infinite $|D_\alpha| = |\alpha|$ (thus for all α , $|D_\alpha| < |D|$);*
2. *if $\alpha < \beta < |D|$, $D_\alpha \subset D_\beta$;*

3. if $\beta < |D|$ is a limit ordinal, then $D_\beta = \bigcup\{D_\alpha \mid \alpha < \beta\}^1$;

4. $D = \bigcup\{D_\alpha \mid \alpha < |D|\}$. □

Remark 3.3.7. Parts (2) and (3) imply that the mapping $|D| \rightarrow \mathbb{P}D$, $\alpha \mapsto D_\alpha$ preserves existing suprema (where we consider the powerset $\mathbb{P}D$ as a poset ordered by subset inclusion). Thus the assumptions of Proposition 3.3.4(2) are satisfied and we may conclude that the chain $\{D_\alpha \mid \alpha < |D|\}$ is well-ordered and limit embedded in $\mathbb{P}D$.

Theorem 3.3.8. *A dcpo L is meet-continuous if and only if it is meet*-continuous.*

Proof. It is trivial that meet-continuity implies meet*-continuity. Conversely, if L is meet*-continuous, we use transfinite induction on the cardinality of the directed set D in the definition of meet-continuity.

If D is finite, and $x \leq \sup D$, then D has a greatest element and the fact that $x \in \overline{\downarrow x \cap \downarrow D}$ is obvious.

Now suppose D is infinite and that y is in the Scott closure of $\downarrow y \cap \downarrow G$ for any $y \in L$ and any directed set G with cardinality smaller than $|D|$ and $y \leq \sup G$. By Theorem 3.3.6 D is the union of a chain $\mathcal{C} = (D_\alpha)_{\alpha < |D|}$ of directed subsets of D , each of which has smaller cardinality than D . The chain $(\sup D_\alpha)_{\alpha < |D|}$ of elements of L is well-ordered because it is a monotone image of the cardinal $|D|$. It is also limit embedded because of Remark 3.3.7 above and the fact that the supremum operation (from the set of directed subsets of D , ordered by inclusion, to L) is Scott-continuous. Now, if $x \leq \sup D = \sup\{\sup D_\alpha \mid \alpha < |D|\}$, then x is in the Scott closure of $\downarrow x \cap \downarrow\{\sup D_\alpha \mid \alpha < |D|\}$ since L is meet*-continuous. For every Scott-open set U , if $x \in U$, then $U \cap \downarrow x \cap \downarrow\{\sup D_\alpha \mid \alpha < |D|\} \neq \emptyset$, which means that there exists $y \in U$ such that $y \leq x$ and $y \leq \sup D_\alpha$ for some $\alpha < |D|$. By the induction

¹This is not stated in [Mar76, Theorem 1] but appears in the proof.

hypothesis, $y \in \overline{\downarrow y \cap \downarrow D_\alpha}$, whence $U \cap \downarrow y \cap \downarrow D_\alpha \neq \emptyset$ and therefore $U \cap \downarrow x \cap \downarrow D \neq \emptyset$, so x is indeed in the Scott closure of $\downarrow x \cap \downarrow D$. \square

Note that we actually have also proved the following result which is a refinement of Lemma 2.1.9.

Lemma 3.3.9. *A poset L is a dcpo if and only if $\sup C$ exists for any well-ordered chain C limit embedded in L .* \square

Corollary 3.3.10. *For a dcpo L which has binary infima, the following statements are equivalent:*

1. L is meet-continuous (in the sense of Definition 3.1.3);
2. for every $x \in L$ and every directed subset D of L , $x \wedge \sup\{d \mid d \in D\} = \sup\{x \wedge d \mid d \in D\}$;
3. for every $x \in L$ and every well-ordered chain C limit embedded in L , $x \wedge \sup\{c \mid c \in C\} = \sup\{x \wedge c \mid c \in C\}$.

Proof. The equivalence between (1) and (2) was addressed in Remark 3.1.4. The fact that (2) implies (3) is trivial. To prove (3) implies (1), from Theorem 3.3.8 one only needs to show that L is meet*-continuous. So suppose $x \in L$, C is a well-ordered chain limit embedded in L and $x \leq \sup C$. From (3) one has $\sup\{x \wedge c \mid c \in C\} = x \wedge \sup\{c \mid c \in C\} = x$ which shows that x is in the Scott closure of $\{x \wedge c \mid c \in C\}$. To conclude the proof it suffices to note that $\{x \wedge c \mid c \in C\} \subseteq \downarrow x \cap \downarrow C$. \square

We come to the main result of this section: a new characterisation of meet-continuous dcpos:

Theorem 3.3.11. *Let L be a dcpo. Then the following statements are equivalent:*

1. L is not meet-continuous;
2. L has some $\mathcal{M}(C)$ or $\mathcal{M}(C)_\perp$ (as defined in 3.3.1) as a retract, where C is a well-ordered chain without a top element.

Proof. (2 \Rightarrow 1) The dcpo L cannot be meet-continuous, since otherwise its retracts $\mathcal{M}(C)$ or $\mathcal{M}(C)_\perp$ must be meet-continuous by Proposition 3.1.7, which is absurd.

(1 \Rightarrow 2) Let L be a dcpo which is not meet-continuous. By Theorem 3.3.8 this means that it is not meet*-continuous either, and so there exist an element a and a well-ordered chain C' (limit embedded into L) such that $a \leq \sup C'$, but a is not in $\overline{\downarrow a \cap \downarrow C'}$. Obviously, a is strictly smaller than $\sup C'$ since $a = \sup C'$ implies that $a \in \downarrow a = \overline{\downarrow a \cap \downarrow C'}$; moreover for every $c \in C'$, $a \not\leq c$ and therefore C' does not have a top element. Finally, we can make every $c \in C'$ incomparable to a by throwing away those elements of C' that are below a .

We now distinguish two cases:

Case 1, $\downarrow a \cap \downarrow C' \neq \emptyset$: Then there exist $b \in L$ and $c \in C'$ such that $b \in \downarrow a \cap \downarrow c$. Let C be the set $C' \setminus \downarrow c$ and denote the set $C \cup \{\sup C', a, b\}$ by M and order it by the induced order from L . Obviously, $\sup C' = \sup C$ and M is isomorphic to $\mathcal{M}(C)_\perp$.

Define a function f from L to M :

$$f(x) = \begin{cases} b, & x \in \overline{\downarrow a \cap \downarrow C} \\ a, & x \in \downarrow a \setminus \overline{\downarrow a \cap \downarrow C} \\ \bigwedge \{c \mid x \leq c, c \in C\}, & x \in \downarrow C \setminus \downarrow a \\ \sup C, & x \notin \downarrow C \text{ \& } x \notin \downarrow a \end{cases}$$

Since C is well-ordered, f is well-defined. We first prove that f is monotone, so let $x, y \in L$ with $x \leq y$.

In case $y \in \overline{\downarrow a \cap \downarrow C}$, x is in $\overline{\downarrow a \cap \downarrow C}$ since $\overline{\downarrow a \cap \downarrow C}$ is a lower set, and we see that $f(x) = f(y) = b$.

In case $y \in \downarrow a \setminus \overline{\downarrow a \cap \downarrow C}$, since $x \leq y$, x must be in $\downarrow a \setminus \overline{\downarrow a \cap \downarrow C}$ or in $\overline{\downarrow a \cap \downarrow C}$, and in both cases $f(x) \leq f(y)$.

In case $y \notin \downarrow a$, if $\{c \mid y \leq c, c \in C\} = \emptyset$, then $f(y) = \sup C \geq f(x)$. For $\{c \mid y \leq c, c \in C\} \neq \emptyset$, if $x \in \downarrow a$, then $x \in \downarrow a \cap \downarrow y \subseteq \downarrow a \cap \downarrow C$, and $f(x) = b \leq f(y)$. Otherwise $x \notin \downarrow a$ and $f(x) \leq f(y)$ follows immediately from the fact that $\{c \mid y \leq c, c \in C\} \subseteq \{c \mid x \leq c, c \in C\}$.

This covers all possible cases and we have established that f is monotone. Now we show Scott-continuity. To this end let D be a directed set in L .

In case $f(\sup D) = b$, for every $x \in D$, $b \leq f(x) \leq f(\sup D) = b$ since f is monotone, so $f(\sup D) = \sup f(D) = b$.

In case $f(\sup D) = a$, then $\sup D \leq a$ and $\sup D \notin \overline{\downarrow a \cap \downarrow C}$. Since $\overline{\downarrow a \cap \downarrow C}$ is Scott-closed and D is directed, there exists some $x \in D$ such that $x \in \downarrow a \setminus \overline{\downarrow a \cap \downarrow C}$, so $a = f(x) \leq \sup f(D) \leq f(\sup D) = a$.

In case $f(\sup D) \in C$, this means that $\sup D \in \downarrow C \setminus \downarrow a$. Without loss of generality, we assume $d \in \downarrow C \setminus \downarrow a$ for all $d \in D$. It then follows that $f(d) \in C$ and hence $f(\sup D)$ is an upper bound of $f(D)$ in C . Now let k be any element in C above $f(d)$ for all $d \in D$. From the definition of function f , we have that $d \leq f(d) \leq k$ for all $d \in D$, so $\sup D \leq k$. Note that $k \in C$, one has $f(\sup D) \leq k$. This implies that $f(\sup D)$ is the supremum of $f(D)$ in C . Since C is limit embedded in L , $f(\sup D)$ is also the supremum of $f(D)$ in L . Thus, $f(\sup D) = \sup f(D)$.

In case $f(\sup D) = \sup C$ we have that $\sup D \not\leq a$ and for every $c \in C$, $\sup D \not\leq c$. So given $c \in C$ there exist $x_1, x_2 \in D$ such that $x_1 \not\leq a$ and $x_2 \not\leq c$, and by the directness of D there is some $x \in D$ greater than x_1, x_2 . For this element it holds that $x \not\leq a, x \not\leq c$, and hence $f(x) > c$. So for every $c \in C$ there is some $x \in D$ such that $f(x) > c$. This shows that $\sup f(D)$ must equal $\sup C$.

This covers all cases to be considered and we conclude that f is a Scott-continuous function from L to L . Inspecting the definition we see that the elements of M are fixed under f . Hence M is a retract of L .

Case 2, $\downarrow a \cap \downarrow C' = \emptyset$: In this case let $C = C'$ and N be the set $C \cup \{\sup C, a\}$ with its order inherited from L . Obviously, N is isomorphic to $M(C)$. Define a function g from L to N :

$$g(x) = \begin{cases} a, & x \in \downarrow a \\ \bigwedge \{c \mid x \leq c, c \in C\}, & x \in \downarrow C \setminus \downarrow a \\ \sup C, & x \notin \downarrow C \ \& \ x \notin \downarrow a \end{cases}$$

The same deduction as in Case 1 shows that g is a retraction from L to N . \square

3.4 Dcpo with a core-compact function space and cartesian closed full subcategories of quasicontinuous domains

We have given several characterisations of meet-continuity of dcpos in previous sections. Concretely, these characterisations are listed as Proposition 3.1.5, Lemma 3.2.1, Theorem 3.2.7, Theorem 3.2.9, Theorem 3.3.8 and Theorem 3.3.11, to name a few. Especially, Theorem 3.3.11 offers us an order-theoretic viewpoint of meet-continuity. In this section, we make use of this characterisation to derive meet-continuity of a dcpo from the core-compactness of its function space. This result illustrates a deep interplay between topological and order-theoretical properties of dcpos.

Proposition 3.4.1. *For any well-ordered chain C without a top element, neither the function space $[\mathcal{M}(C)_\perp \rightarrow \mathcal{M}(C)_\perp]$ nor $[\mathcal{M}(C) \rightarrow \mathcal{M}(C)]$ is core-compact.*

Proof. Let C be any well-ordered chain without a top element and c_0 its bottom element. We begin with $D_\perp := [\mathcal{M}(C)_\perp \rightarrow \mathcal{M}(C)_\perp]$ and assume for the sake of a contradiction that it is core-compact. First, one easily sees that D_\perp is a complete lattice since $\mathcal{M}(C)_\perp$ is. It then follows from Theorem 2.5.8 that D_\perp is sober, hence locally compact by Theorem 2.5.16.

Consider the function $a \Rightarrow \perp$ that maps the element a to \perp and keeps everything else fixed. It is clearly Scott-continuous and strictly less than the identity function on $\mathcal{M}(C)_\perp$. By local compactness this implies that we should have a compact saturated neighbourhood K in D_\perp such that $\text{id}_{\mathcal{M}(C)_\perp}$ is in the interior of K and $a \Rightarrow \perp \notin K$. Let $K' := \{f \in K \mid f \leq \text{id}_{\mathcal{M}(C)_\perp}\}$. Clearly, K' is not empty (since $\text{id}_{\mathcal{M}(C)_\perp} \in K$), and for each $f \in K'$ we must have $f(a) = a$ as otherwise we would have $f \leq a \Rightarrow \perp$ and $a \Rightarrow \perp \in K$. Now \top can only be mapped to a or to itself by such an f . In the former case, some $c \in C$ would also have to be mapped to a to ensure continuity but this would violate the condition $f \leq \text{id}_{\mathcal{M}(C)_\perp}$; so $f(\top) = \top$ is the only possibility that remains. In other words, each such f continuously maps the infinite well-ordered chain $C \cup \{\perp, \top\}$ into itself, keeping both \perp and \top fixed. Note that K' is compact since it is the intersection of the compact set K and the closed set $\downarrow \text{id}_{\mathcal{M}(C)_\perp}$. We now show that K' does not “isolate” $\text{id}_{\mathcal{M}(C)_\perp}$ against directed suprema from below. (For the argument that follows it may be useful to keep Figure 3.2 in mind.)

Consider the function $g: \mathcal{M}(C)_\perp \rightarrow \mathcal{M}(C)_\perp$ defined by $g(x) = \min\{f(x) \mid f \in K'\}$. As argued above, $\{f(x) \mid f \in K'\} = \{a\}$ when $x = a$, and $\{f(x) \mid f \in K'\} \subseteq C \cup \{\perp, \top\}$ otherwise. It follows that g is well-defined and monotone. We now show that it is in fact Scott-continuous. Note that a is fixed by g . We proceed

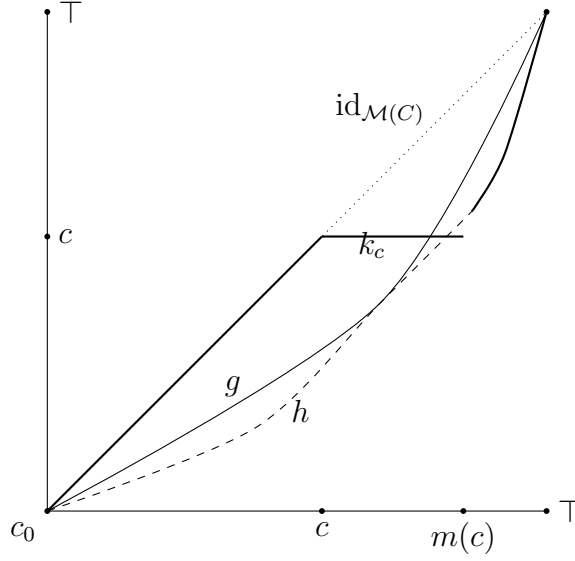


FIGURE 3.2: The situation in the proof of Theorem 3.4.2.

by showing that g continuously maps $C \cup \{\perp, \top\}$ into $C \cup \{\perp, \top\}$. To this end, let $x_0 \in C \cup \{\perp, \top\}$ and choose a basic Scott-open neighbourhood of $g(x_0)$ of the form $\uparrow c$, where $c \in C \cup \{\perp\}$. For every $f \in K'$, $f(x_0) \in \uparrow c$. Then there is an open neighbourhood U_f of x_0 and an open neighbourhood V_f of f such that $f(x) \in \uparrow c$ for all $(f, x) \in V_f \times U_f$. This is because $\mathcal{M}(C)_\perp$ is core-compact, hence $\Sigma([\mathcal{M}(C)_\perp \rightarrow \mathcal{M}(C)_\perp] \times \mathcal{M}(C)_\perp) = \Sigma([\mathcal{M}(C)_\perp \rightarrow \mathcal{M}(C)_\perp]) \times \Sigma(\mathcal{M}(C)_\perp)$ from Theorem 2.5.7, and by Proposition 2.2.7 the evaluation mapping $\text{eval} : (f, x) \mapsto f(x) : [\mathcal{M}(C)_\perp \rightarrow \mathcal{M}(C)_\perp] \times \mathcal{M}(C)_\perp \rightarrow \mathcal{M}(C)_\perp$ is Scott-continuous. By compactness of K' , a finite number of the V_f are covering K' . Let U be the intersection of the corresponding finitely many U_f . Then U is a neighbourhood of x_0 such that $f(x) \in \uparrow c$ for all $f \in K'$ and $x \in U$. Hence, $g(x) \in \uparrow c$ for all $x \in U$. So we have proved that g is Scott-continuous. (This technique is from K. Keimel's remarks [Kei84].)

Now we present a directed set of functions with supremum $\text{id}_{\mathcal{M}(C)_\perp}$ but none of them is in K . This will be a contradiction to the assumption that K is a neighbourhood of $\text{id}_{\mathcal{M}(C)_\perp}$. To this end, consider the Scott-continuous function $h : \mathcal{M}(C) \rightarrow \mathcal{M}(C)$

defined on a and the compact elements of $C \cup \{\top\}$ by

$$h(x) = \begin{cases} \perp, & x = \perp; \\ a, & x = a; \\ c_0, & x = c_0; \\ g(c), & x = c + 1. \end{cases}$$

It follows that g and h agree for limit ordinals and $h(\top) = g(\top) = \top$, but there are also infinitely many inputs where h is strictly less than g ; more precisely, for any $e \in C$, there exists a $d \in C, d \geq e$ such that $h(d + 1) < g(d + 1)$. Indeed, suppose there exists some $e \in C$ such that $h(d + 1) = g(d + 1)$ for all $d \geq e$. Because $h(d + 1) = g(d)$, it then follows that $g(d) = g(d + 1)$ when $d \geq e$. Using transfinite induction and the fact that g is Scott-continuous, we get that $g(x) = g(y)$ for all $x, y \geq e$. In particular, we obtain $g(e) = g(\top) = \top$. However, g is below $\text{id}_{\mathcal{M}(C)_\perp}$ and this implies $\top = g(e) \leq e$, which is not possible since C does not have a top element.

From $h(\top) = \top$ and Scott-continuity we get that for any $c \in C$, there exists $m > c$ such that $h(m) > c$. Define $m(c)$ to be the least element of $\{m \in C \mid h(m) > c\}$. We use this to define a family \mathcal{K} of functions $k_c: \mathcal{M}(C)_\perp \rightarrow \mathcal{M}(C)_\perp$ indexed by the elements of C and defined by

$$k_c(x) = \begin{cases} x, & x \leq c; \\ c, & c < x \leq m(c); \\ h(x), & \text{otherwise.} \end{cases}$$

It is clear that each k_c is Scott-continuous as it is pieced together from Scott-continuous functions on Scott-closed subsets. It is also clear that the supremum

of \mathcal{K} is the identity on $\mathcal{M}(C)_\perp$, but unfortunately, \mathcal{K} may not be directed. This is only a small hindrance, however, because D_\perp is complete and we can enrich \mathcal{K} with all finite suprema. Notice that for any non-empty finite subset $F \subseteq_{\text{fin}} C$, the supremum $\sup_{c \in F} k_c$ is equal to h on $\uparrow \max\{m(c) + 1 \mid c \in F\}$, hence from the last paragraph, $\sup_{c \in F} k_c$ cannot be greater than g . This, then, yields a directed set with supremum $\text{id}_{\mathcal{M}(C)_\perp}$ no member of which is above g and therefore not above an element of K' . Since all of this takes place in $\downarrow \text{id}_{\mathcal{M}(C)_\perp}$, none of them exceeds any of the other members of K either. Thus we have given a counterexample to the claim that K is a Scott neighbourhood of $\text{id}_{\mathcal{M}(C)_\perp}$ and this contradiction shows that the assumption that the function space D_\perp is core-compact must have been wrong.

The argument for $D := [\mathcal{M}(C) \rightarrow \mathcal{M}(C)]$ is similar but easier because any order-preserving function below $\text{id}_{\mathcal{M}(C)}$ must map a to a and \top to \top . Since $\mathcal{M}(C)$ is join-complete, D is then join-complete as well. Theorem 2.5.8 suffices to bridge the gap between core-compactness and local compactness in this case. \square

Theorem 3.4.2. *Given a dcpo L , if the function space $[L \rightarrow L]$ is core-compact, then L must be meet-continuous.*

Proof. Assume that L is not meet-continuous, then by Theorem 3.3.11, L has some $\mathcal{M}(C)$ or $\mathcal{M}(C)_\perp$ as a retract, where C is a well-ordered chain without a top element. So $[\mathcal{M}(C) \rightarrow \mathcal{M}(C)]$ or $[\mathcal{M}(C)_\perp \rightarrow \mathcal{M}(C)_\perp]$ is a retract of $[L \rightarrow L]$ from Proposition 2.7.10. Since core-compactness is preserved by retractions from Proposition 2.7.9, we know that either $[\mathcal{M}(C) \rightarrow \mathcal{M}(C)]$ or $[\mathcal{M}(C)_\perp \rightarrow \mathcal{M}(C)_\perp]$ is core-compact. However, from Proposition 3.4.1 this is impossible. \square

As an application of the previous theorem, we arrive at the main result of this section, which states that any cartesian closed full subcategory of quasicontinuous domains actually consists of continuous ones entirely.

Theorem 3.4.3. *Let \mathbf{C} be a cartesian closed full subcategory of \mathbf{qCONT} , the category of quasicontinuous domains with Scott-continuous functions as morphisms. Then every object in \mathbf{C} is continuous.*

Proof. Assume L is a quasicontinuous domain in \mathbf{C} which is not continuous. By Theorem 3.1.12, L is not meet-continuous, so by the preceding theorem its function space is not core-compact hence not quasicontinuous by Remark 2.5.2, so can't be an object of \mathbf{C} . However, it follows from Lemma 2.2.9 that the function space is the exponential object in any cartesian closed full subcategory of \mathbf{DCPO} . This contradiction shows that L must be continuous. \square

As introduced in Section 2.8.2, the maximal cartesian closed full subcategories of the category \mathbf{CONT} of continuous domains were fully classified by A. Jung in [Jun89, Jun90b]. In the pointed case, they consist of continuous L-domains or FS-domains (see Theorem 2.8.19). The preceding theorem can be read as saying that these are also the maximal cartesian closed full subcategories of \mathbf{qCONT}_\perp . As a consequence of this result, if we consider the Jung-Tix problem in the category \mathbf{qCONT}_\perp , then the possible candidates are actually again \mathbf{RB}_\perp and \mathbf{FS}_\perp , same as in the continuous setting.

Chapter 4

The dichotomy theorem for locally compact sober dcpos

We have seen from the last chapter that any cartesian closed full subcategory of quasicontinuous domains consists of continuous ones entirely, and this result destroys our hope of finding an answer to the Jung-Tix problem in the category \mathbf{qCONT} beyond the scope of \mathbf{FS} .

In this chapter, we aim to extend this result to a even larger category. In particular, since quasicontinuous domains are locally compact sober dcpos (see Theorem 2.4.11), we ask ourself whether any cartesian closed full subcategory of locally compact sober dcpos consists of continuous domains as well. We are not able to answer this question in this chapter, however, we prove a similar but generalised version of Jung's dichotomy lemma (Lemma 2.8.17) in the setting of locally compact sober dcpos. In the pointed case, we show that any cartesian closed full subcategory of locally compact sober dcpos consists of stably compact dcpos entirely or every object in it has complete principal ideals. Since Jones and Plotkin's powerdomain \mathcal{P} ruins

lattice-like structure, this dichotomy result shows that any potential answer to the Jung-Tix problem in the category of locally compact sober dcpos must be some full subcategory consisting of stably compact dcpos entirely.

In order to prove our generalised dichotomy theorem, we first investigate coherence of dcpos in the realm of local compactness and sobriety. Two characterisation theorems of coherent dcpos are given. One of them is valid for all well-filtered dcpos. To be more specific, we show that one can check coherence of well-filtered dcpos by showing the compactness of the intersections of any two principal filters.

Following this work, we prove that those dcpos that are of our interest are actually bicomplete. This is achieved by characterising bicompleteness (Theorem 4.2.11) and proving that with meet-continuity the function space of any non-bicomplete dcpo fails to be core-compact.

Recall that by Theorem 3.4.2 any dcpo with a core-compact function space is actually meet-continuous. We prove our dichotomy theorem by showing that a function space in the category of locally compact sober dcpos is meet-continuous only if either its input dcpo is coherent or its output dcpo has complete principal ideals.

4.1 Coherence and Lawson-compact dcpos

Coherence of topological spaces, as defined in Definition 2.2.11, means that the intersection of any two compact saturated compact subsets is again compact. As can be seen from Theorems 2.8.25, 2.8.26 and 2.8.31 the probabilistic powerdomain \mathcal{P} lives well with coherence in different settings. In this section, we take a closer look at coherence of dcpos. In particular, we prove that a well-filtered dcpo is coherent if and only if the intersection of any two principal filters is compact.

For a topological space X , in Section 2.5 we have denoted the set of all compact saturated sets of X by $\mathcal{Q}(X)$. Furthermore, for the argument that follows, we consider a topology, the so-called *upper Vietoris topology* v , on $\mathcal{Q}(X)$ generated by the sets

$$\square U = \{K \in \mathcal{Q}(X) \mid K \subseteq U\},$$

where U ranges over the open subsets of X . We use $\mathcal{Q}_v(X)$ to denote the resulting topological space. For a dcpo L , we use $\mathcal{Q}_v(L)$ to denote $\mathcal{Q}_v((L, \sigma(L)))$.

Lemma 4.1.1. *Let L be a well-filtered dcpo. Then L is coherent if and only if $\uparrow x \cap \uparrow y$ is compact for all $x, y \in L$.*

Proof. If L is coherent, it is obvious that $\uparrow x \cap \uparrow y$ is compact for all $x, y \in L$, since $\uparrow x, \uparrow y$ are compact saturated.

For the reverse, suppose $\uparrow x \cap \uparrow y$ is compact for all $x, y \in L$. We proceed to prove that for any compact saturated sets $A, B \subseteq L$, $A \cap B$ is compact in L . To this end, fix some element $a \in L$; we define a function f from L to $\mathcal{Q}_v(L)$ by sending an element x to the compact saturated set $\uparrow x \cap \uparrow a$. We claim that f is continuous. Indeed, for every Scott-open subset $U \subseteq L$, $f^{-1}(\square U) = \{x \mid \uparrow x \cap \uparrow a \subseteq U\}$ is obviously an upper set. Let $D \subseteq L$ be a directed subset with $\sup D \in f^{-1}(\square U)$, then one has $\uparrow(\sup D) \cap \uparrow a \subseteq U$, that is, $\bigcap_{d \in D} (\uparrow d \cap \uparrow a) \subseteq U$. Note that L is well-filtered and $\{\uparrow d \cap \uparrow a \mid d \in D\}$ is a filtered family of compact saturated sets by assumption, so we have some $d \in D$ such that $\uparrow d \cap \uparrow a \subseteq U$, i. e., $d \in f^{-1}(\square U)$. Hence f is continuous. Since f is continuous, for the given compact saturated subset $A \subseteq L$, $f(A) = \{\uparrow x \cap \uparrow a \mid x \in A\}$ is a compact subset of $\mathcal{Q}_v(L)$. We now claim that the union of $f(A)$, which is just $A \cap \uparrow a$, is compact in L . Indeed, for any compact subset \mathcal{C} of $\mathcal{Q}_v(L)$, let $\{U_\alpha\}$ be a directed family of open sets of L covering $\bigcup \mathcal{C}$. By compactness, every element K of \mathcal{C} is already covered by one U_α ; in other words, $K \in \square U_\alpha$. It

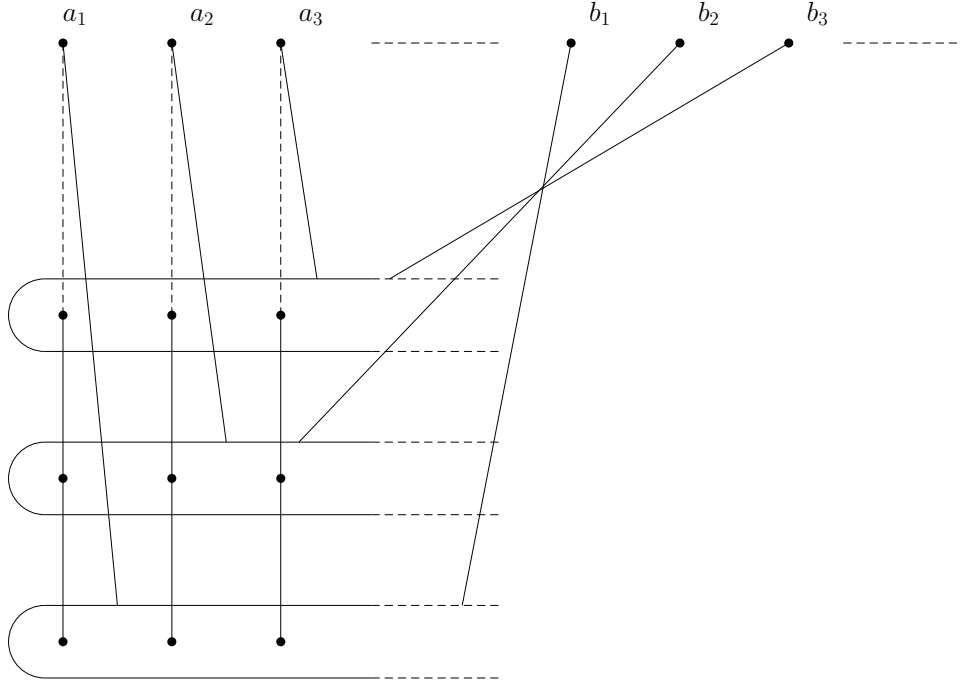


FIGURE 4.1: Figure in Example 4.1.2.

follows that $\{\sqcup U_\alpha\}$ is a directed family covering \mathcal{C} , and now the compactness of \mathcal{C} tells us that $\mathcal{C} \subseteq \sqcup U_\alpha$ for some α . Hence $\bigcup \mathcal{C} \subseteq U_\alpha$ for this α . (This argument is similar to the one employed by Andrea Schalk in [Sch93, Chapter 7] for showing that $\bigcup: \mathcal{Q}_v(\mathcal{Q}_v(X)) \rightarrow \mathcal{Q}_v(X)$ is well-defined.)

Now for such A the above argument enables us to define another function g from L to $\mathcal{Q}_v(L)$ as: $g(x) = \uparrow x \cap A$ for every $x \in L$. A similar deduction shows that g is continuous. So for the compact saturated subset B of L , $g(B)$ is compact in $\mathcal{Q}_v(L)$, and again the union of $g(B)$, which is $A \cap B$, is compact in L . So L is coherent. \square

We would like to remark that this result does not hold in general.

Example 4.1.2. Recall Johnstone’s non-sober dcpo $\mathcal{J} = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$. The partial order on \mathcal{J} is defined by $(m, n) \leq (m', n')$ iff either $m = m'$ and $n \leq n' \leq \infty$ or $n' = \infty$ and $n \leq m'$. Let $B = \{b_i \mid i \in \mathbb{N}\}$, where $b_i, i \in \mathbb{N}$ are pairwise distinct, and let \mathcal{W} be the disjoint union of \mathcal{J} and B . We define a partial order on \mathcal{W} as follows:

- $(m, n) \leq (m', n')$ in \mathcal{W} iff $(m, n) \leq (m', n')$ in \mathcal{J} ;
- $(m, n) \leq b_i$ iff $n \leq i$.

For simplicity, we write (i, ∞) in \mathcal{J} as a_i for all $i \in \mathbb{N}$. The poset \mathcal{W} is depicted in Figure 4.1.

As can be easily seen from the figure that \mathcal{W} is a dcpo. We show that \mathcal{W} is a non-coherent dcpo. Take $K_1 = \{a_i \mid i \text{ is odd}\} \cup B$ and $K_2 = \{a_i \mid i \text{ is even}\} \cup B$. Obviously, K_1, K_2 are saturated. Moreover, both K_1 and K_2 are compact. Indeed, take any Scott-open cover \mathcal{U} of K_1 , that is, $K_1 \subseteq \bigcup \mathcal{U}$. Then some Scott-open set in \mathcal{U} , say U , covers a_1 . Since $a_1 = \bigvee_{j \in \mathbb{N}} (1, j)$, there exists some big enough natural number j_0 such that $(1, j_0) \in U$. Then we know that $a_i, b_i \in U$ for all i larger than j_0 . So U covers all but finite elements of K_1 , and it follows that K_1 is compact. The same argument shows that K_2 is compact. Now we consider the intersection of K_1 and K_2 , which is equal to B , and we see that B is not a compact subset since each $b_i \in B$ is a compact element. The compactness of each b_i , for $i \in \mathbb{N}$, comes from the fact that it is a maximal element and that no infinite chain is below b_i . So \mathcal{W} is not coherent. The intersection of any two principal filters in \mathcal{W} , however, is compact. This is because either the intersection is empty, or a principal filter, or contains some maximal element a_i and is contained in the top layer.

So we obtain a dcpo \mathcal{W} in which the intersection of any two principal filters is compact, whereas \mathcal{W} itself is not coherent. As a by-product, by Lemma 4.1.1, \mathcal{W} cannot be well-filtered.

The following fact about core-compact complete lattices is essentially due to G. Gierz and K.H. Hofmann [GH77]; we collect it here as a corollary to Lemma 4.1.1.

Corollary 4.1.3. *For a complete lattice L , the following statements are equivalent:*

1. L is core-compact, i. e., $\sigma(L)$ is a continuous lattice;
2. $(L, \sigma(L))$ is stably compact.

Proof. The only interesting part is that 1 implies 2. Suppose that L is a complete lattice and $\sigma(L)$ is continuous, then $(L, \sigma(L))$ is a sober space by Theorem 2.5.8 and hence locally compact from Theorem 2.5.16. Since sober spaces are well-filtered from Proposition 2.5.11, and for any $x, y \in L$, $\uparrow x \cap \uparrow y = \uparrow(x \vee y)$ is compact, L is coherent by Lemma 4.1.1. Finally, L is obviously compact in the Scott topology since it has a least element. \square

Lemma 4.1.1 also affords us a better understanding of Lawson-compact dcpos. Since the Lawson topology is finer than the Scott topology, every Lawson-compact dcpo is compact in the Scott topology. We first give a straightforward description of dcpos which are compact in the Scott topology.

Proposition 4.1.4. *Let L be a dcpo. Then $\Sigma(L)$ is compact if and only if L is finitely grounded, that is, there exists a finite subset $F \subseteq L$ such that $L = \uparrow F$.*

Proof. We prove the “only if” direction. Assume that L is compact in the Scott topology. By Proposition 2.2.19 we know that L can be written as $\uparrow M$, where M is the set of all minimal elements of L . Moreover, M must be finite; otherwise, the family $\{M \setminus F \mid F \subseteq_{\text{fin}} M\}$ is a filtered set of non-empty Scott-closed sets with an empty intersection, which contradicts compactness. \square

Recall that the patch topology on L arises by taking all Scott-closed sets together with all compact saturated sets as a subbasis for the closed sets. The following theorem characterising Lawson-compactness is a generalisation of Theorem 2.8.24 which is stated for quasicontinuous domains.

Theorem 4.1.5. *Let L be a well-filtered dcpo. Then the following statements are equivalent:*

1. L is patch-compact, i. e., L is compact in the patch topology;
2. L is Lawson-compact;
3. L is compact and $\uparrow x \cap \uparrow y$ is compact for every $x, y \in L$;
4. L is finitely grounded and $\uparrow x \cap \uparrow y$ is compact for every $x, y \in L$;
5. L is compact and coherent.

Proof. (1 \Rightarrow 2) That 1 implies 2 is true for all dcpos since the patch topology is finer than the Lawson topology.

(2 \Rightarrow 3) It is obvious that L is compact since the Lawson topology is finer than the Scott topology. For every $x, y \in L$, $\uparrow x \cap \uparrow y$ is Lawson closed therefore it is Lawson-compact, thus Scott compact.

(3 \Rightarrow 4) This is from Proposition 4.1.4.

(4 \Rightarrow 5) This is from Proposition 4.1.4 and Lemma 4.1.1.

(5 \Rightarrow 1) We use the Alexander Lemma for compactness. Let $\{K_\alpha\}_{\alpha \in A}$ be a family of compact saturated subsets and $\{C_\beta\}_{\beta \in B}$ a family of closed subsets, and we assume that their union $\{K_\alpha\}_{\alpha \in A} \cup \{C_\beta\}_{\beta \in B}$ has the finite intersection property, that is, $\bigcap_{\alpha \in F \subseteq A} K_\alpha \cap \bigcap_{\beta \in G \subseteq B} C_\beta \neq \emptyset$ for all finite F and G . We now prove that $\bigcap_{\alpha \in A} K_\alpha \cap \bigcap_{\beta \in B} C_\beta \neq \emptyset$. To this end, we enrich $\{K_\alpha\}_{\alpha \in A}$ and $\{C_\beta\}_{\beta \in B}$ with all finite intersections of their members, respectively, and denote the resulting families by \mathcal{K} and \mathcal{C} , respectively. We can also assume that \mathcal{K} and \mathcal{C} are not empty by adding L to each of them. Now we know that \mathcal{C} is a filtered family of closed sets. Since L is coherent, \mathcal{K} is a non-empty filtered family of compact saturated sets. Moreover, by assumption, $K \cap C \neq \emptyset$ for all $K \in \mathcal{K}$ and $C \in \mathcal{C}$. By Proposition 2.5.14, we know

that $\bigcap \mathcal{K} \cap \bigcap \mathcal{C} \neq \emptyset$. Hence the intersection $\bigcap_{\alpha \in A} K_\alpha \cap \bigcap_{\beta \in B} C_\beta$, which is equal to $\bigcap \mathcal{K} \cap \bigcap \mathcal{C}$, is not empty. \square

Recently, X. Xi and J. Lawson [XL17] proved that all Lawson-compact dcpos are actually well-filtered in the Scott topology.

Theorem 4.1.6. [XL17, Corollary 3.2] *Let L be a complete lattice or a bounded-complete dcpo. Then L equipped with the Scott topology is well-filtered. More generally, any dcpo P with compact Lawson topology has a well-filtered Scott topology.* \square

With the aid of the previous theorem, Theorem 4.1.5 holds without assuming that L is well-filtered.

Theorem 4.1.7. [XL17, Theorem 4.2] *Let L be a dcpo. Then the following statements are equivalent:*

1. L is patch-compact, i. e., L is compact in the patch topology;
2. L is Lawson-compact;
3. L is well-filtered, compact and $\uparrow x \cap \uparrow y$ is compact for every $x, y \in L$;
4. L is well-filtered, finitely grounded and $\uparrow x \cap \uparrow y$ is compact for every $x, y \in L$;
5. L is well-filtered, compact and coherent. \square

Corollary 4.1.8. *Every bounded-complete lattice L is coherent.*

Proof. By Theorem 4.1.6, we know that L is well-filtered. For every $x, y \in L$ the intersection of $\uparrow x$ and $\uparrow y$, which is $\uparrow(x \vee y)$ or empty, is always compact, so the statement follows from Lemma 4.1.1. \square

For locally compact sober spaces, we have the following characterisation for coherence via the notion of *relative compactness*.

Definition 4.1.9. Let X be a topological space. Given subsets A and B with $A \subseteq B$, A is said to be *relatively compact* in B if every open cover of B admits a finite subcover of A .

For two open subsets U, V of X , one easily sees that U is relatively compact in V if and only if $U \ll V$ in $\mathcal{O}(X)$.

Lemma 4.1.10. *Let X be a locally compact sober space. Then the following statements are equivalent:*

1. L is coherent;
2. for any compact saturated subsets A, B and Scott-open sets U, V with $A \subseteq U, B \subseteq V$, $A \cap B$ is relatively compact in $U \cap V$;
3. for $U, V, W \in \mathcal{O}(X)$, if $U \ll V$ and $U \ll W$, then $U \ll V \cap W$.

The equivalence between 1 and 3 can also be found as Lemma 5.2.24 and Lemma 8.3.32 in [GL13].

Proof. (1 \Rightarrow 2) This is obvious since by coherence $A \cap B$ is compact.

(2 \Rightarrow 3) Let U, V, W be open subsets of X and $U \ll V, W$. Since L is locally compact, there exist compact saturated subsets C and K such that $U \subseteq C \subseteq V$ and $U \subseteq K \subseteq W$. Hence we have that $U \subseteq C \cap K \subseteq V \cap W$ and that $C \cap K$ is relatively compact in $V \cap W$. Then it is easy to see that U is also relatively compact in $V \cap W$ and hence $U \ll V \cap W$.

(3 \Rightarrow 1) For compact saturated subsets A and B , consider the family $\mathcal{F} := \{U \cap V \mid$

$A \subseteq U$ & $B \subseteq V$, $U, V \in \mathcal{O}(X)$ of open sets. We claim that \mathcal{F} is a Scott-open filter of $\mathcal{O}(X)$. Obviously, \mathcal{F} is a filter of open sets. Let $\{U_i \mid i \in I\}$ be a directed family of open sets of X and $\bigcup_{i \in I} U_i \in \mathcal{F}$. This means that we have some open sets U, V with $A \subseteq U, B \subseteq V$, and $U \cap V \subseteq \bigcup_{i \in I} U_i$. Since X is locally compact, $\mathcal{O}(X)$ is a continuous lattice. We can find open sets U', V' such that $A \subseteq U' \ll U$ and $B \subseteq V' \ll V$. From the assumption, we know that $U' \cap V' \ll U \cap V$. Remember that $U \cap V \subseteq \bigcup_{i \in I} U_i$, so we have some $i \in I$ such that $U' \cap V' \subseteq U_i$. This means that $U_i \in \mathcal{F}$ and hence \mathcal{F} is a Scott-open filter of opens. Now the sobriety of X and the Hofmann-Mislove Theorem tell us that the intersection of \mathcal{F} , which equals $A \cap B$, is compact. \square

In continuous domains, property M is introduced as a characterisation of Lawson-compactness. A continuous domain L is said to satisfy *property M* if for any $x_1, y_1, x_2, y_2 \in L$ with $y_1 \ll x_1$ and $y_2 \ll x_2$, there exists a finite set $F \subseteq L$ such that $\uparrow x_1 \cap \uparrow x_2 \subseteq \uparrow F \subseteq \uparrow y_1 \cap \uparrow y_2$. Such a property is useful in proving that certain domains are Lawson-compact, for example, FS-domains, bi-finite domains (see [GHK⁺03, Proposition III-5.14]), and also useful in constructing functions on domains at the element-level (e.g., [Jun89, Lemma 4.23]). The following observation is similar to Lemma 4.1.10 and we record it here as a rephrasing of property M.

Proposition 4.1.11. *Let L be a continuous domain. Then the following statements are equivalent:*

1. L satisfies property M;
2. for any $x, y \in L$ and any Scott-open sets U, V with $x \in U, y \in V$, $\uparrow x \cap \uparrow y$ is relatively compact in $U \cap V$;
3. for any $x, y \in L$, $\uparrow x \cap \uparrow y$ is compact;

4. L is coherent.

The equivalence between 1 and 4 is Jung's order-theoretic characterisation of coherence and it can be found at [Jun89, Lemma 4.18] or at [GL13, Exercise 8.3.33].

Proof. (1 \Rightarrow 2) Assume that L satisfies property M. Since L is continuous, we could find $x' \in U$ and $y' \in V$ such that $x' \ll x$ and $y' \ll y$. By property M, there exists a finite set F such that $\uparrow x \cap \uparrow y \subseteq \uparrow F \subseteq \uparrow x' \cap \uparrow y' \subseteq U \cap V$. Then obviously, $\uparrow x \cap \uparrow y$ is relatively compact in $U \cap V$.

(2 \Rightarrow 3) This follows from a similar argument as in Lemma 4.1.10 by using the fact that every continuous domain is locally supercompact and sober (Theorem 2.3.14).

(3 \Rightarrow 4) This is from Lemma 4.1.1 since every continuous domain is sober, hence well-filtered.

(4 \Rightarrow 1) Given $x_1, y_1, x_2, y_2 \in L$ with $y_1 \ll x_1$ and $y_2 \ll x_2$. Since L is continuous, $\uparrow y_1$ and $\uparrow y_2$ are Scott-open; and by Proposition 2.3.8 $\uparrow y_1 \cap \uparrow y_2 = \bigcup \{ \uparrow x \mid \uparrow x \subseteq \uparrow y_1 \cap \uparrow y_2 \}$. From the assumption we have that $\uparrow x_1 \cap \uparrow x_2$ is a compact subset in $\uparrow y_1 \cap \uparrow y_2$, hence there exists some finite subset F of $\uparrow y_1 \cap \uparrow y_2$ such that $\uparrow x_1 \cap \uparrow x_2 \subseteq \uparrow F$. Hence $\uparrow x_1 \cap \uparrow x_2 \subseteq \uparrow F \subseteq \uparrow F \subseteq \uparrow y_1 \cap \uparrow y_2 \subseteq \uparrow y_1 \cap \uparrow y_2$. \square

4.2 Bicompleteness of sober dcpos

Recall that a dcpo L is said to be bicomplete if its order dual L^{op} is also a dcpo. As can be seen easily, an alternative description of L being bicomplete is that every filtered subset in L has an infimum.

Proposition 4.2.1. *Every bicomplete dcpo is grounded. Moreover, if L is a bicomplete dcpo, then for every subset $A \subseteq L$ and x an upper bound of A , i.e., $x \in A^u$, there exists a minimal upper bound of A such that $m \leq x$.*

Proof. This is just Proposition 2.2.18 on L^{op} by noticing that $A^u = \bigcap_{x \in A} \uparrow x$ is Scott-closed in L^{op} . \square

Definition 4.2.2. Let L be a poset. We say that L has *property m* if for each non-empty finite set $A \subseteq L$, for any $x \in A^u$ there is a minimal upper bound y of A which lies below x .

Corollary 4.2.3. *Every bicomplete dcpo has property m .*

Proof. This is obvious from Proposition 4.2.1. \square

Proposition 4.2.4. *Every coherent dcpo has property m .*

Proof. Let L be a coherent dcpo and A be a non-empty finite subset of L with an upper bound. Then the set A^u of upper bounds of A is compact since $A^u = \bigcap_{x \in A} \uparrow x$ and L is coherent. Then from Proposition 2.2.19 we know that L has property m . \square

Proposition 4.2.5. *Every compact and coherent dcpo is bicomplete.*

Proof. Let L be a compact and coherent dcpo and G a filtered subset of it. Since L is compact, we know that the set $A = \bigcap_{x \in G} \downarrow x$ of lower bounds of G is not empty, and obviously A is Scott-closed. If A does not have a greatest element, then there exist at least two maximal elements in A , say a and b . By coherence of L we know that $\uparrow a \cap \uparrow b$ is compact and $G \subseteq \uparrow a \cap \uparrow b$. Since $\uparrow a \cap \uparrow b \cap \downarrow x \neq \emptyset$ for every $x \in G$, by compactness of $\uparrow a \cap \uparrow b$ we then have that $A \cap \uparrow a \cap \uparrow b$ is not empty. However, this is absurd. So A has a greatest element, and this element is the infimum of G . \square

Corollary 4.2.6. *Every coherent pointed dcpo is bicomplete.* \square

Bicompleteness is preserved by retractions.

Proposition 4.2.7. *Let L be a bicomplete dcpo and M is a retract of L . Then M is bicomplete.*

Proof. Let f and g be the corresponding retraction and section. Assume that C is a filtered set in M , then $g(C)$ is a filtered subset of L . Since L is bicomplete, $\inf g(C)$ exists. One easily sees that $f(\inf g(C))$ is the infimum of C in M . \square

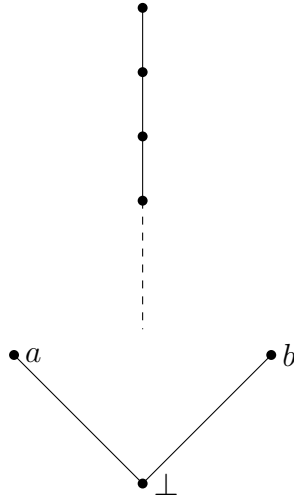
In the sequel, bicompleteness will be considered over sober dcpos. In particular, we give our characterisation theorem of bicompleteness for meet-continuous sober dcpos which, from Theorem 3.4.2, are in our interest.

Example 4.2.8. The following examples are not bicomplete.

- Let \mathbb{N} be the poset of natural numbers with the usual numbering order and \mathbb{N}^{op} its dual poset. Then \mathbb{N}^{op} is a non-bicomplete dcpo.
- Let $\mathcal{K}(\mathbb{N}^{op})$ be the dcpo obtained by adding two incomparable elements a, b below the chain \mathbb{N}^{op} . Then $\mathcal{K}(\mathbb{N}^{op})$ is a grounded dcpo which is not bicomplete.
- Let $\mathcal{K}(\mathbb{N}^{op})_{\perp}$ be the dcpo obtained by adding a least element \perp to $\mathcal{K}(\mathbb{N}^{op})$. Then $\mathcal{K}(\mathbb{N}^{op})_{\perp}$ is a pointed dcpo which is not bicomplete.

Inspired by these three examples, we give more general constructions of non-bicomplete dcpos.

Definition 4.2.9. For every downward well-ordered chain C without a bottom element, we define the poset $\mathcal{K}(C) = C \cup \{a, b\}$, where a and b are not in C and the order on $\mathcal{K}(C)$ is: $x \leq y$ iff $x = y = a$; $x = y = b$; $x \in \{a, b\}, y \in C$; or $x, y \in C$, $x \leq y$ in C . Define $\mathcal{K}(C)_{\perp}$ to be the lifting of $\mathcal{K}(C)$ by adding a least element \perp .

FIGURE 4.2: $\mathcal{K}(\mathbb{N}^{op})_{\perp}$ in Example 4.2.8.

The following proposition is obvious.

Proposition 4.2.10. *Let C be a downward well-ordered chain without a bottom element. Then C , $\mathcal{K}(C)$ and $\mathcal{K}(C)_{\perp}$ are algebraic domains, and none of them is bicomplete. \square*

We now give our characterisation theorem of bicompleteness.

Lemma 4.2.11. *Let L be a sober dcpo. If every minimal element in L is compact, then the following statements are equivalent:*

1. L is not bicomplete;
2. L has some C , $\mathcal{K}(C)$ or $\mathcal{K}(C)_{\perp}$ as a retract, where C is a downward well-ordered chain without a bottom element.

Proof. That 2 implies 1 follows from Proposition 4.2.7.

The interesting part is that 1 implies 2. Assume that L is not bicomplete, then we can find some chain C in L such that C does not have an infimum in L . Moreover,

from Lemma 3.3.9 this C can be chosen in such a way that it is downward well-ordered in the induced order from L .

Let A be the set of lower bounds of C in L . Then $A = \bigcap_{x \in C} \downarrow x$ is obviously a Scott-closed subset of L . Equip $C \cup A$ with the induced order from L and define a function $r : L \rightarrow C \cup A$ by

$$r(x) = \begin{cases} x, & x \in A; \\ \bigwedge \{c \in C \mid x \leq c\}, & \text{otherwise.} \end{cases}$$

Since C is downward well-ordered, the function r is well-defined and monotone. Moreover, r is Scott-continuous. Indeed, for any directed subset D of L , $\sup r(D) \leq r(\sup D)$ since r is monotone. For the reverse, the non-trivial case is that $\sup r(D) \in C$. Without loss of generality, we assume that $r(D) \subseteq C$. Since C is downward well-ordered, $r(D)$ has a greatest element, say $r(d)$ for some $d \in D$. Then $r(d) = r(x) \geq x$ for all $x \in \uparrow d \cap D$. Hence $\sup D \leq r(d)$; therefore, $r(\sup D) \leq r(d)$ since $r(d) \in C$. So we have proved that r is Scott-continuous. It then follows that r is a retraction from L to $C \cup A$ with the corresponding section being the inclusion map of $C \cup A$ into L .

We now distinguish two cases:

Case 1: A is empty. Then clearly C is a retract of L .

Case 2: A is not empty. In this case we can assume that every element of A is above some minimal element in A since otherwise, we can find some descending chain in A without any lower bounds. Since every chain has a well-ordered cofinal subset, this enables us to find in A a downward well-ordered chain without any lower bounds as well, and this will lead us to Case 1.

Since A , the set of lower bounds of C , is a Scott-closed non-empty subset, and since C does not have an infimum, we have that A has at least two maximal elements, say

a and b . We further consider two subcases:

Subcase 2.1: Every minimal element of A is below exactly one maximal element in A .

In this subcase, we define a function g on $C \cup A$ as:

$$g(x) = \begin{cases} x, & x \in C; \\ a, & x \in \downarrow a; \\ b, & \text{otherwise.} \end{cases}$$

It is easy to check that g is a Scott-continuous retraction on $C \cup A$ with image $\{a, b\} \cup C$, which is isomorphic to $\mathcal{K}(C)$. Then $g \circ r$ is the desired Scott-continuous retraction showing that $\{a, b\} \cup C$ is a retract of L .

Subcase 2.2: There exists some minimal element $m \in A$ such that more than one maximal element of A is above it.

Then we consider set $\overline{\uparrow m \cap A}$, the Scott closure of $\uparrow m \cap A$ in L . Because $\overline{\uparrow m \cap A}$ has more than one maximal element, it is not irreducible in the sober dcpo L . This implies that we have two Scott-open subsets U, V of L such that they intersect with $\overline{\uparrow m \cap A}$ respectively, but $U \cap V \cap \overline{\uparrow m \cap A} = \emptyset$ ¹. Since U, V are Scott-open and they intersect with $\overline{\uparrow m \cap A}$, then they also intersect with $\uparrow m \cap A$. Fix some points $c \in U \cap \uparrow m \cap A$ and $d \in V \cap \uparrow m \cap A$ (see Figure 4.3). Now we can see that $\{C, c, d, m\}$ is a copy of $\mathcal{K}(C)_\perp$ inside L . Moreover, we show it is a retract of $C \cup A$. Indeed, consider the function h defined on $C \cup A$ as follows:

$$h(x) = \begin{cases} x, & x \in C; \\ c, & x \in U \cap \uparrow m \cap A; \\ d, & x \in V \cap \uparrow m \cap A; \\ m, & \text{otherwise.} \end{cases}$$

¹Note that U and V may intersect in A .

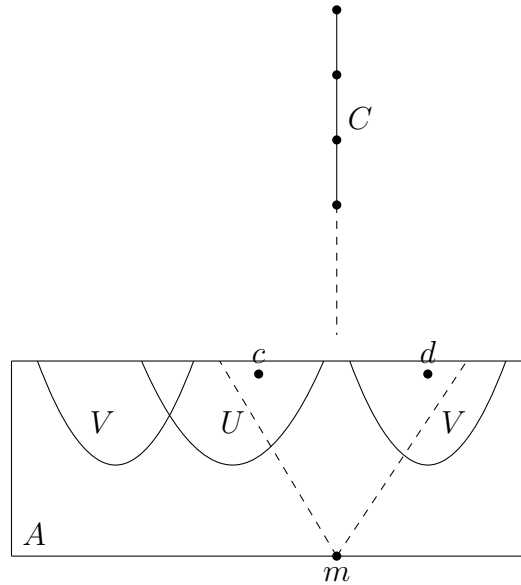


FIGURE 4.3: The situation in the proof of Lemma 4.2.11.

Since in our assumption, the minimal element m is compact, $\uparrow m$ is Scott-open. Now to check that h is a retraction is just routine, and in this case, L has $\mathcal{K}(C)_\perp$ as a retract witnessed by $h \circ r$. \square

The following theorems are straightforward consequences of the previous lemma.

Theorem 4.2.12. *Let L be a meet-continuous sober dcpo. If L is not bicomplete, then L has C , $\mathcal{K}(C)$ or $\mathcal{K}(C)_\perp$ as a Scott-continuous retract, where C is a downward well-ordered chain without a bottom element.*

Proof. From Corollary 3.2.4, in a meet-continuous dcpo, every minimal element (if they exist) is compact. Then the statement follows from the previous lemma. \square

Theorem 4.2.13. *Let L be a pointed sober dcpo. If L is not bicomplete, then L has $\mathcal{K}(C)_\perp$ as a Scott-continuous retract, where C is a downward well-ordered chain without a bottom element.*

Proof. This is Subcase 2.2 in the proof of Lemma 4.2.11. \square

4.3 Sober dcpos with a core-compact function space

Proposition 4.3.1. *For any downward well-ordered chain C without a bottom element, none of the function spaces $[C \rightarrow C]$, $[\mathcal{K}(C) \rightarrow \mathcal{K}(C)]$, or $[\mathcal{K}(C)_\perp \rightarrow \mathcal{K}(C)_\perp]$ is core-compact.*

Proof. We first show that $[C \rightarrow C]$ is not core-compact. Since this function space is join-complete, by Corollary 2.5.17, we only need to show that it is not locally compact. More precisely, we prove that the identity map id_C does not have any compact neighbourhoods. By way of contradiction, suppose that W is a compact neighbourhood of id_C . Then for each $x \in C$, the set $\{g(x) \mid g \in W\}$ is compact since the evaluation function $\text{eval}: [C \rightarrow C] \times C \rightarrow C$ is continuous and $\{g(x) \mid g \in W\}$ is the continuous image of the set $W \times \{x\}$ which is compact in $\Sigma([C \rightarrow C] \times C)$ by Theorem 2.5.7 and Proposition 3.3.2. Moreover, $\{g(x) \mid g \in W\}$ has a least element since it is compact and C is a chain.

Consider the function $f: C \rightarrow C$ defined by $f(x) = \min\{g(x) \mid g \in W\}$. As argued above, f is well-defined. Obviously, f is monotone, and Scott-continuous since every element in C is compact. Since W is a Scott neighbourhood of id_C and $W \subseteq \uparrow f$, we have $f \ll \text{id}_C$ from Proposition 2.3.7.

We proceed by showing that f cannot be way-below id_C . Consider the successor function τ on C , defined by $\tau(c) = c + 1$. Remember that C is downward well-ordered, so $c + 1 < c$. The functions

$$g_c(x) = \begin{cases} \tau \circ f(x), & x \leq c; \\ x, & \text{otherwise.} \end{cases}$$

approximate id_C but none of them dominates f .

This contradiction shows that W is not a Scott neighbourhood of id_C . So $[C \rightarrow C]$ is not core-compact.

Note that all of the above also holds in $\downarrow\text{id}_C$, so $\downarrow\text{id}_C$ as a dcpo is not core-compact. Hence $[\mathcal{K}(C) \rightarrow \mathcal{K}(C)]$ is not core-compact since its retract $\downarrow\text{id}_{\mathcal{K}(C)}$, which is isomorphic to $\downarrow\text{id}_C$, is not core-compact.

Finally, we prove $[\mathcal{K}(C)_\perp \rightarrow \mathcal{K}(C)_\perp]$ is not core-compact by showing that its principal ideal $\downarrow\text{id}_{\mathcal{K}(C)_\perp}$ is not core-compact. To this end, consider the set $A := \{f \in \downarrow\text{id}_{\mathcal{K}(C)_\perp} \mid f(a) = a \ \& \ f(b) = b\}$. One easily sees that A is Scott-open in $\downarrow\text{id}_{\mathcal{K}(C)_\perp}$ and A is isomorphic to $\downarrow\text{id}_C$. So A is not core-compact. Hence $\downarrow\text{id}_{\mathcal{K}(C)_\perp}$ is not core-compact, since in a core-compact dcpo every Scott-open set is a core-compact dcpo in the induced order by Corollary 2.7.7. \square

We arrive at our main result in this section.

Theorem 4.3.2. *Let L be a sober dcpo with a core-compact function space $[L \rightarrow L]$. Then L is bicomplete.*

Proof. Suppose that L is not bicomplete. Since $[L \rightarrow L]$ is core-compact, L must be meet-continuous by Lemma 3.4.2. By Theorem 4.2.12, L has C , $\mathcal{K}(C)$ or $\mathcal{K}(C)_\perp$ as a retract, where C is some downward well-ordered chain without a bottom element. Hence either $[C \rightarrow C]$, $[\mathcal{K}(C) \rightarrow \mathcal{K}(C)]$ or $[\mathcal{K}(C)_\perp \rightarrow \mathcal{K}(C)_\perp]$ is a retract of $[L \rightarrow L]$. This implies that one of these function spaces must be core-compact. However, this cannot be true as we saw in Proposition 4.3.1 that none of them is core-compact. \square

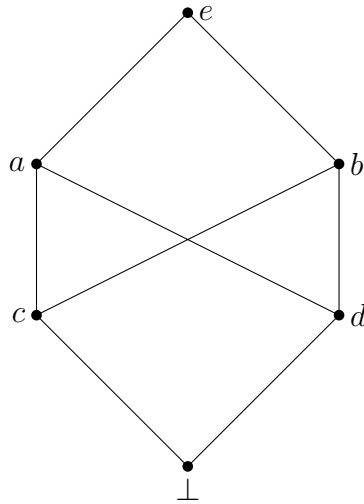


FIGURE 4.4: A typical non-L-dcpo $\mathbf{X}_{\perp}^{\top}$.

4.4 The dichotomy theorem

In this section, we give the main result of this chapter: a dichotomy theorem about locating cartesian closed full subcategories of locally compact sober dcpos. The result states that any cartesian closed full subcategory of pointed locally compact sober dcpos consists entirely of coherent dcpos or of L-dcpo. In Section 4.1 we have given equivalent descriptions of coherence (Lemma 4.1.10) in the setting of locally compact sober dcpos. To obtain our dichotomy result, we need more information of L-dcpo. Recall that an L-dcpo is a dcpo within which every principal ideal is a complete lattice in the induced order. A typical non-L-dcpo $\mathbf{X}_{\perp}^{\top}$ is depicted in Figure 4.4. From Theorem 4.3.2 we know that our dcpos of interest are actually bicomplete; therefore we start with an equivalent description of bicomplete L-dcpo.

Theorem 4.4.1. *Let L be a pointed sober dcpo which is bicomplete. Then L is not an L-dcpo if and only if L has $\mathbf{X}_{\perp}^{\top}$ (defined in Figure 4.4) as a retract.*

Proof. The “if” direction is obvious. We prove the non-trivial part.

Let L be a bicomplete sober dcpo with a least element \perp . If L is not an L-dcpo,

then we have some $e \in L$ such that $\downarrow e$ is not complete. Since L is bicomplete, filtered infima exist in $\downarrow e$. This means that there exist elements $a, b \in \downarrow e$ such that a, b have no infimum in $\downarrow e$, since otherwise $\downarrow e$ would be complete. Consider the closed set $\downarrow a \cap \downarrow b$. It is not empty since $\perp \in \downarrow a \cap \downarrow b$, then it has at least two maximal elements. The sobriety of L now tells us that $\downarrow a \cap \downarrow b$ is not irreducible, so there exist two Scott-open sets U and V intersecting $\downarrow a \cap \downarrow b$, respectively, with $U \cap V \cap \downarrow a \cap \downarrow b = \emptyset$. Choose some element c in $U \cap \downarrow a \cap \downarrow b$ and some d in $V \cap \downarrow a \cap \downarrow b$, respectively. We define a function $r: L \rightarrow L$ as follows:

$$r(x) = \begin{cases} e, & x \notin \downarrow a \cup \downarrow b; \\ a, & x \in \downarrow a \setminus \downarrow b; \\ b, & x \in \downarrow b \setminus \downarrow a; \\ c, & x \in U \cap \downarrow a \cap \downarrow b; \\ d, & x \in V \cap \downarrow a \cap \downarrow b; \\ \perp, & \text{otherwise.} \end{cases}$$

It is clear that r is a retraction on L with image $\{a, b, c, d, e, \perp\}$ which is a copy of $\mathbf{X}_{\perp}^{\top}$ inside L . □

We now come to a theorem which generalises Lemma 2.8.17.

Theorem 4.4.2. *Let D be a locally compact sober dcpo and E a pointed bicomplete sober dcpo. If D is not coherent and E is not an L-dcpo, then the function space $[D \rightarrow E]$ is not meet-continuous.*

Proof. Assume that $[D \rightarrow E]$ is meet-continuous although neither E is an L-dcpo nor D is coherent. From Theorem 4.4.1 we know that $[D \rightarrow \mathbf{X}_{\perp}^{\top}]$ (see Figure 4.4 for $\mathbf{X}_{\perp}^{\top}$) is also meet-continuous since it is a retract of $[D \rightarrow E]$.

Since D is not coherent, Lemma 4.1.10 implies that there are compact saturated subsets A, B and Scott-open sets U, V of D such that $A \subseteq U, B \subseteq V$, but $A \cap B$ is not relatively compact in $U \cap V$. Thus, there exists a directed family $\{U_i \mid i \in I\}$ of open sets such that $U \cap V = \bigcup_{i \in I} U_i$, but U_i fails to cover $A \cap B$ for every $i \in I$. Define a function f from D to \mathbf{X}_\perp^\top as follows:

$$f(x) = \begin{cases} c, & x \in U \setminus V; \\ d, & x \in V \setminus U; \\ b, & x \in U \cap V; \\ \perp, & \text{otherwise.} \end{cases}$$

Moreover, for every $U_i, i \in I$, we define a function g_i as follows:

$$g_i(x) = \begin{cases} c, & x \in U \setminus V; \\ d, & x \in V \setminus U; \\ e, & x \in U_i; \\ a, & (U \cap V) \setminus U_i; \\ \perp, & \text{otherwise.} \end{cases}$$

It is easy to verify that f and $g_i, i \in I$, are Scott-continuous, and the set $G = \{g_i \mid i \in I\}$ is directed with its supremum above f . Note that

$$f \in N(A \rightarrow \uparrow c) \cap N(B \rightarrow \uparrow d)$$

and

$$N(A \rightarrow \uparrow c) \cap N(B \rightarrow \uparrow d) \cap \downarrow f \subseteq \{h \in [D \rightarrow \mathbf{X}_\perp^\top] \mid h(A \cap B) = \{b\}\}.$$

Since c, d are compact in \mathbf{X}_\perp^\top , from Proposition 2.5.6 $N(A \rightarrow \uparrow c) \cap N(B \rightarrow \uparrow d)$ is a Scott-open neighbourhood of f . Moreover, for each $i \in I$, $(A \cap B) \setminus U_i \neq \emptyset$, there is some $x \in (A \cap B) \setminus U_i$. Then $g_i(x) = a$, and therefore we have

$$N(A \rightarrow \uparrow c) \cap N(B \rightarrow \uparrow d) \cap \downarrow f \cap \downarrow G = \emptyset.$$

Hence f is not in $\overline{\downarrow f \cap \downarrow G}$, the Scott closure of $\downarrow f \cap \downarrow G$. This implies that the function space $[D \rightarrow \mathbf{X}_\perp^\top]$ is not meet-continuous. A contradiction. \square

Let **LcS** be the category of locally compact sober dcpos and **SCD** be the category of stably compact dcpos. Then our dichotomy theorem for locally compact sober dcpos reads as follows.

Theorem 4.4.3. *Let \mathbf{C} be a cartesian closed full subcategory in \mathbf{LcS}_\perp . Then either \mathbf{C} is included in \mathbf{SCD}_\perp , or every object in \mathbf{C} is an L-dcpo.*

Proof. Let L be any dcpo in \mathbf{C} . Then the function space $[L \rightarrow L]$ is in \mathbf{C} from the cartesian closedness of \mathbf{C} . It is obvious that both L and $[L \rightarrow L]$ are compact, locally compact and sober in the Scott topology. By Theorem 3.4.2 and Theorem 4.3.2 they are also meet-continuous and bicomplete.

If we assume that L is neither coherent nor has complete principal ideals, then from Theorem 4.4.2 the function space $[L \rightarrow L]$ is not meet-continuous. This contradiction implies that \mathbf{C} is contained in $\mathbf{SCD}_\perp \cup \mathbf{L}_\perp$. Moreover, \mathbf{C} should be entirely contained in one of them, since otherwise we could find in \mathbf{C} a non-coherent dcpo M and a dcpo N which is not an L-dcpo, and apply Theorem 4.4.2 again to conclude that $[M \rightarrow N]$ is not meet-continuous. \square

In the category of pointed continuous domains, the category of pointed L-domains is cartesian closed. Moreover, A. Jung proved that a pointed continuous domain with

a stably compact function space must be an FS-domain (see Theorem 2.8.18 and Theorem 4.1.7). Hence one gets a full classification of continuous domains: there exist exactly two maximal cartesian closed full subcategories of pointed continuous domains: \mathbf{FS}_\perp and \mathbf{cL}_\perp .

In our case, however, we do not know whether a similar result can be obtained. Since every dcpo in our interest is meet-continuous and bicomplete (see Theorems 3.4.2, 4.3.2), specifically, we ask the following:

Question 4.4.4. Is the category of locally compact sober and meet-continuous L-dcpo cartesian closed?

What we do know is the following fact:

Fact 4.4.5. *The category of stably compact and meet-continuous dcpo is not cartesian closed.*

Consider the function space $[L \rightarrow L]$ of Plotkin's ladder L in Figure 2.6. L is a Lawson-compact algebraic domain, hence it is stably compact and meet-continuous. In Example 2.8.23 we have seen that $[L \rightarrow L]$ is not continuous. We claim that $[L \rightarrow L]$ is not even meet-continuous. Actually, we can show that for any coherent dcpo D , if $[D \rightarrow D]$ is meet-continuous, then $K(D)$, the set of compact elements of D , does not contain a copy of M , where M is the set of compact elements in Plotkin's ladder L , i.e., $M = L \setminus \{\top\}$. To formalise this idea, we start from the following definition.

Definition 4.4.6. Given a poset L and $A \subseteq L$, we define:

- $U^0(A) = A$,

- $U^{n+1}(A) = \{x \in L \mid x \text{ is a minimal upper bound for some finite subset of } U^n(A)\},$
- $U^\infty(A) = \bigcup_{n \in \mathbb{N}} U^n(A).$

L is said to have *property U^∞* if for any finite subset A of $K(L)$, $U^\infty(A)$ is finite.

Proposition 4.4.7. *Let L be a coherent dcpo. If the function space $[L \rightarrow L]$ is meet-continuous, then L has property U^∞ .*

Proof. By way of contradiction, we assume that the property U^∞ in L does not hold, this is, there exists a finite subset $A \subseteq L$ of compact elements such that $U^\infty(A)$ is infinite. Let $B_n = U^{n+1}(A) \setminus U^n(A)$. We distinguish two cases.

Case 1: Some $U^{n+1}(A)$ is infinite while $U^n(A)$ is finite. This means there exists a finite subset F of $U^n(A)$ such that F has infinitely many minimal upper bounds. Notice that L is meet-continuous by Corollary 2.7.11 and Proposition 3.1.7. It follows from Corollary 3.2.5 and by induction that those minimal upper bounds must be compact elements; thus the principal filters generated by these compact elements form an open cover of the set $\bigcap \{\uparrow x \mid x \in F\}$. However, finitely many of them fail to cover $\bigcap \{\uparrow x \mid x \in F\}$. This is a contradiction to the coherence of L .

Case 2: B_n is finite and non-empty for every n . Note that $B_{n+1} \subseteq \uparrow B_n$, so $\{B_n\}_{n \in \mathbb{N}}$ is a directed family of finite sets in the Smyth preorder. We apply Rudin's Lemma to find a directed subset D of $U^\infty(A)$ such that $D \cap B_n \neq \emptyset$ for every n and D is infinite since $B_i \cap B_j = \emptyset$ for any distinct $i, j \in \mathbb{N}$. Let $\sup D = t$. Define a function f as follows:

$$f(x) = \begin{cases} t, & x \notin \downarrow t; \\ x, & x \in \downarrow t. \end{cases}$$

Furthermore, for every element $d \in D$, define a function g_d :

$$g_d(x) = \begin{cases} t, & x \notin \downarrow t; \\ d, & x \in \downarrow t. \end{cases}$$

It is trivial to check that f and $g_d, d \in D$, are Scott-continuous and $f \leq \sup_{d \in D} g_d$. We proceed by showing that f is not in the Scott closure of $\downarrow f \cap \downarrow \{g_d \mid d \in D\}$, which will lead to a contradiction to the meet-continuity of $[L \rightarrow L]$. To this end, consider the Scott-open set $\bigcap_{c \in C} N(\uparrow c \rightarrow \uparrow c)$, where $C = \downarrow t \cap A$. Notice that $f \in \bigcap_{c \in C} N(\uparrow c \rightarrow \uparrow c)$, by induction we know that every $x \in U^\infty(C)$ hence every $d \in D$ is fixed by every Scott-continuous function in $\downarrow f \cap \bigcap_{c \in C} N(\uparrow c \rightarrow \uparrow c)$. This is because $D \subseteq U^\infty(C)$. Remember that D is infinite, so any function below $g_d, d \in D$ cannot keep the whole of D intact. Hence we have

$$\downarrow f \cap \downarrow \{g_d \mid d \in D\} \cap \bigcap_{c \in C} N(\uparrow c \rightarrow \uparrow c) = \emptyset.$$

This entails that $f \notin \overline{\downarrow f \cap \downarrow \{g_d \mid d \in D\}}$.

Now both cases lead to contradictions, so L must have property U^∞ . \square

We update our knowledge and ask the following question.

Question 4.4.8. Let \mathbf{C} be a full subcategory of **DCPO** with every object L in \mathbf{C} satisfying:

- L is stably compact;
- L is meet-continuous;
- L has property U^∞ .

Is the category \mathbf{C} cartesian closed? If it is not, what can we say about cartesian closed full subcategories of \mathbf{C} ?

4.5 Dcpo without a bottom

In this section, we consider cartesian closed full subcategories of general locally compact sober dcpos. Similar to the work in [Jun89, Chapter 3], we can get four full subcategories such that any cartesian closed subcategory is contained in one of them entirely.

The following lemma is an essential observation for our discussion.

Lemma 4.5.1. *Let L, N be grounded dcpos and M_L, M_N the sets of minimal elements of L and N , respectively. If both L and the function space $[L \rightarrow N]$ are meet-continuous, then either L is finitely grounded, that is, M_L is finite; or for any $m, n \in M_N$, $\uparrow m \cap \uparrow n = \emptyset$, i.e., N is a disjoint sum of pointed dcpos.*

Proof. We prove this by contradiction. Assume that M_L is infinite and there exist $m, n \in M_N, d \in N$ such that $d \in \uparrow m \cap \uparrow n$. Let c_m, c_d be the constant functions from L to N with images m and d , respectively. Then one easily sees that $c_m \leq c_d$ and that c_m is a minimal element in $[L \rightarrow N]$. Since the function space is meet-continuous, from Corollary 3.2.4 c_m is a compact element of $[L \rightarrow N]$. We show the contradiction by proving that c_m cannot be compact. To this end for any finite subset $F \subseteq_{\text{fin}} M_L$ we define a function $f_F : L \rightarrow N$ as:

$$f_F(x) = \begin{cases} d, & x \in \uparrow F; \\ n, & \text{otherwise.} \end{cases}$$

Remember that L is meet-continuous, so every element in M_L is compact, hence $\uparrow F$ is Scott-open for all $F \subseteq_{\text{fin}} M_L$. This implies that f_F is Scott-continuous for any $F \subseteq_{\text{fin}} M_L$. Moreover, we see that the set $\{f_F \mid F \subseteq_{\text{fin}} M_L\}$ is directed in the function space and $\bigvee \{f_F \mid F \subseteq_{\text{fin}} M_L\} = c_d$, hence $c_m \leq c_d = \bigvee \{f_F \mid F \subseteq_{\text{fin}} M_L\}$. However, one sees that no such f_F is above c_m , and this violates the compactness of c_m . \square

We now investigate dcpo whose function spaces are finitely grounded. Obviously, these dcpo must be finitely grounded themselves, i.e., they have finitely many minimal elements. Moreover, we will see that the set of points “generated” by these finitely many minimal elements is also finite.

Definition 4.5.2. For a poset L , we define the *root* of L , written $rt(L)$, to be the set $U^\infty(\emptyset)$. We call L *well-rooted* if $U^\infty(\emptyset)$ is finite, consists of compact elements and for every $x \in L$, the set $\downarrow x \cap U^\infty(\emptyset)$ has a largest element.

Lemma 4.5.3. *Let L be a meet-continuous dcpo with property m . If the function space $[L \rightarrow L]$ is finitely grounded, then L is well-rooted.*

Proof. Obviously L itself is finitely grounded. Since L is meet-continuous, M_L is a set of compact elements by Corollary 3.2.4 and every element in $U^\infty(\emptyset)$ is also compact by Corollary 3.2.5.

We now prove that $U^\infty(\emptyset)$ is finite. To this end, we consider for each $x \in U^\infty(\emptyset)$ the function $r_x: L \rightarrow L$ defined in Example 2.7.3. Since $r_x = \text{id}_L$ on $\downarrow x$, any Scott-continuous function f below r_x keeps every minimal element in $\downarrow x \cap U^\infty(\emptyset)$ fixed. Then by induction every element in $\downarrow x \cap U^\infty(\emptyset)$, and hence x itself, is fixed by f . Note that $[L \rightarrow L]$ is finitely grounded, we find for each $x \in U^\infty(\emptyset)$ a minimal function g_x below r_x , and hence $g_x(x) = x$. Now for different elements $a, b \in U^\infty(\emptyset)$, we have minimal functions g_a, g_b below r_a, r_b , respectively. Without loss of generality

we assume $b \not\leq a$. Then we have $g_a(b) \leq r_a(b) = a \not\leq b = g_b(b)$, which implies that $g_a \neq g_b$. Thus $U^\infty(\emptyset)$ must be finite; otherwise by this method we could find infinitely many minimal functions in $[L \rightarrow L]$, which contradicts finite groundedness.

Finally, since L has property m , for any $x \in L$, the set $\downarrow x \cap U^\infty(\emptyset)$ is directed. Remember that $U^\infty(\emptyset)$ is finite, hence the finite directed set $\downarrow x \cap U^\infty(\emptyset)$ has a largest element. \square

Proposition 4.5.4. *Let L be a dcpo. If the second-order function space $[[L \rightarrow L] \rightarrow [L \rightarrow L]]$ is bicomplete and meet-continuous, then either L is well-rooted or L is a disjoint sum of pointed dcpos.*

Proof. Obviously, both L and $[L \rightarrow L]$ are meet-continuous and bicomplete. In particular $[L \rightarrow L]$ is grounded and L has property m .

Assume for sake of a contradiction that neither L is a disjoint sum of pointed dcpos nor well-rooted. It follows easily that $[L \rightarrow L]$ is not a disjoint sum of pointed dcpos either. Moreover, from Lemma 4.5.3 we know that $[L \rightarrow L]$ cannot be finitely grounded. Then by Lemma 4.5.1 we know the second-order function space $[[L \rightarrow L] \rightarrow [L \rightarrow L]]$ is not meet-continuous. A contradiction. \square

Proposition 4.5.5. *Let L be a sober dcpo with its function space $[L \rightarrow L]$ core-compact and meet-continuous. Then either $\uparrow a$ is coherent for all compact $a \in L$, or $\uparrow a$ is an L -dcpo for all compact $a \in L$.*

Proof. First, since L is a retract of $[L \rightarrow L]$ by Corollary 2.7.11, L is core-compact by Proposition 2.7.9 hence locally compact by Theorem 2.5.16. L is also bicomplete by Theorem 4.3.2. From Example 2.7.4, we know that for any compact elements a and b , the principal filters $\uparrow a, \uparrow b$ are retracts of L , hence $\uparrow a, \uparrow b$ are also locally compact, sober and bicomplete.

Moreover, by Proposition 2.7.10 and Proposition 3.1.7, we know that both the function spaces $[\uparrow a \rightarrow \uparrow b]$ and $[\uparrow b \rightarrow \uparrow a]$ are meet-continuous. Note that $\uparrow a, \uparrow b$ are pointed as dcpo, and so the contrapositive statement of Theorem 4.4.2 gives us the desired conclusion. \square

The following result, which is the main theorem of this section, is a combination of the previous ones.

Theorem 4.5.6. *Let \mathbf{C} be a cartesian closed subcategory of locally compact sober dcpo. If L is an object in \mathbf{C} , then L satisfies at least one of the following properties:*

1. *L is a disjoint sum of pointed L -dcpo;*
2. *L is a disjoint sum of pointed stably compact dcpo;*
3. *L is well-rooted and $\uparrow a$ is coherent for any compact element $a \in L$;*
4. *L is well-rooted and $\uparrow a$ is an L -dcpo for any compact element $a \in L$.*

Proof. It follows from Lemma 2.2.9 that function spaces are exponentiable objects in \mathbf{C} , hence for any object M in \mathbf{C} , $[M \rightarrow M]$ is also locally compact and sober. By Theorem 3.4.2 we have that M is meet-continuous, and by Theorem 4.3.2 that M is bicomplete.

For the dcpo L in \mathbf{C} , from cartesian closedness we know that both $[L \rightarrow L]$ and $[[L \rightarrow L] \rightarrow [L \rightarrow L]]$ are in \mathbf{C} , hence both of them are meet-continuous and bicomplete. Now we apply Proposition 4.5.4 to obtain that either L is well-rooted or L is a disjoint sum of pointed dcpo. We apply Proposition 4.5.5 to get that either all $\uparrow a$ are coherent, or all $\uparrow a$ are L -dcpo (for a compact in L). Finally we combine these results and hence obtain four possibilities. Note that in a disjoint sum of pointed dcpo, minimal elements are compact, and we finish the proof. \square

Within the category of locally compact, sober, bicomplete and meet-continuous dcpos with Scott-continuous functions as morphisms, we further define the following full subcategories:

- **dL**: the category of disjoint sum of pointed L-dcpo;
- **wL**: the category of well-rooted dcpos with principal open filters being L-dcpo;
- **dSCD**: the category of disjoint sum of pointed stably compact dcpos;
- **wSCD**: the category of well-rooted dcpos with principal open filters being coherent.

The examples in Figure 4.5, which are from [Jun89, Chapter 3], show that none of these four categories is contained in the union of the other three. Theorem 4.5.6 reads as, every cartesian closed full subcategory \mathbf{C} in \mathbf{LcS} is contained in the union of **dL**, **wL**, **dSCD** and **wSCD**. Moreover, we will see in the next statement that \mathbf{C} is entirely contained in one of these four categories.

Theorem 4.5.7. *Let \mathbf{C} be a cartesian closed full subcategory of \mathbf{LcS} . Then \mathbf{C} is contained in one the four categories: **dL**, **wL**, **dSCD** and **wSCD**.*

Proof. By Theorem 4.5.6 we know that \mathbf{C} is contained in $\mathbf{dL} \cup \mathbf{wL} \cup \mathbf{dSCD} \cup \mathbf{wSCD}$. For any two dcpos L, M in \mathbf{C} , both $[L \rightarrow M]$ and $[M \rightarrow L]$ are in \mathbf{C} . Then by Lemma 4.5.1 and Lemma 4.5.3 they are both in $\mathbf{dL} \cup \mathbf{dSCD}$ or in $\mathbf{wL} \cup \mathbf{wSCD}$, which implies that \mathbf{C} is entirely contained in $\mathbf{dL} \cup \mathbf{dSCD}$ or in $\mathbf{wL} \cup \mathbf{wSCD}$. Similarly, by Theorem 4.4.2, Proposition 4.5.5 and Proposition 2.7.10, \mathbf{C} is entirely contained in $\mathbf{dSCD} \cup \mathbf{wSCD}$ or in $\mathbf{dL} \cup \mathbf{wL}$. Combining these two situations, we obtain that \mathbf{C} is entirely contained in one of these categories. □

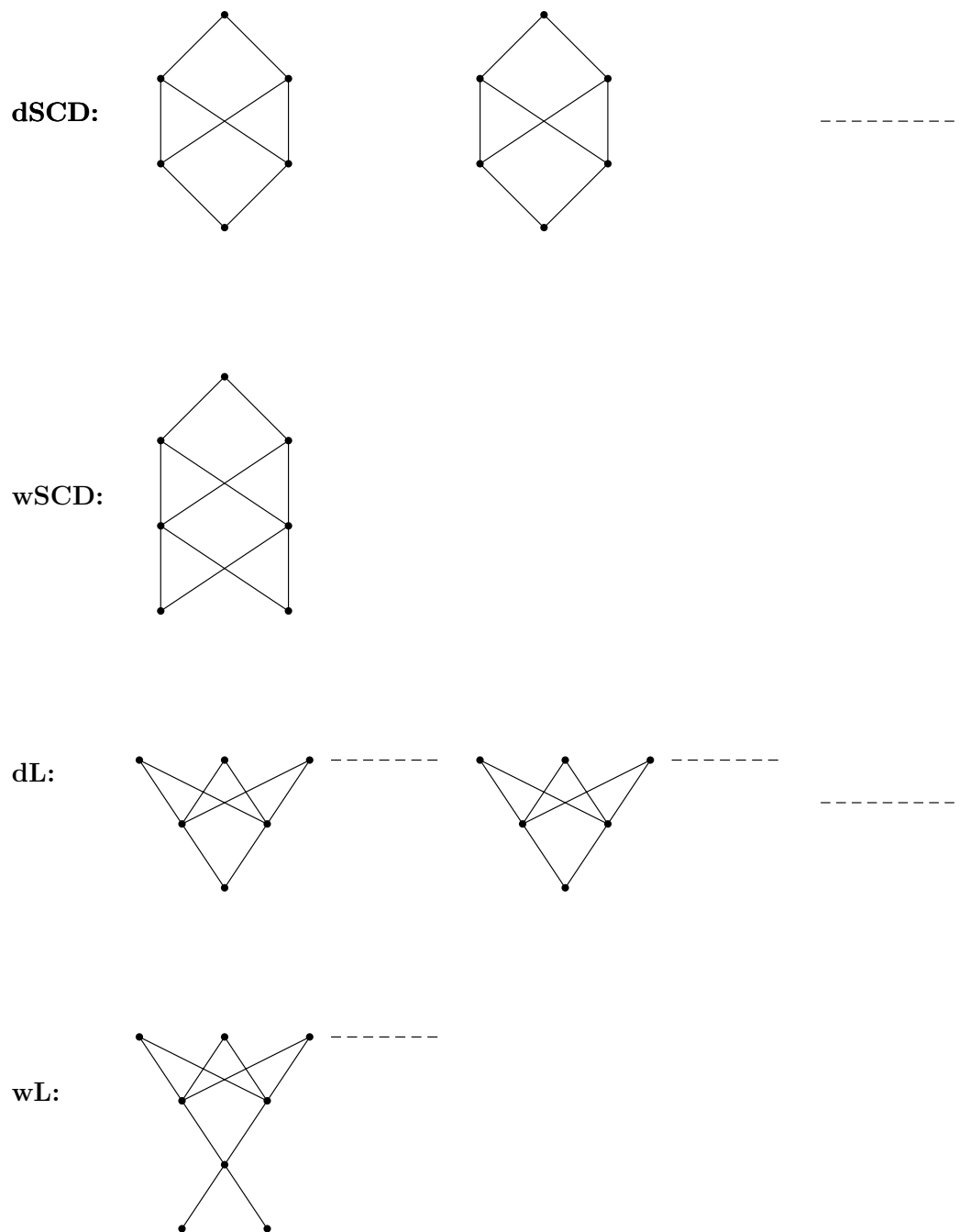


FIGURE 4.5: Locally compact sober dcpo's which are contained in exactly one of the categories **dL**, **wL**, **dSCD** and **wSCD**.

We end this section by asking the following question:

Question 4.5.8. In order to extend our theory to a more general setting, the category of core-compact dcpo's, we ask whether over dcpo's we could obtain sobriety

from core-compactness. Since all properties we considered should be reconciled with the function space construction, this means we can assume meet-continuity as well by Theorem 3.4.2. Our question is then asked as follows:

Let L be a core-compact and meet-continuous dcpo. Is L sober? If $[L \rightarrow L]$ is core-compact and meet-continuous, is L a sober dcpo?

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