Quantitative Continuous Domains

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Abstract

In this dissertation, we explore the problem of defining distance in weakly separated topological spaces focusing primarily on continuous posets in their Scott topology.

In the first part of the dissertation, we define a distance to be a two-argumented, real-valued mapping and study the role of various axioms that we subsequently identify and add to our setup. The novelty of our approach stems from the fact that in general we discard reflexivity of the distance.

In the second part, we unify theories of partial metrics and measurements in the framework of continuous posets. We present various representation theorems including a complete characterization of partial metrizability of algebraic posets. We prove existence of measurements and partial metrics for diverse classes of domains.

Lastly, we apply the theory developed so far to two kinds of problems. Firstly, we perform constructions of distance maps on the product, a function space and an inverse limit of domains. Secondly, we investigate the problem of modelling first-countable topological spaces by domains. In particular, we present representation theorems for metric and ultrametric spaces and propose a domain-theoretic construction of Choquet completion for a wide class of topologies.

Rodzicom poświęcam.

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Chapter 1

Introduction

There are two fundamental concepts in the theory of computation, namely *approximation* and *convergence*. Upon an agreement between researchers that is known nowadays as *Church's thesis*, computation must always process a finite amount of information and be completed in a finite number of steps. As a result we are able to compute only with *approximants* of "real" objects and, moreover, we are concerned only with processes that terminate, or, in other words, *converge*. Abstracting away from actual models of computation proposed over the last century, we are left with a set P of objects of computation, ordered by the amount of information they represent, an approximation relation $\ll$ between them and the notion of a convergence that is known to be best expressed in terms of a suitable topology on P. That is, we are left with a *domain of computation*. The theory of domains originated with Dana Scott's attempt to give a precise mathematical model of the untyped λ-calculus. Until 1969, however, no such model had been found; the reason being that any model would need to provide a unified treatment of objects and functions between them, whereas it was already known to Cantor that for every set A containing more than one element, the set A^A of endofunctions has a cardinality strictly bigger than the set A itself. At the time it was not at all clear if the self-application in the λ-calculus would not lead to Russelian paradoxes in the model. It was an ingenious idea of Scott to take instead of a set A a domain P and consider just the *continuous* endofunctions. His insight resulted in showing the existence of a space that was isomorphic with its own function space. In his Turing Award Lecture from 1976 Scott writes:

> *I remember quite clearly that the logician Andrzej Mostowski, who was also visiting Oxford at the time, simply did not believe that the kind of function spaces I defined had a constructive description. But when I saw they actually did, I began to suspect that the*

possibilities of using function spaces might just be more surprised than we had supposed.

And that was indeed the case: soon it was clear that in certain categories of domains function spaces are no different than "simple" objects with respect to both the structure and the cardinality and hence a consistent model of the λ-calculus and programming languages in general can be constructively presented. Since that time Domain Theory gained a *momentum* and matured enough to be awarded the status of an independent mathematical discipline. It is nowadays applied in other areas of Science including mathematical analysis, theory of fractals and dynamical processes, topology and still most prominently in different aspects of theoretical computer science.

Not surprisingly, Domain Theory is still explained to beginners in terms of *information*. Given two elements of a domain P with $x \sqsubseteq y$ one says that *y is at least as informative as x* and *consistent* with x. Consequently, a directed set in P is a collection of elements representing consistent information, possibly being a result of some computational process; functions between domains are the ones that preserve the information content; *continuous* mappings are even more specialized: they respect convergence of directed sets to their suprema. This "informatic" point of view was first formalised by means of information systems by Scott [Sco82a, Sco82b]. Work on information systems was continued by Larsen and Winskel [LW84]. Today, it is well-known that information systems are equivalent to a certain class of domains, known as Scott domains. Various researchers have been investigating generalizations (e.g. work on Information Categories by Edalat and Smyth [ES93]) and special cases of Scott's idea (e.g. formal spaces by Fourman and Grayson [FG82]).

The informatic intuition outlined above proves useful when one tries to study limits of applicability of the theory. Ingenuity is not necessary to notice that domains, as just certain partial orders, cannot be used in more refined, *quantitative* reasoning about computation. The contrast between the needs of computer science and the modelling power that domain theory can offer became even more visible when in the early Eighties another approach to modelling computational processes had been proposed. In their influential paper [dBZ82] de Bakker and Zucker presented a model of concurrent processes based on metric spaces. Their work was later further generalized by America and Rutten [AR89] who considered a general problem of solving recursive domain equations in the category of metric spaces.

Since that time much scientific effort has been spent on trials to unify both domain-theoretic and metric approaches to semantics of programming languages. The task turned out to be difficult. In the introduction to his paper [Smy88] Smyth writes:

> *The metric space approach to semantics is at odds with the approach via Scott domains, simply because metric spaces are necessarily Hausdorff, whereas the natural topology of a domain, in which there are elements representing partial information, is non-Hausdorff.*

In his attempt of reconciling domains with metric spaces, Smyth proposed a framework based on quasi-metrics and quasi-uniformities. Both of these quantitative structures differ from their "classical" counterparts by discarding symmetry. A simple observation shows that every preordered set $\langle A, \sqsubseteq \rangle$ can be given a quasi-metric $q\colon A \times A \to [0, \infty)$ by declaring for every two elements $x, y \in A$:

$$q(x,y) = \begin{cases} 0 & \text{if } x \sqsubseteq y \\ 1 & \text{otherwise} \end{cases} \tag{1.1}$$

and a quasi-uniformity induced in a standard way by the mapping q. However, in Smyth's opinion, in order to accommodate semantic domains used in computer science, a reformulation of basic definitions involving limits and completeness is necessary. Consequently, in the search for *the* class of domains of computation, he suggested bicomplete totally bounded quasi uniform spaces [Smy91]. Smyth indeed reworked the definition of completeness (that is named Smyth-completeness since then) in his next paper [Smy94] that was inspired by a theory of completeness for *syntopologies* in the sense of Császár [Csá63]. In the same paper, Smyth introduced so called *topological quasi-uniform spaces*, in which the quasi-uniform structure is linked to an auxiliary topology by some additional axioms. In fact, the need for additional topology arises in the study of examples motivated by computer science. More precisely, the topology induced by a quasi-uniform structure on the underlying set is usually the Alexandroff topology, whereas the natural topology on domains is the Scott topology. Smyth's insight immediately inspired many further studies in the field including [Sün93, Kün93, Sün95, Sün97], to name a few.

Another important attempt to unify partial orders and metric spaces emerged in work on so called *continuity spaces* introduced by Kopperman in his influential paper [Kop88]. The idea was to use a non-symmetric distance that takes values in a set with a rich order structure. In fact, Kopperman demonstrated that all topologies arise from such generalized metrics. Consequently, in [Fla97] Flagg refined Kopperman's idea and proposed the concept of a ν-continuity space. He suggested that the appropriate space of values of the generalized distance should be a *value quantale*, that is, a completely distributive quantale ν in which the set of elements that are approximated by 0 is a filter. Soon after that, both authors published a joint paper [FK97] summarizing their research.

Interestingly, Flagg's approach [Fla97] is based on a simple but extremely fruitful observation that turned out to be a cornerstone in research on the unification of order and distance. In his paper [Law73], Lawvere demonstrated that both preordered sets and generalized metric spaces, with a whole range of other structures, can be described in an elegant and homogeneous manner using the language of *enriched category theory*. Enriched categories introduced by Eilenberg and Kelly in [EK66] differ from "ordinary" categories by allowing the Hom functor to map into other categories than **Set**. Consequently, categories enriched over the two point lattice **2** are precisely preorders, quasi-pseudometric spaces (i.e. real-valued distances satisfying the reflexivity and triangle inequality axioms) coincide with categories enriched over the interval $[0, \infty]$, and Flagg's ν-continuity spaces are exactly categories enriched in a value quantale ν.

The categorical perspective has also been adopted in the independent study conducted by Wagner in his PhD dissertation [Wag94] and later in [Wag97]. He has proposed a general framework for solving recursive domain equations, thereby unifying original attempts of Scott and de Bakker and Zucker that in the Eighties seemed to be fundamentally different.

As one may expect, Smyth's and Lawvere's ideas have been finally combined together. Rutten, Bonsangue and van Breugel in a series of papers [Rut95, BvBR95, BvBR96, Rut97, BvBR98] showed, among other things, how to construct completions and powerdomains in their setting known as *generalized metric spaces* (*gms*, for short). Independently from the research group mentioned above, Flagg and Sünderhauf [FS] introduced and studied a concept of a topological ν-continuity space under the banner of *quantitative domain theory*. Their efforts concentrated on comparing different versions of completions that are available for quantitative domains. Their work has been complemented by Künzi and Schellekens in [KS]. Finally, Flagg, Sünderhauf and Wagner in the recent article [FSW] constructed ideal completions of arbitrary continuity spaces, defined a general version of the Scott topology and gave a representation theory for algebraic quantitative domains.

It is worth to mention that an excellent survey on the history of the attempts to unify ordered and metric structures, and more generally - on the role of non-symmetric distances in topology - is given by Künzi in [Kün].

In this dissertation we adopt a fundamentally different point of view on what the quantitative domain theory should be. One can argue that elegance of the enriched category theory approach is deceptive since the simplicity of Scott's original idea of a domain has been lost in the process of generalization and abstraction. In fact, the class of objects considered in generalized metric spaces by Rutten and others, as well as in quantitative domain theory as understood by Flagg, Sünderhauf and Wagner, is horrendously large as by assumption it contains *all* preorders (recall the formula (1.1) above). One can argue in favour of the presented unification theory that we can restrict our attention to certain subclasses of the quantitative objects, such as compact T_0 continuity spaces (called ν-domains in [FK97]). However, this category is still big enough to accommodate all continuous dcpos and let us stress that the construction of an appropriate enrichment is formidably complex. It seems that the unification theory, while being a definitive intellectual achievement, may still require a lot of simplifications and further insight to be of a practical significance for the theoretical computer science community.

In contrast, we propose a quantitative domain theory that builds on the familiar and well-understood framework of continuous posets (we usually restrict our attention to countably based ones). Our domains are equipped with a real-valued distance that induces the Scott topology (hence there is no need to introduce an additional topology, as in the case of topological ν-continuity spaces). Our approach has roots in work on *partial metrics* introduced by Matthews in [Mat92], studied further in [O'N95a, O'N95b, Hec99, Scha, KV92] and on *measurements* defined by Martin in [Mar00]. Our dissertation is devoted to the study of both these structures in detail.

The definition of a partial metric is based on the idea that points do not necessarily have zero distance from themselves.[1] Consequently, appropriate changes must be applied to the remaining metric axioms. In spite of being symmetric, partial metrics are capable of capturing order and the Scott topology on a domain. Partial metrics allow us to formalize a common intuition that the maximal elements of a domain of computation are "ideal" or "complete", whereas the approximating ones are "partial" or "incomplete", the reason being that in many cases there is a natural construction of a partial metric distance on the underlying domain that assigns zero to maximal elements and a positive real number to the remaining elements.

Similarly, measurements have been designed especially to formalize the idea of "information content" of the elements that is to be understood as the assignment of a value to each element, which allows for the distinction of the comparable ones and is available for every element on the domain. Consequently, a measurement is in particular a strictly anti-monotone real-valued map, which assigns a real number to any element on the domain.

[1] Consider, as an example, the distance between subsets of a metric space $\langle X, \rho \rangle$ given by $d(A,B) := \sup\{\rho(x,y) \mid x \in A, y \in B\}$ for any $A, B \subseteq X$. Informally, subsets are interpreted as "partially specified points".

For example, a natural measurement for the poset of natural numbers ω is a mapping $\mu\colon \omega \to [0,\infty)^{op}$ defined as $\mu n = 1/n$.

As the major achievement of this thesis we regard the unification of the theories of measurements and partial metrics in the framework of continuous posets. In fact, the self-distance of a partial metric compatible with the Scott topology is a measurement. Conversely, in many cases, the distance can be recovered from a measurement by a standard construction. The correspondence proves to have fruitful consequences. In the quest for finding a minimal set of axioms for a distance on continuous posets, we introduce various generalizations of partial metrics and demonstrate the equivalence with appropriate classes of measurements. In particular, we solve the problem of characterizing all partial metrics compatible with the Scott topology on algebraic dcpos. Moreover, we use measurements as vehicles to build distances on products and function spaces. Finally, we engage in the subject of modelling topological spaces by domains and give a detailed account of the applicability of the theory of measurements and partial metrics to this task.

Among the things that we were not able to achieve is a complete characterization of partial metrizability of the Scott topology on continuous posets. Another deficiency in comparison to the unification theory of preorders and partial metrics is the lack of a treatment of powerdomains, whereas for example in the case of quantitative domains in the sense of Flagg, Sünderhauf and Wagner, some elegant constructions have been presented in [Sün00].

We however hope that the results of our dissertation will be a starting point for a sufficiently general but simple theory of objects that are suitable for modelling various aspects of computation and useful in modern applications of domain theory to general topology and mathematical analysis.

Overview of the Chapters

This dissertation is divided into four chapters. After the introduction, in the second chapter, we outline a general theory of distance that, in contrast to other existing approaches in general topology, discards the reflexivity axiom. Our *distance space* is thus a set equipped with a real-valued mapping with a possibly non-trivial self-distance. Subsequently, we add more and more axioms to the framework and distinguish two classes of *partial symmetric* and *partial semimetric* spaces. We focus on the study of the topological and order-theoretic properties of the latter class; further, we bring our attention to distances that satisfy the *local axiom of triangle*. We prove that spaces equipped with such distances are quasi-developable and that the famous Archangelskij metrization theorem generalizes to our setting. We close the chapter with a few general observations concerning the self-distance of partial semimetrics that will prove useful in the further development of the theory in the framework of continuous posets.

The third chapter is the core of our dissertation. We focus on distance maps on continuous posets that induce the Scott topology. We prove that measurements in the sense of Martin correspond to partial semimetrics establishing the link between two theories that were so far unrelated. Next, we identify two subclasses of measurements, namely the ones that satisfy the *weak modularity law* and the *local axiom of triangle*. We relate the former ones to *stable partial metrics*. Furthermore, we show that measurements of the latter type characterize partial metrics on algebraic posets. This characterization provides a solution to an open problem in the partial metric theory posed by Heckmann in [Hec99]. In the second part of the chapter we focus on topological properties of the metric induced by a partial metric. First of all, we prove that it is located between Lawson and Martin topologies in the lattice of all topologies on the underlying poset. Secondly, we investigate completeness of the induced metric. As it happens, we are able to derive strong results whenever we assume bounded-completeness of the domain. We prove that the metric induced by a stable partial metric is complete if and only if the associated measurement is cocomplete (preserves infima of filtered subset). Next, we improve our result by showing that an arbitrary partial metric on a bounded-complete dcpo induces a completely metrizable topology.

In the last part of the third chapter, we give several constructions of partial metrics for the Scott topology on continuous posets. In general we demand that for a constructed partial metric, the induced metric is complete and the derived measurement satisfies the *kernel condition* (that is, the preimage of 0 coincides with the subset of maximal elements of the poset). It appears that this ideal situation is rare and we can only establish it for the class of Scott domains and countably based continuous lattices.

Relaxing our demands, we get some further interesting existence results. Firstly, we prove that every countably based algebraic domain which satisfies the Lawson condition [Law97b] admits an appropriate partial metric and a measurement with the kernel condition. On the other hand, dropping the requirement on the measurement, we demonstrate that every countably based algebraic domain admits a partial metric such that the induced metric topology agrees with the Lawson topology. In this case, Lawson-compactness of the domain implies completeness of the induced metric. Finally, generalizing the last construction from the algebraic to the continuous case, we show that the induced metric topology may no longer agree with the Lawson topology but is complete whenever the domain is bounded-complete.

We locate the strength of our results more exactly by giving various examples and counterexamples.

The last and fourth chapter is devoted to the study of two areas of application of quantitative continuous domains. First, we investigate cartesian products, inverse limits and function spaces of partially metrized domains. Following Spreen [Spr01] we show how to build a measurement on the respective constructs from the measurements on the parts and how to extend it to a canonical partial metric. In the second part of the chapter we turn to the problem of modelling topological spaces as subsets of maximal elements of continuous posets. We are often able to match structural properties of the modelling poset to properties of the modelled space. Thus, measurements correspond to developability, partial metrics to metrizability, a tree structure of the domain to ultrametrizability, etc. We also observe that completeness of models is tightly related to Choquet completeness of spaces. We investigate the latter in more detail and propose a domain-theoretic construction of Choquet completion for a wide class of topologies. In particular, our results generalize a domain-theoretic construction of the completion for metric spaces proposed by Heckmann in [Hec99]. As a byproduct we introduce the notion of an *abstract measurement* that is, in our opinion, a suitable generalization of the notion of a measurement on a continuous dcpo to the setting of abstract bases. We conclude the chapter and the whole thesis with a diagram, which summarizes our knowledge about modelling topological spaces by continuous posets with measurements.

Parts of the dissertation have been published previously in two technical reports [Was00, Was01b] and in a conference paper [Was01a].

Chapter 2

Outline of a general theory of distance

The aim of this chapter is to develop a simple and sound framework that allows for considerations of quantitative properties of certain weakly separated topological spaces. The motivation for the work stems from Quantitative Domain Theory, of which the theory of measurements and the theory of partial metrics are examples. We adhere to a common intuition in Domain Theory that properties of elements of topological spaces are represented by open sets (see [Smy92]). In this context, one may speak of quantitative properties whenever there exists a mapping from the space to some space of *quantities*, usually taken to be the set of real numbers $\mathbb{R}$. Therefore, in our framework we study topological spaces equipped with real-valued functions, which we will call *distance spaces*. It should be mentioned that studies of distance spaces and even more general ones have been conducted before; examples are V-spaces [Sie52], neighbourhood spaces [Smy95], closure spaces and semi-pseudometrics [Čec66] and a wide range of other generalized metric spaces including the non-symmetric ones [Kün]. The studies vary with respect to a minimal set of imposed axioms. Notably, however, a so called reflexivity axiom seems to be present in each of the mentioned frameworks. In contrast, our approach discards reflexivity and focuses on the self-distance of elements, which becomes a rich source of information about the space.

Our primary goal is to understand the role of various axioms identified in theories of distance. Our study is by no means complete; rather, it is guided by examples from Domain Theory. In short, one could say that we examine properties of measurements and partial metrics on continuous domains in order to precisely state their dependence on the axioms. In this way, we arrive at a notion of a *partial semimetric*, which is essentially a two-argumented version of a measurement in an arbitrary topological space.

We focus on the so called *local axiom of triangle* (*local triangle inequality* or simply **(L)**), which can be seen as a generalization of the classical triangle inequality, sharp triangle inequality of partial metrics and one of the two axioms of quasi-uniform spaces. We sketch the relationship of distance with local triangle inequality to the notion of *syntopology* (cf. Császár [Csá63], Smyth [Smy94]).

There are two central results in this chapter: a generalization of the famous Archangelskij Metrization Theorem (Theorem 2.15) to the framework of partial semimetric spaces with **(L)** (Theorem 2.28) and the quasi-developability of such spaces (Theorem 2.29). However, this chapter is intended to be an introductory study of the origins of some phenomena in Quantitative Domain Theory and hence we sketch directions of research rather than engage in proving many new results.

One of the shortcomings of our presentation is a lack of *bitopological* treatment of distance spaces, which are naturally equipped with two topologies. We believe that research on duality in distance spaces will significantly contribute to their understanding, however, we leave this as a subject of future study.

2.1 The distance topology and its dual

First, let us consider the definition of a distance space, its intrinsic topologies and some natural axioms that can be introduced in distance spaces.

Definition 2.1. *A* distance *on a set X is a map $\rho\colon X \times X \to [0, \infty)$. A pair $\langle X, \rho\rangle$ is called a* distance space.

A distance function assigns to each element x of X a filter $\mathcal{N}_x$ of subsets of X by taking $U \in \mathcal{N}_x$ if and only if there exists $\varepsilon > 0$ such that $B_\rho(x, \varepsilon) \subseteq U$. The set

$$B_\rho(x, \varepsilon) := \{y \in X \mid \rho(x, y) < \rho(x, x) + \varepsilon\}$$

is called a *ball centered at x with radius ε*. In the same way one forms a collection $\mathcal{N}_x^{op}$ by replacing the ball in the definition of $\mathcal{N}_x$ by a *dual ball*

$$B_\rho^{op}(x, \varepsilon) := \{y \in X \mid \rho(x, y) < \rho(y, y) + \varepsilon\}.$$

The collection

$$\tau_\rho := \{U \mid \forall x \in U.\, U \in \mathcal{N}_x\}$$

is a topology on X called the *distance topology on X*. Dually, the family of sets

$$\tau_\rho^{op} := \{V \mid \forall x \in V.\, V \in \mathcal{N}_x^{op}\}$$

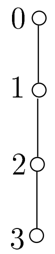


Figure 2.1: A non-metric distance.

constitutes a *dual (distance) topology* on X.

In general the collection $\mathcal{B}_x$ of balls centered at x is *not* a neighbourhood base at x and the balls are *not* open themselves.

Sometimes we will find useful to consider, instead of balls, so called *c-balls* and *dual c-balls* defined for $x \in X$ and $\varepsilon > 0$ respectively as:

$$N_\rho(x, \varepsilon) := \{y \in X \mid \rho(x, y) \leq \rho(x, x) + \varepsilon\}$$

and

$$N_\rho^{op}(x, \varepsilon) := \{y \in X \mid \rho(x, y) \leq \rho(y, y) + \varepsilon\}.$$

The set X together with an operation $\mathcal{N}\colon X \to \mathcal{P}(\mathcal{P}(X))$, which assigns to each point the filter $\mathcal{N}_x$, is an example of a *neighbourhood space* in the sense of [Smy95]. The distance topology just defined is in fact one of the natural topologies for neighbourhood spaces, studied in more detail by Smyth (cf. [Smy95], Proposition 2.10).

Example 2.2. To illustrate the difference between arbitrary distance spaces and metric ones, consider the four-element chain as shown in Fig. 2.1 (numbers denote respective self-distances; we set the distances $\rho(x, y)$ and $\rho(y, x)$ between points x, y to be $\max\{\rho(x, x), \rho(y, y)\}$. The distance is symmetric and satisfies the triangle inequality. Note that open sets are upper; in particular, the only open set containing the bottom element (denoted $\perp$) is the whole space. Indeed, the distance between $\perp$ and any other element of the space is 3 and hence any ball containing $\perp$ must be the whole space. In contrast, any metric on a finite space induces the discrete topology.

Note that in this particular example the balls are open themselves. In arbitrary distance spaces it is usually not the case.

2.2 Basic properties of distance spaces

For subsets A, B of X and an element $x \in X$ we define:

$$\rho(x, B) := \inf\{\rho(x, b) \mid b \in B\}$$

and

$$\rho(A, B) := \inf\{\rho(a, b) \mid a \in A, b \in B\}.$$

Proposition 2.3. *In a distance space $\langle X, \rho\rangle$ a subset $H \subseteq X$ is closed iff for all $x \notin H$ we have $\rho(x, H) > \rho(x, x)$.*

Proof. Straight from the definition of the distance topology we infer that a subset H of X is closed with respect to τ_ρ iff for all $x \notin H$ there exists $\varepsilon > 0$ such that for all $y \in H$ we have $\rho(x, y) \geq \rho(x, x) + \varepsilon$. This is however equivalent to saying that $\rho(x, H) \geq \rho(x, x) + \varepsilon > \rho(x, x)$. $\square$

In any topological space $\langle X, \tau\rangle$ the relation

$$x \sqsubseteq_\tau y \text{ iff } \forall U \in \tau.\ x \in U \text{ implies } y \in U,$$

where $x, y \in X$, is a preorder called the *specialisation preorder*. As a corollary of the preceding result we obtain the following interesting property of the specialisation preorder $\sqsubseteq_\rho$ of the distance topology:

Corollary 2.4. *In a distance space $\langle X, \rho\rangle$ the following are equivalent:*

1. $x \sqsubseteq_\rho y$;
2. $x = y$ *or* $\forall \varepsilon > 0.\ \exists z \neq x.\ (\rho(x, z) < \rho(x, x) + \varepsilon$ *and* $z \sqsubseteq_\rho y)$.

Proof.

$$\begin{array}{rll} x \not\sqsubseteq_\rho y & \text{iff} & x \notin \mathbf{cl}\{y\} \\ & \text{iff} & \exists \varepsilon > 0.\ \rho(x, \mathbf{cl}\{y\}) \geq \rho(x, x) + \varepsilon \\ & \text{iff} & \exists \varepsilon > 0.\ \forall z \in B_\rho(x, \varepsilon).\ z \notin \mathbf{cl}\{y\} \\ & \text{iff} & \exists \varepsilon > 0.\ \forall z \in B_\rho(x, \varepsilon).\ z \not\sqsubseteq_\rho y. \end{array}$$

Note that we have used Proposition 2.3 in the second equivalence.

Conversely, assume $x \neq y$ and let U be an open set in X. Then $x \in U$ implies that there exists $\varepsilon > 0$ such that $x \in B_\rho(x, \varepsilon) \subseteq U$. By assumption, there is $z \in B_\rho(x, \varepsilon) \subseteq U$ with $z \sqsubseteq y$. But open sets are upper with respect to the specialisation preorder and so $y \in U$. $\square$

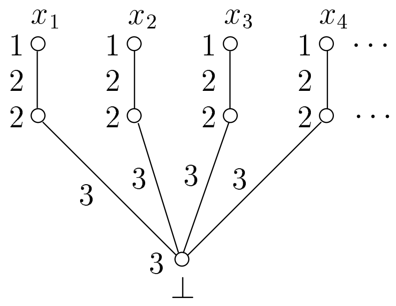


Figure 2.2: (cv) does not hold in general.

Proposition 2.5. *Let $\langle X, \rho \rangle$ be a distance space, (x_n) be a sequence of elements of X and $x \in X$. Then*

$$\rho(x, x_n) \to \rho(x, x) \text{ implies } x_n \to_{\tau_\rho} x.$$

Proof. Let U be any open set around x. Then there exists $\varepsilon > 0$ such that $x \in B_\rho(x, \varepsilon) \subseteq U$. Suppose that $\rho(x, x_n) \to \rho(x, x)$. Then there exists $N_\varepsilon \in \omega$ such that for all $n \geq N_\varepsilon$, $|\rho(x, x_n) - \rho(x, x)| < \varepsilon$. It is equivalent to saying that either $0 \leq \rho(x, x_n) - \rho(x, x) < \varepsilon$ or $0 \geq \rho(x, x_n) - \rho(x, x) > -\varepsilon$ for all $n \geq N_\varepsilon$. In either case, we are able to conclude that $x_n \in B_\rho(x, \varepsilon) \subseteq U$ for all $n \geq N_\varepsilon$. Therefore, $x_n \to_{\tau_\rho} x$. □

Note that the corresponding equivalence:

(cv) $\rho(x, x_n) \to \rho(x, x)$ iff $x_n \to_{\tau_\rho} x$.

(the tag **(cv)** stands for "convergence") may not hold in arbitrary partial symmetric spaces. Fig. 2.2 provides a counterexample: we assume that all of the distances between elements, which are not given on the picture are 5. One can check that the only open set containing the bottom element is the whole space and hence every sequence converges to it. On the other hand, for the sequence (x_n) we have $\rho(\perp, x_i) \to 5$, for $i = 1, 2, 3, ...$, while $\rho(\perp, \perp) = 3$.

It is known that **(cv)** holds if the T_2 axiom is assumed (cf. [Gru84], Lemma 9.3).

2.3 Distance axioms

On a distance space a number of further axioms can be introduced. We consider *symmetry* of the distance map

$$\forall x, y \in X.\ \rho(x, y) = \rho(y, x) \qquad \textbf{(symm)}$$

and the axiom of *small self-distances*

$$\forall x, y \in X.\ \rho(x,x) \leq \rho(x,y) \text{ and } \rho(x,x) \leq \rho(y,x), \tag{ssd}$$

which obviously reduces to

$$\forall x, y \in X.\ \rho(x,x) \leq \rho(x,y) \tag{2.1}$$

in presence of symmetry. The latter condition is a generalization of the *reflexivity axiom*

$$\forall x, y \in X.\ \rho(x,x) = 0 \tag{r}$$

known from the theory of metric spaces.

In Topology, consult for example [Gru84], a *symmetric* is a distance map, which satisfies **(symm)** and **(r)**. On the other hand, a symmetric distance function satisfying the *sharp triangle inequality*

$$\forall x, y, z \in X.\ \rho(x,y) \leq \rho(x,z) + \rho(z,y) - \rho(z,z) \tag{$\Delta^\sharp$}$$

is named a *weak partial metric* [Hec99]. A *partial metric* [Mat92] is a weak partial metric with **(ssd)**. Lastly, one considers mostly distance topologies, which satisfy the T_0 separation axiom. An elegant formulation of the $(\mathbf{T_0})$ axiom in terms of the distance mapping will be given later for a wide class of spaces (cf. Corollary 2.12).

The following definition is a compromise on the existing terminology.

Definition 2.6. *A distance function $\rho\colon X \times X \to [0, \infty)$ on a set X is a* partial symmetric *whenever it satisfies* **(symm)**, **(ssd)** *and the distance topology τ_ρ is T_0.*

2.4 Partial semimetric spaces

In this section we introduce a particularly interesting subclass of partial symmetric spaces. Both partial metric spaces considered by Matthews [Mat92], O'Neill [O'N95a, O'N95b], Schellekens [Scha, Schb], Heckmann [Hec99] and Waszkiewicz [Was01a] and semimetric spaces in Topology are our major examples of partial semimetric spaces.

Definition 2.7. *Let $\langle X, \rho \rangle$ be a partial symmetric space. The map ρ is a* partial semimetric *if for any $x \in X$, the collection $\{\mathbf{int}(B_\rho(x,\varepsilon)) \mid \varepsilon > 0\}$ is a base for the filter $\mathcal{N}_x$.*

In other words, we require that the family $\{B_\rho(x,\varepsilon) \mid x \in X, \varepsilon > 0\}$ forms a (not necessarily open) neighbourhood base at x with respect to the distance topology τ_ρ on X.

To summarize the key differences between "classical" distance maps and ours, we collect their properties in the following table. A '+' indicates that the respective condition is satisfied. All maps in the table satisfy **(symm)**, **(ssd)** and **($\mathbf{T_0}$)**.

Distance	Balls form a neighbourhood base	**(r)**
partial symmetric		
partial semimetric	+	
symmetric		+
semimetric	+	+

Proposition 2.8. *Let $\langle X, \rho\rangle$ be a distance space. The following are equivalent:*

1. *$\{B_\rho(x,\varepsilon) \mid \varepsilon > 0\}$ forms a (not necessarily open) neighbourhood base at x;*

2. *$\forall x \in X. \forall H \subseteq X.\ \rho(x,H) \leq \rho(x,x)$ iff $x \in \mathbf{cl}\{H\}$.*

Proof. ($\Rightarrow$) Let $x \in X$ and $\varepsilon > 0$.

$$\begin{aligned} x \in \mathbf{cl}(H) \quad &\text{iff} \quad \forall \varepsilon > 0.\ B_\rho(x,\varepsilon) \cap H \neq \emptyset \\ &\text{iff} \quad \forall \varepsilon > 0.\ \exists z \in H.\ \rho(x,z) < \rho(x,x) + \varepsilon \\ &\text{iff} \quad \rho(x,H) \leq \rho(x,x). \end{aligned}$$

Conversely, Since $X \setminus B_\rho(x,\varepsilon) = \{z \mid \rho(x,z) \geq \rho(x,x) + \varepsilon\}$, we have $\rho(x, X \setminus B_\rho(x,\varepsilon)) \geq \rho(x,x) + \varepsilon > \rho(x,x)$. Therefore, $x \notin \mathbf{cl}(X \setminus B_\rho(x,\varepsilon))$ and so $x \in \mathbf{int}(B_\rho(x,\varepsilon))$. □

As an immediate corollary, we note that partial semimetrizability can be introduced by an appropriate closure operator familiar from the theory of metric spaces. It was stated in [Gru84], Theorem 9.7 for semimetrics.

Corollary 2.9. *Let $\langle X, \rho\rangle$ be a symmetric distance space. The following are equivalent:*

1. *ρ is a partial semimetric.*

2. *$\forall x \in X. \forall H \subseteq X.\ \rho(x,H) = \rho(x,x)$ iff $x \in \mathbf{cl}(H)$.*

One can check that if the distance topology is Hausdorff, then X is partially semimetrizable iff it is partially symmetrizable and first countable (see the discussion before Definition 9.5 of [Gru84]). In the lack of separation axioms, the property **(cv)** is necessary for this characterization to hold.

Proposition 2.10. *Let $\langle X, \rho \rangle$ be a partial symmetric space. Then the following are equivalent:*

1. *The map ρ is a partial semimetric;*

2. *X is first countable and* **(cv)** *holds.*

Proof. If ρ is a partial semimetric, then the collection $\{\mathbf{int}(B_\rho(x, 1/2^{n+1})) \mid n \in \omega\}$ is a neighbourhood base at $x \in X$, which amounts to first countability of X. Now, if for some sequence (x_n) of elements of X and some $x \in X$, we have $x_n \to_{\tau_\rho} x$, then for every $\varepsilon > 0$, (x_n) is cofinally in $\mathbf{int}(B_\rho(x, \varepsilon))$ and so in $B_\rho(x, \varepsilon)$. Therefore, using the **(ssd)** axiom,

$$\forall \varepsilon > 0.\ \exists N \in \omega.\ \forall n \geq N.\ \ |\rho(x, x_n) - \rho(x, x)| < \varepsilon.$$

That is, $\rho(x, x_n) \to \rho(x, x)$.

Conversely, if X is first countable and **(cv)** holds, then a point $x \in X$ is in $\mathbf{int}(B_\rho(x, \varepsilon))$ for any $\varepsilon > 0$. Suppose not; then there exists $\varepsilon_0 > 0$ and a sequence (x_n) with $x_n \to_{\tau_\rho} x$ such that $x_n \notin B_\rho(x, \varepsilon_0)$. (Choose x_n to be in the n^{th} member of a decreasing countable base for x and not in $B_\rho(x, \varepsilon_0)$.) So $\rho(x, x_n) \not\to \rho(x, x)$, what contradicts **(cv)**. Now, if $x \in \mathbf{int}(B_\rho(x, \varepsilon))$, then the collection $\{B_\rho(x, \varepsilon) \mid \varepsilon > 0\}$ forms a neighbourhood base at x. $\square$

In a partial semimetric space the specialisation preorder reflects the distance in a simple way.

Proposition 2.11. *Let $\langle X, \rho \rangle$ be a partial semimetric space. Then for all $x, y \in X$,*

$$x \sqsubseteq_{\tau_\rho} y \text{ iff } \rho(x, y) = \rho(x, x).$$

Proof.

$$\begin{aligned} x \sqsubseteq_{\tau_\rho} y \quad & \text{iff} \quad x \in \mathbf{cl}(\{y\}) \\ & \text{iff} \quad x \in \{z \mid \rho(z, y) = \rho(z, z)\} \\ & \text{iff} \quad \rho(x, y) = \rho(x, x). \end{aligned}$$

We use Proposition 2.8(2) and **(ssd)** in the second equivalence. $\square$

As a result we are able to characterize the $(\mathbf{T_0})$ axiom in terms known from the theory of partial metric spaces.

Corollary 2.12. *The distance topology in a partial semimetric space* $\langle X, \rho\rangle$ *is* T_0 *iff*

$$x = y \text{ iff } \rho(x, y) = \rho(x, x) = \rho(y, y).$$

□

2.5 Derived distance functions and their topologies

Convention	In the rest of this chapter we assume that the $(\mathbf{T_0})$ axiom holds in all distance spaces that we consider.

In a distance space $\langle X, \rho\rangle$ one can introduce a number of other maps, which are derived from the distance. Whenever the map ρ satisfies **(ssd)**, it is convenient to form the corresponding *quasi-distance* $q_\rho\colon X \times X \to [0, \infty)$ by

$$q_\rho(x, y) := \rho(x, y) - \rho(x, x).$$

Whenever possible, we will drop the index ρ from q_ρ for the sake of clarity. The **(ssd)** axiom assures that the function q is well-defined. The usefulness of the quasi-distance stems from the fact that it satisfies the reflexivity axiom **(r)** and still induces the same topology as the distance ρ. Whenever the map ρ satisfies more axioms, the name for the quasi-distance q will change accordingly. For instance, a *quasi-semimetric* is a quasi-distance formed from a partial semimetric.

The quasi-distance has a dual, namely the map $q_\rho^{op}\colon X \times X \to [0, \infty)$ defined by $q_\rho^{op}(x, y) := \rho(x, y) - \rho(y, y)$. The dual quasi-distance induces the dual distance topology.

The symmetrization of the quasi-distance and its dual is a map $d_\rho\colon X \times X \to [0, \infty)$,

$$\forall x, y \in X.\ d_\rho(x, y) := q_\rho(x, y) + q_\rho^{op}(x, y) = 2\rho(x, y) - \rho(x, x) - \rho(y, y).$$

The function d_ρ is always symmetric and reflexive and hence is a symmetric. It is called the *induced symmetric*. We do not know whether it is true in general that the induced symmetric derived from a partial semimetric is a *semimetric*. However, there are two notable cases when the prefix "*semi-*" is retained: firstly, whenever a partial semimetric topology is Hausdorff, secondly, in presence of the *local axiom of triangle* studied in next section (the latter claim follows from Proposition 2.16 and the observation that a partial semimetric is a semimetric if and only if it satisfies the reflexivity axiom).

Lastly, for a distance function, the *self-distance mapping* (called also the *weight function*) is non-trivial in general and, as we shall see in Section 2.10, exhibits many interesting properties.

Proposition 2.13. *For a distance space $\langle X, \rho\rangle$ with* **(ssd)***, the induced topology τ_{d_ρ} is the join of the distance topology τ_ρ and its dual τ_ρ^{op}. Moreover, τ_{d_ρ} is semimetrizable whenever both τ_ρ and τ_ρ^{op} are partially semimetrizable.*

Proof. Both statements follow from the fact that

$$B_{d_\rho}(x,\varepsilon) \subseteq B(x,\varepsilon) \cap B_\rho^{op}(x,\varepsilon) \subseteq B_{d_\rho}(x,2\varepsilon)$$

for any $x \in X$ and $\varepsilon > 0$. $\square$

Proposition 2.14. *For a sequence (x_n) and an element $z \in X$ in a partial semimetric space $\langle X, \rho\rangle$, the following are equivalent:*

(a) $\lim d_\rho(x_n, z) = 0$*;*

(b) $\lim \rho(x_n, z) = \rho(z,z)$ *and* $\lim \rho(x_n, z) = \lim \rho(x_n, x_n)$*;*

(c) $(x_n) \to_{\tau_\rho} z$ *and* $\lim \rho(x_n, z) = \lim \rho(x_n, x_n)$.

Proof. Note that all limits are taken with respect to the Euclidean topology on the real line. The equivalence of (a) and (b) follows easily since

$$d_\rho(x_n, z) = (\rho(x_n, z) - \rho(z,z)) + (\rho(x_n, z) - \rho(x_n, x_n))$$

and both terms on the right-hand side of the equation are non-negative by **(ssd)**. The equivalence of (b) and (c) is clear by Proposition 2.10. $\square$

2.6 The local axiom of triangle

In this section we study a certain condition on convergence of sequences in a distance space $\langle X, \rho\rangle$, namely:

$$q_\rho(x, y_n) \to 0,\ q_\rho(y_n, z_n) \to 0 \text{ imply } q_\rho(x, z_n) \to 0, \tag{\textbf{L}}$$

for any two sequences $(y_n), (z_n)$ and and element x from X. The property **(L)** of a distance space is called here the *local axiom of triangle* and if it holds, we call X a *distance space that satisfies local triangle inequality* or a *distance space with* **(L)**.

We give several characterization of the local triangle inequality. We show that the condition implies partial semimetrizability of the underlying topology. We discuss its dependence on the other axioms and the similarity to the triangle inequality. For symmetric spaces in Topology a similar condition was recognised as early as 1910 in the work of Fréchet [Fré10] and further investigated in [Chi17, Chi18]. The local axiom of triangle was defined by Niemytzki [Nie27] and analysed in [Wil31]. A more modern-style proof of metrizability of a topological space, which admits a symmetric satisfying local triangle inequality is given by Archangelskij [Arc66] and explained in detail in [Gru84]. The precise formulation of the Archangelskij metrization theorem follows:

Theorem 2.15 (Archangelskij). *Let the set X be a T_1 topological space symmetrizable with respect to the symmetric d. If for all $x \in X$ and all sequences $(y_n), (z_n) \subseteq X$ we have*

$$d(x, y_n) \to 0 \text{ and } d(y_n, z_n) \to 0 \text{ imply } d(x, z_n) \to 0,$$

then X is metrizable. $\square$

In Section 2.8 we prove a counterpart of the Archangelskij theorem, which works in partial symmetric spaces. Here, we demonstrate some useful characterizations of the local axiom of triangle and its basic implications.

In fact any partial symmetric with property **(L)** is a partial semimetric.

Proposition 2.16. *Let $\langle X, \rho \rangle$ be a partial symmetric space. If the condition* (**L**) *holds in X, then the map ρ is a partial semimetric.*

Proof. For any $x \in X$ denote by $\mathcal{B}_x$ the collection of all ρ-balls centered at x. For all $x \in X$ the family $\mathcal{B}_x$ constitutes a neighbourhood at x if and only if for all $x \in X$ and for all $B \in \mathcal{B}_x$ there exists $C \in \mathcal{B}_x$ such that if $y \in C$, there is some $D \in \mathcal{B}_y$ with $D \subseteq B$. (See for example [Wil70], Theorem 4.5 p.33.) Equivalently, we have

$$\begin{aligned}&\forall x \in X.\ \forall \varepsilon > 0.\ \exists \delta_1 > 0.\ \forall y \in X.\ \exists \delta_2 > 0.\ \forall z \in X.\\ &q_\rho(x, y) < \delta_1,\ q_\rho(y, z) < \delta_2 \text{ imply } q_\rho(x, z) < \varepsilon,\end{aligned}$$

or in other words,

$$\begin{aligned}&\forall x \in X.\ \forall \varepsilon > 0.\ \forall (y_n), (z_m) \subseteq X.\\ &\lim_n q_\rho(x, y_n) = 0,\ \lim_n \lim_m q_\rho(y_n, z_m) = 0 \text{ imply } \lim_m q_\rho(x, z_m) = 0,\end{aligned}$$

which is in general a weaker condition than **(L)**.

Alternative proof: By Corollary 2.9 it is enough to show that for any subset H of X, the set $H' := \{x \mid q_\rho(x, H) = 0\}$ is closed. Suppose not. Then $q_\rho(x, H') = 0$ for some $x \notin H'$. By definition of the distance between a point and a set, there exist sequences $(y_n) \subseteq H'$ and $(z_n) \subseteq H$ such that $q_\rho(x, y_n) \to 0$ and $q_\rho(y_n, z_n) \to 0$. By assumption, $q_\rho(x, z_n) \to 0$. Hence $q_\rho(x, H) = 0$ and consequently, $x \in H'$, a contradiction. □

Proposition 2.17. *For a distance space $\langle X, \rho\rangle$, the following are equivalent:*

(1) *ρ satisfies* (L)*;*

(2) $\forall x \in X.\ \forall \varepsilon > 0.\ \exists \delta > 0.\ \forall y, z \in X.$
$\rho(x, z) < \rho(x, x) + \delta, \rho(z, y) < \rho(z, z) + \delta$ imply $\rho(x, y) < \rho(x, x) + \varepsilon$.

Proof.

(2)

iff

$\forall x \in X.\ \forall \varepsilon > 0.$
$\neg[\exists (y_k), (z_k) \subseteq X.\ \rho(x, z_k) \to \rho(x, x), \rho(z_k, y_k) \to \rho(z, z), \rho(x, y_k) \geq \rho(x, x) + \varepsilon]$,

iff

$\forall x \in X.$
$\neg[\exists (y_k), (z_k) \subseteq X.\ \rho(x, z_k) \to \rho(x, x), \rho(z_k, y_k) \to \rho(z, z), \rho(x, y_k) \nrightarrow \rho(x, x)]$,

iff

$\forall x \in X.\ \forall (y_k), (z_k) \subseteq X.$
$\rho(x, z_k) \to \rho(x, x), \rho(z_k, y_k) \to \rho(z, z)$ imply $\rho(x, y_k) \to \rho(x, x)$

iff

(1).

□

Therefore, we can easily show that every partial metric satisfies the local axiom of triangle. From the following result it is also obvious that the converse claim does not hold in general.

Corollary 2.18. *Let $\langle X, p\rangle$ be a partial metric space. Then the map p satisfies* **(L)**.

Proof. We will prove a more general statement. We claim that a mapping $\rho\colon X \times X \to [0, \infty)$ satisfies $(\Delta^\sharp)$ if and only if

$$\forall x, y, z \in X.\ \forall \varepsilon_1, \varepsilon_2 > 0. \quad \rho(x, z) < \rho(x, x) + \varepsilon_1 \text{ and } \rho(z, y) < \rho(z, z) + \varepsilon_2 \text{ imply } \rho(x, y) < \rho(x, x) + \varepsilon_1 + \varepsilon_2.$$

The formula above implies **(L)** (it can be easily seen by comparing it to the one in Proposition 2.17.(2).)

To prove the claim, let $x, y, z \in X$, $\varepsilon_1, \varepsilon_2 > 0$ and assume the hypothesis of the formula above. Then by the sharp triangle inequality we have

$$\rho(x, y) \leq \rho(x, z) + \rho(z, y) - \rho(z, z) < \rho(x, x) + \varepsilon_1 + \varepsilon_2,$$

as required. Conversely, take any $\varepsilon > 0$. We have $\rho(x, z) < \rho(x, x) + (\rho(x, z) - \rho(x, x) + \varepsilon)$ and $\rho(z, y) < \rho(z, z) + (\rho(z, y) - \rho(z, z) + \varepsilon)$. By assumption, $\rho(x, y) < \rho(x, x) + (\rho(x, z) - \rho(x, x) + \varepsilon) + (\rho(z, y) - \rho(z, z) + \varepsilon) = \rho(x, z) + \rho(z, y) - \rho(z, z) + 2\varepsilon$. Hence the sharp triangle inequality follows from the arbitrariness of ε and the claim is now proved. $\square$

The following result generalizes a similar result for symmetric spaces obtained by H.W. Martin in [Mar72].

Proposition 2.19. *Let $\langle X, \rho \rangle$ be a partial symmetric space. Then the following are equivalent:*

(1) *The map ρ satisfies* **(L)***;*

(2) *If K is compact, H is closed and $K \cap H = \emptyset$, then $q_\rho(K, H) > 0$.*

Proof. (1)⇒(2) We will show that $q_\rho(K, H) = 0$ implies $K \cap H \neq \emptyset$. Suppose $q_\rho(K, H) = 0$. Hence there is a sequence (k_n) of elements of K such that $q_\rho(k_n, H) < 1/2^{n+1}$ for every $n \in \omega$. This means that there also exists a sequence (h_n) of elements from H such that $q_\rho(k_n, h_n) \to 0$. By compactness of K, there exists a convergent subsequence (k_{n_m}) of (k_n) with $k_{n_m} \to x \in K$. Proposition 2.16 guarantees that the map ρ is a partial semimetric and hence $q_\rho(x, k_{n_m}) \to 0$. Then for the corresponding subsequence (h_{n_m}) of h_n we have $q_\rho(k_{n_m}, h_{n_m}) \to 0$. By assumption, $q_\rho(x, h_{n_m}) \to 0$ and hence $x \in \mathbf{cl}H = H$. This means that $K \cap H \neq \emptyset$.

(2)⇒(1) Conversely, for any two sequences $(x_n), (y_n)$ and an element x of X, suppose that $q_\rho(x, x_n) \to 0$ and $q_\rho(x_n, y_n) \to 0$. Define a compact subset K of X and a closed subset H of X in the following way:

$$K := \{x\} \cup \{x_n \mid n \in \omega\},$$

$$H := \mathbf{cl}_{\tau_\rho}\{y_n \mid n \in \omega\}.$$

There are two cases to consider: either $K \cap H = \emptyset$ or $K \cap H \neq \emptyset$. In the former case, by assumption (2), we infer that there exists $\varepsilon > 0$ such that $q_\rho(K, H) > \varepsilon$. Therefore,

$$\inf_n\{q_\rho(x_n, y_n)\} \geq q_\rho(K, H) > \varepsilon,$$

which is a impossible since $q_\rho(x_n, y_n) \to 0$. In the latter case we distinguish three simple sub-cases. If no x_n's belong to $K \cap H$, then $x \in K \cap H$ and hence $(y_n) \to_{\tau_\rho} x$, which by Proposition 2.10 is equivalent to saying that $q_\rho(x, y_n) \to 0$. If infinitely many x_n's belong to $K \cap H$, then since H is closed, also $x \in K \cap H$ and then by the same argument as above, $q_\rho(x, y_n) \to 0$. Finally, if a finite number of x_n's belong to $K \cap H$, say $x_1, ..., x_{k-1} \subseteq K \cap H$, then consider the remaining sequence $(x_n)_{n \geq k}$ and repeat the proof with $K := \{x\} \cup \{x_n \mid n \geq k\}$. $\square$

2.7 Syntopology in distance spaces

Let $\langle X, \rho \rangle$ be a distance space. For any $A, C \in \mathcal{P}(X)$ and $\varepsilon > 0$ let us introduce the following relation

$$A <_\varepsilon C \text{ iff } \exists V \in \tau_\rho.\ A \subseteq V \subseteq B_\rho(V, \varepsilon) \subseteq C. \tag{2.2}$$

For any $x \in X$, we abbreviate $\{x\} <_\varepsilon A$ to $x <_\varepsilon A$. Note that in any partial semimetric space we have $x \in U \in \tau_\rho$ if and only if $x <_\varepsilon U$ for some $\varepsilon > 0$.

In fact, the above system of stratified relations has many interesting structural properties, which are collected in the definition of *syntopology* introduced by Császár [Csá63] and exploited in detail by Smyth [Smy94].

Definition 2.20. *A* syntopology*, or* syntopological space *is a set X together with a family $\{<_i \mid i \in I\}$ of binary relations over $\mathcal{P}(X)$, such that each relation $<_i$ satisfies the following conditions:*

(O_1) $\emptyset <_i \emptyset, X <_i X$,

(O_2) $A <_i B$ *implies* $A \subseteq B$,

(O_3) $A \subseteq A' <_i B' \subseteq B$ *implies* $A <_i B$,

(P) $\forall \mathcal{A} \subseteq \mathcal{P}(X).\ \forall A \in \mathcal{A}.\ A <_i B$ *implies* $\bigcup \mathcal{A} <_i B$,

(Q) $A <_i B$ *and* $A <_i B'$ *implies* $A <_i B \cap B'$,

(S) $\forall i, j \in I.\ \exists k \in I.\ <_k$ *is finer than* $<_i$ *and* $<_j$,

(int) $\forall i \in I.\ \exists j \in I.\ \forall A, C \in \mathcal{P}(X).\ \exists D \in \mathcal{P}(X).\ A <_i C$ *implies* $A <_j D <_j C$.

For any syntopology $\mathcal{S} = (<_i)_{i \in I}$ we have the natural topology $\tau_\mathcal{S}$ given by:

$$U \in \tau_\mathcal{S} \text{ iff } \forall x \in U.\ \exists i \in I.\ \{x\} <_i U.$$

Definition 2.21. *We will say that a distance space X is* interpolative *whenever the syntopological axiom* **(int)** *is satisfied, where the index set I ranges over positive real numbers.*

Proposition 2.22. *An interpolative distance space gives rise to a syntopology with the family of binary relations $\{<_\varepsilon \mid \varepsilon > 0\}$ over $\mathcal{P}(X)$ defined as in formula* (2.2)*. The topology $\tau_{\mathcal{S}}$ induced by the syntopology agrees with the distance topology τ_ρ.* $\square$

Note that the agreement of the topologies implies that any interpolative distance space satisfies the following condition:

$$\forall x \in X.\ \forall x \in U \in \tau_\rho.\ \exists \varepsilon > 0.\ \exists V \in \tau_\rho.\ x \in V \subseteq B_\rho(V, \varepsilon) \subseteq U. \tag{\textbf{U}}$$

The following is now immediate:

Proposition 2.23. *If a distance space $\langle X, \rho \rangle$ is interpolative, then the map ρ is a partial semimetric with the local triangle inequality.* $\square$

Proof. Suppose that $K \subseteq U$ for some compact and open subsets of X, respectively. By **(U)**, we can choose a family of open sets $\{V_k \mid k \in K\}$ with $K \subseteq \bigcup_{k \in K} V_k \subseteq \bigcup_{k \in K} B_\rho(V_k, \varepsilon_k) \subseteq U$. Hence, $K \subseteq \bigcup_{i=0}^n V_{k_i} \subseteq B_\rho(\bigcup_{i=0}^n V_{k_i}, \min \varepsilon_{k_i}) \subseteq U$, by compactness of K. Therefore, $B_\rho(K, \varepsilon) \subseteq U$ for $\varepsilon := \min \varepsilon_{k_i}$. $\square$

It is worth noting that the conditions **(U)** and **(L)** agree on any continuous poset with a measurement, where τ_ρ is chosen to be the Scott topology (see Chapter 3, page 43).

Unfortunately, in the lack of other axioms, the meaning of **(int)** in interpolative distance spaces is very unclear. On the other hand, **(int)** always holds in the presence of the sharp triangle inequality **($\Delta^\sharp$)** and hence partial metrics are important examples of interpolative distance.

Proposition 2.24. *Partial metrics are interpolative.*

Proof. Let X be a set with a partial metric $\rho\colon X \times X \to [0, \infty)$. By [Hec99], Proposition 2.11, balls in X are open in τ_ρ. Using the sharp triangle inequality one can prove that for all $\varepsilon > 0$ and all $U \in \tau_\rho$, there exists $\eta > 0$ with $B_\rho(B_\rho(U, \eta), \eta) \subseteq B_\rho(U, \varepsilon)$ and the sets involved are open by the remark above. Consequently, it follows that $B_\rho(B_\rho(B_\rho(U, \eta), \delta), \delta) \subseteq B_\rho(U, \varepsilon)$ for some $\delta > 0$. Since we can assume $\eta > \delta$, we have $B_\rho(B_\rho(B_\rho(U, \delta), \delta), \delta) \subseteq B_\rho(U, \varepsilon)$ Now, if $A, C \in \mathcal{P}(X)$ with $A <_\varepsilon C$, that is, $A \subseteq B_\rho(U, \varepsilon) \subseteq C$, then setting $D := B_\rho(B_\rho(U, \delta), \delta)$ gives $A <_\delta D <_\delta C$, as required. $\square$

In fact, open balls in a partial metric space form a base of a *quasi-uniformity* on X. Recall (cf. [FL82]) that a quasi-uniformity on a set X is a filter $\mathcal{U}$ of binary relations on X such that

1. Every element of $\mathcal{U}$ contains the diagonal.

2. $\forall U \in \mathcal{U}.\ \exists V \in \mathcal{U}.\ V \circ V \subseteq U$, where $\circ$ is the usual relational product.

Now, if X is equipped with a partial metric $p\colon X \times X \to [0, \infty)$, then the collection $\mathcal{B} = \{B_p(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$ is a basis for $\mathcal{U}$. Observe that the second defining condition of $\mathcal{U}$ is equivalent to

$$\begin{aligned}&\forall \varepsilon > 0.\ \exists \delta > 0.\ \forall x, y, z \in X.\\ &p(x, z) < p(x, x) + \delta, p(z, y) < p(z, z) + \delta \text{ imply } p(x, y) < p(x, x) + \varepsilon,\end{aligned} \tag{2.3}$$

which differs from the local triangle inequality only by the order of quantifiers. We repeat **(L)** below for comparison:

$$\begin{aligned}&\forall x \in X.\ \forall \varepsilon > 0.\ \exists \delta > 0.\ \forall y, z \in X.\\ &p(x, z) < p(x, x) + \delta, p(z, y) < p(z, z) + \delta \text{ imply } p(x, y) < p(x, x) + \varepsilon.\end{aligned}$$

Therefore, in arbitrary partial semimetric spaces with **(L)**, the filter basis $\mathcal{B}$, as defined above, may not be a basis for a quasi-uniformity. That is the reason of why we cannot use results about quasi-uniform spaces directly in our considerations of distance spaces.

The syntopological point of view on distance spaces allows us to express some of their desirable properties with help of the system of stratified relations (2.2). For example, (cf. Lemma 3.45) in any partial metric space $\langle X, \rho\rangle$ dual c-balls are closed in the distance topology. We characterize the above mentioned property as follows:

Proposition 2.25. *Let $\langle X, \rho\rangle$ be a distance space. Then the following are equivalent:*

1. $\forall x \in X.\ \forall \varepsilon > 0.\ N^{op}(x, \varepsilon)$ *is closed in* τ.

2. $\forall x, y \in X.\ q_\rho(x, y) = \inf\{\delta > 0 \mid x <_\delta U \in \tau_\rho \Rightarrow y \in U\}$.

Proof. Adapted from [FS]. Let $x, y \in X$ and $\varepsilon > 0$ with $q_\rho(x, y) < \varepsilon$. Suppose that $x <_\varepsilon U$ for some $U \in \tau_\rho$. Then there exists $V \in \tau_\rho$ with $x \in B_\rho(V, \varepsilon) \subseteq U$ and so $y \in B_\rho(x, \varepsilon) \subseteq B_\rho(V, \varepsilon) \subseteq U$. Therefore,

$$\forall x, y \in X.\ \forall \varepsilon > 0.\ \ q_\rho(x, y) \leq \varepsilon \ \text{ implies } \ \varepsilon \in \{\delta > 0 \mid x <_\delta U \Rightarrow y \in U\}. \tag{2.4}$$

Next, suppose that $\varepsilon \in \{\delta > 0 \mid x <_\delta U \Rightarrow y \in U, U \in \tau_\rho\}$. For the sake of contradiction, assume that $q_\rho(x,y) > \varepsilon$. This implies that $x \in X \setminus N^{op}(y,\varepsilon)$. Since $X \setminus N^{op}(y,\varepsilon)$ is open, we have $x <_\varepsilon B_\rho(X \setminus N^{op}(y,\varepsilon),\varepsilon)$. By assumption, $y \in B_\rho(X \setminus N^{op}(y,\varepsilon),\varepsilon)$, a contradiction proving that $q_\rho(x,y) \leq \varepsilon$. Hence, we have shown that

$$\forall x, y \in X.\ \forall \varepsilon > 0.\ \varepsilon \in \{\delta > 0 \mid x <_\delta U \Rightarrow y \in U\} \ \text{ implies } \ q_\rho(x,y) \leq \varepsilon. \tag{2.5}$$

The equations (2.4) and (2.5) together give

$$\forall x, y \in X.\ \forall \varepsilon > 0.\ q_\rho(x,y) = \inf\{\delta > 0 \mid x <_\delta U \Rightarrow y \in U\}, \tag{2.6}$$

as required.

Conversely, assume (2.6). If $y \in X \setminus N^{op}(x,\varepsilon)$, then $q_\rho(y,x) > \varepsilon$. By interpolation, there exists $\varepsilon' > 0$ with $q_\rho(y,x) > \varepsilon' > \varepsilon$. Therefore, $\varepsilon' \notin \{\delta > 0 \mid x <_\delta U \Rightarrow y \in U\}$. Thus, there are open sets $U, V \in \tau_\rho$ with $y \in V \subseteq B_\rho(V,\varepsilon') \subseteq U$ and $x \notin U$. We conclude that $y \in V \subseteq X \setminus N^{op}(x,\varepsilon)$ and hence $N^{op}(x,\varepsilon)$ is closed. □

Therefore, we conclude that

Proposition 2.26. *Every partial metric $\rho\colon X \times X \to [0,\infty)^{op}$ on a set X is of the form:*

$$\forall x, y \in X.\ \rho(x,y) = \inf\{\delta > 0 \mid x <_\delta U \Rightarrow y \in U\} + \rho(x,x). \tag{2.7}$$

Proof. Use Proposition 2.25 together with Lemma 3.45, page 70. □

Remark A natural notion of completion of a syntopological space was proposed in [Smy94] and is known in the literature as the *Smyth-completion* (or *S-completion*). We do not study Smyth's construction in the case of general interpolative distance spaces. Instead, we consider the Smyth-completion of partial metrics on continuous posets in Chapter 3.

2.8 Metrizability of the induced symmetric space

In this section we present the first of the two major results of this chapter. We prove a counterpart for the Archangelskij metrization theorem (Theorem 2.15) working in partial symmetric spaces.

Lemma 2.27. *The induced symmetric space $\langle X, d_\rho\rangle$ satisfies* **(L)** *whenever $\langle X, \rho\rangle$ does.*

Proof. Suppose that for some sequences $(y_n), (z_n) \subseteq X$ and an element $x \in X$ we have $\lim d_\rho(x, y_n) = 0$ and $\lim d_\rho(y_n, z_n) = 0$. Then by Proposition 2.14.(b), one sees that $\lim \rho(x, y_n) = \rho(x, x)$, $\lim \rho(y_n, z_n) = \lim \rho(y_n, y_n)$ and $\rho(x, x) = \lim \rho(z_n, z_n)$. The two former equalities imply that $\lim \rho(x, z_n) = \rho(x, x)$ by **(L)** for the map ρ. Together with the third equality, we get $\lim d_\rho(x, z_n) = 0$ by yet another application of Proposition 2.14. $\square$

Theorem 2.28. *For any partial symmetric space $\langle X, \rho\rangle$ with* **(L)** *the induced topology $\tau_\rho \vee \tau_\rho^{op}$ is metrizable.*

Proof. By Lemma 2.27 and Proposition 2.16, the induced symmetric is a semimetric and satisfies **(L)**.

We will show that $\langle X, d_\rho\rangle$ is Hausdorff. Since it is first-countable by Proposition 2.10, it will be enough to demonstrate that limits of sequences are unique in X. For, let (y_n) be a sequence of elements of X and suppose that (y_n) has two limits x and z in X. Then again by Proposition 2.10 applied to the mapping d_ρ we have $d_\rho(x, y_n) \to 0$ and $d_\rho(z, y_n) \to 0$. Using **(L)** we conclude that $d_\rho(x, z) = 0$. The last equality is equivalent to $\rho(x, z) = \rho(x, x)$ and $\rho(x, z) = \rho(z, z)$. The characterization of the order in partial semimetric spaces from Proposition 2.11 implies that $x \sqsubseteq_{\tau_\rho} z$ and $z \sqsubseteq_{\tau_\rho} x$. Hence, $x = z$ using the T_0 axiom.

Taking all of the proved properties together, one can see that $\langle X, d_\rho\rangle$ is a Hausdorff semimetric space, which satisfies the local axiom of triangle. Therefore, the Archangelskij Theorem applies and we conclude that $\langle X, d_\rho\rangle$ is metrizable. $\square$

2.9 Quasi-developability of a distance space

As an application of the metrizability theorem proved in the last section, we consider the problem of quasi-developability of partial symmetric spaces. Let us first introduce the necessary terminology.

Let $\langle X, \tau\rangle$ be a topological space, $x \in X$ and $\mathcal{C}$ be any collection of subsets of X. Denote $\mathbf{card}\{C \in \mathcal{C} \mid x \in C\}$ by $\mathrm{ord}(x, \mathcal{C})$.

A sequence $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \ldots$ of collections of open subsets of a topological space $\langle X, \tau\rangle$ is called a *quasi-development* for X provided that if $x \in U \in \tau$, then there exists $n \in \omega$ such that $x \in \mathbf{St}(x, \mathcal{G}_n) \subseteq U$, where $\mathbf{St}(x, \mathcal{G}_n) := \bigcup\{U \in \mathcal{G}_n \mid x \in U\}$. A topological space $\langle X, \tau\rangle$ is *quasi-developable* if it admits a quasi-development.

Proposition 1 of [KV92] states that every partial metric space is quasi-developable. Since we have proved in Corollary 2.18 that every partial metric satisfies **(L)** and noted that the converse does not hold in general, the following theorem improves Künzi and Vajner's result. We adapt the proof idea and notation from [KV92].

Theorem 2.29. *Let $\langle X, \rho\rangle$ be a partial symmetric space with* **(L)**. *Then the topological space $\langle X, \tau_\rho\rangle$ is quasi-developable.*

Proof. For simplicity, denote the induced semimetric by d, instead of the standard d_ρ. For each $k, n \in \omega$, set

$$A_{nk} := \{x \in X \mid \rho(x, x) \in [(k-1)2^{-n}, k2^{-n})\}.$$

Let $\bigcup_{t\in\omega} \mathcal{B}_t$ be a base for the metrizable induced topology τ_d such that each collection $\mathcal{B}_t$ is discrete (that is, every $x \in X$ has an open neighbourhood that intersects at most one element of $\mathcal{B}_t$). For each $t, l \in \omega$, define

$$C_{tl} := \{x \in X \mid B_d(x, 2^{-l}) \text{ intersects at most one element of } \mathcal{B}_t\}.$$

For each $k, l, t \in \omega$ set

$$\mathcal{R}_{klt} := \{\mathbf{int} B_\rho(B \cap C_{tl} \cap A_{(l+1)k}, 2^{-(l+1)}) \mid B \in \mathcal{B}_t\}.$$

We claim that $\bigcup_{k,l,t} \mathcal{R}_{klt}$ is a quasi-development for τ_ρ.

For an arbitrary $x \in X$ and a natural number h, denote by $m := m(x, h)$ the natural number chosen according to Proposition 2.17 (that is, for $x \in X$ and $\varepsilon := 2^{-h}$ we take $\delta := 2^{-m}$). By Proposition 2.16, the map ρ is a partial semimetric. Therefore, $x \in \mathbf{int} B_\rho(x, 2^{-m})$. Using the fact that $\tau_\rho \subseteq \tau_d$, there exists $t_0 \in \omega$ such that for some $B_0 \in \mathcal{B}_{t_0}$ we have $x \in B_0 \subseteq \mathbf{int} B_\rho(x, 2^{-m}) \subseteq B_\rho(x, 2^{-m})$. Furthermore, since $\mathcal{B}_{t_0}$ is discrete, there is $l_0 \in \omega$ with $l_0 \geq m$ such that

$$B_d(x, 2^{-l_0}) \cap B \neq \emptyset \text{ and } B \in \mathcal{B}_{t_0} \text{ imply } B = B_0. \tag{†}$$

Finally, there exists $k_0 \in \omega$ such that $x \in A_{(l_0+1)k_0}$.

Note first that $x \in B_0 \cap C_{t_0 l_0} \cap A_{(l_0+1)k_0}$. Hence

$$x \in \mathbf{int} B_\rho(x, 2^{-(l_0+1)}) \subseteq \mathbf{int} B_\rho(B_0 \cap C_{t_0 l_0} \cap A_{(l_0+1)k_0}, 2^{-(l_0+1)}).$$

We claim that

$$B_\rho(B_0 \cap C_{t_0 l_0} \cap A_{(l_0+1)k_0}, 2^{-(l_0+1)}) \subseteq B_\rho(x, 2^{-h}),$$

which will show that $\bigcup_{k,l,t} \mathcal{R}_{klt}$ is a basis for τ_ρ. Let $y \in B_\rho(B_0 \cap C_{t_0 l_0} \cap A_{(l_0+1)k_0}, 2^{-(l_0+1)})$. Then $y \in B_\rho(B_0, 2^{-(l_0+1)})$. That is, $\rho(z, y) < \rho(z, z) + 2^{-(l_0+1)}$ for some $z \in B_0$. Hence $\rho(z, y) < \rho(z, z) + 2^{-m}$. On the other hand, since $B_0 \subseteq B_\rho(x, 2^{-m})$, we have $\rho(x,z) < \rho(x,x) + 2^{-m}$. From the last two inequalities we conclude that $\rho(x,y) < \rho(x,x) + 2^{-h}$, again using Proposition 2.17. That is, we have shown that $y \in B_\rho(x, 2^{-h})$.

It remains to prove that $\mathbf{ord}(x, \mathcal{R}_{k_0(l_0+1)t_0}) = 1$. Let $x \in B_\rho(B_0 \cap C_{t_0(l_0+1)} \cap A_{(l_0+2)k_0}, 2^{-(l_0+2)})$. Then there is $y \in B_0 \cap C_{t_0(l_0+1)} \cap A_{(l_0+2)k_0}$ such that $\rho(y,x) < \rho(y,y) + 2^{-(l_0+2)}$.

We will demonstrate that

$$\rho(x,y) < \rho(x,x) + 2^{-(l_0+1)}.$$

If $\rho(x,x) > \rho(y,y)$, then $\rho(x,y) = \rho(y,x) < \rho(x,x) + 2^{-(l_0+1)}$, using the symmetry of ρ. Otherwise, since $x, y \in A_{(l_0+2)k_0}$, we have $\rho(y,y) < \rho(x,x) + 2^{-(l_0+2)}$ and so $\rho(x,y) = \rho(y,x) < \rho(y,y) + 2^{-(l_0+2)} < \rho(x,x) + 2^{-(l_0+1)}$.

Hence $d(x,y) = 2\rho(x,y) - \rho(x,x) - \rho(y,y) < 2^{-l_0}$, which means that $y \in B_d(x, 2^{-l_0}) \cap B$. By (†), we thus have $B = B_0$. We have shown that $\mathbf{ord}(x, \mathcal{R}_{k_0(l_0+1)t_0}) = 1$. Now, it is immediate to see that the collection $\bigcup_{k,l,t} \mathcal{R}_{klt}$ is a quasi-development for τ_ρ. □

2.10 The self-distance mapping in distance spaces

The self-distance map $\mu_\rho \colon X \to [0,\infty)^{op}$ (called also a *weight function*) associated with a partial semimetric ρ proves to be an important object to study. We start with some basic properties of the mapping. Later, we will see that in Domain Theory weight functions correspond to certain *measurements* of Martin [Mar00]. In Section 3.2.5 we use measurements to build partial metrics on algebraic domains.

Note that the codomain of the self-distance map is the set of non-negative real numbers with the opposite to the natural order.

In this section we consider the interplay between self-distance maps and the specialisation order in partial semimetric spaces.

Proposition 2.30. *The self-distance map associated with a partial semimetric ρ is monotone and strictly monotone with respect to specialisation orders of its domain and codomain.*

Proof. For any $x, y \in X$, $x \sqsubseteq_{\tau_\rho} y$ is equivalent to $\rho(x,y) = \rho(x,x)$ by Proposition 2.11. But by **(ssd)**, $\rho(y,y) \leq \rho(x,y) = \rho(x,x)$. That is, $\mu_\rho y \leq \mu_\rho x$. Hence the map μ_ρ is monotone.

Here, strict monotonicity is the condition

$$\forall x, y \in X.\ (x \sqsubseteq_\rho y \text{ and } \mu_\rho x = \mu_\rho y) \text{ imply } x = y$$

and one can note that this condition is an equivalent formulation of the T_0 axiom of the space (using Proposition 2.11). $\square$

For the self-distance map μ_ρ associated with a partial symmetric ρ define

$$\mu_\rho(x,\varepsilon) := \{y \in X \mid y \sqsubseteq_{\tau_\rho} x \wedge \mu_\rho y < \mu_\rho x + \varepsilon\}. \tag{2.8}$$

We say that $\mu(x,\varepsilon)$ is the set of elements of X which are *ε-close to x*.

Lemma 2.31. *Let $\langle X, \rho\rangle$ be a partial semimetric space. Then*

$$\forall \varepsilon > 0.\, \forall x \in P. \quad \mu_\rho(x,\varepsilon) \subseteq B_\rho(x,\varepsilon).$$

Proof. Suppose $z \in \mu_\rho(x,\varepsilon)$. Since $z \sqsubseteq_\rho x$, we have $\rho(x,z) = \mu_\rho z$ by semimetrizability. Therefore, $\rho(x,z) = \mu_\rho z < \mu_\rho x + \varepsilon$. That is, $z \in B_\rho(x,\varepsilon)$. $\square$

Proposition 2.32. *Let $\langle X, \rho\rangle$ be a partial semimetric space, $x \in X$ and S be any subset of X. If $s \sqsubseteq_\rho x$ for all $s \in S$ and $\mu_\rho x = \inf\{\mu s \mid s \in S\}$, then x is the supremum of S.*

Proof. Let $x \in U \in \tau_\rho$ and let u be any upper bound of S. Since U is open, there exists $\varepsilon > 0$ such that $x \in B_\rho(x,\varepsilon) \subseteq U$. By assumption, there exists $s \in S$ with $\mu_\rho s < \mu_\rho x + \varepsilon$. Since $s \sqsubseteq_\rho x$, $s \in \mu_\rho(x,\varepsilon)$. By Lemma 2.31, $s \in B_\rho(x,\varepsilon)$ and so $s \in U$. But the latter set is upper, and therefore $u \in U$. We have shown that for any $U \in \tau_\rho$, $x \in U$ implies $u \in U$ and hence $x \sqsubseteq_\rho u$ follows. This means that $x = \bigsqcup S$. $\square$

Many more results can be obtained if we consider a self-distance mapping in a space, which is a continuous poset with respect to the specialisation order and such that the distance topology coincides with the Scott topology. In this particular case, the weight map is a measurement, which induces the Scott topology everywhere, in the language of Martin [Mar00].

2.10.1 Stability condition for partial semimetrics

It happens that there exists a class of partial semimetric spaces, where the distance topology can be recovered from the self-distance map. Continuous posets in their Scott topology are our major examples of such spaces. We develop the basics here and continue the theme in Chapter 3.

Definition 2.33. *Let $\langle X, \tau_\rho\rangle$ be a partial symmetric space. We say that the map ρ is* stable *if for all $x, y \in X$ we have*

$$\rho(x,y) := \inf\{\mu_\rho z \mid z \sqsubseteq_{\tau_\rho} x, y\}.$$

Lemma 2.34. *Let $\langle X, \rho\rangle$ be a partial semimetric space. Then*

$$\forall\varepsilon > 0.\, \exists\delta > 0.\, \forall x \in P. \quad \mu_\rho(x, \delta) \subseteq \mathbf{int}(B_\rho(x, \varepsilon)).$$

Proof. Let $x \in X$ and $\varepsilon > 0$. Then by definition, there exists $\delta > 0$ such that $x \in B_\rho(x, \delta) \subseteq \mathbf{int}(B_\rho(x, \varepsilon))$. By Lemma 2.31, $\mu_\rho(x, \delta) \subseteq \mathbf{int}(B_\rho(x, \varepsilon))$. □

Lemma 2.35. *Let $\langle X, \rho\rangle$ be a stable partial semimetric space. Then for every $x \in X$ and $\varepsilon > 0$ we have*

$$B_\rho(x, \varepsilon) \subseteq {\uparrow}\mu_\rho(x, \varepsilon),$$

where ${\uparrow}\mu_\rho(x, \varepsilon) := \{y \in X \mid \exists z \in \mu_\rho(x, \varepsilon).\ z \sqsubseteq_\rho y\}$.

Proof. Let $x \in X$, $\varepsilon > 0$ and $y \in B_\rho(x, \varepsilon)$. Then $\rho(x, y) < \mu_\rho x + \varepsilon$. By stability, there exists $z \sqsubseteq_\rho x, y$ with $\mu_\rho z < \mu_\rho x + \varepsilon$ and hence $z \in \mu_\rho(x, \varepsilon)$. Then, $y \in {\uparrow}\mu_\rho(x, \varepsilon)$ follows. □

Theorem 2.36. *Let (X, ρ) be a stable partial semimetric space. Then*

$$\{{\uparrow}\mu_\rho(x, \varepsilon) \mid \varepsilon > 0\}$$

is a neighbourhood base at x in τ_ρ.

Proof. Let $x \in X$ and $\varepsilon > 0$. Then $B_\rho(x, \varepsilon) = {\uparrow}\mu_\rho(x, \varepsilon)$ by Lemmata 2.31 and 2.35. □

Chapter 3

Distance for continuous domains

Over recent years, a number of attempts have been made to equip semantic domains with a notion of distance between points. The goal of this research has been twofold: firstly, to unify order-theoretic structures with metric spaces as semantic domains, and secondly, to provide a mechanism for making *quantitative* statements about programs, such as speed of convergence or complexity of algorithms. Following the second aim, the field is now known as *Quantitative Domain Theory.*

Research in this area has led to a number of concepts which generalise the classical notion of a metric space to an ordered setting. For example, Smyth [Smy88] introduced quasi-metrics, which by virtue of being non-symmetric encode topology and order on a domain at the same time. Partial metrics were introduced by Matthews [Mat92] and studied further by O'Neill [O'N95a, O'N95b, O'N98]. Although they are symmetric, they are still capable of capturing order. This work was further extended by Heckmann in [Hec99], leading to a concept of weak partial metrics.

More recently, Keye Martin introduced the idea of a *measurement* on a domain [Mar00]. At first glance, measurements are quite different from distance functions because they take only one argument. The original motivation of Martin was to make precise (in a quantitative way) the idea that elements at the top of a domain are totally specified whereas everything else is partial or an approximation. Consequently, he speaks of measuring the "indefiniteness" of a domain element, and seeks functions which assign zero exactly to the maximal elements. For reasons, which will become clear later on, we call this requirement the "kernel condition". Martin's work deals seriously with the second aim mentioned above, and he has undertaken several case studies of evaluating concrete algorithms [Mar00, Marc].

Both measurements and partial metrics are sound candidates for quantitative measures on domains. However, they are so for different reasons.

Measurements introduce quantity in a simple, intuitive and natural way. Definitions are supported firmly by many examples from Domain Theory; Martin shows that it is possible to capture the notion of an algorithm as a map between domains and in this scenario measurements are used to reason about the complexity of algorithms [Mar01]. On the other hand, partial metrics seem to be the most obvious extension of metrics to the "non-Hausdorff world". They retain symmetry and a form of triangle inequality, which refines the classical one. In fact, thanks to a theorem by Matthews [Mat92], we know that partial metrics correspond precisely to *weighted quasi-metrics*, which is a proper subclass of the horrendously large class of quasi-metrics. Partial metrics, unlike general quasi-metrics seem to be especially well-suited for a distance on domains since we can ask for a form of "kernel condition" to hold, namely, we require that the maximal elements of a domain are exactly these ones with zero self-distance.

One of the main conceptual achievements of this thesis is a demonstration that measurements on continuous domains are in one-to-one correspondence with partial semimetrics for the Scott topology. Further, we observe that respective subclasses of measurements and partial semimetrics also coincide. The following table summarizes the situation (all distance maps are assumed to induce the Scott topology):

Notion	1-1 corresponds to:	Domain	Section
measurement	Scott-continuous partial semimetric	continuous poset	3.2.1
Lebesgue measurement	Scott-continuous Lebesgue partial semimetric	continuous poset	3.2.2
weakly modular measurement	stable partial metric	continuous poset	3.2.3
Lebesgue measurement	partial metric	algebraic dcpo	3.2.5

Our observations have interesting implications for Quantitative Domain Theory. First of all, we no longer face a choice between measurements and partial metrics as the latter ones are just more specialised manifestations of the former. On the other hand, certain measurements can now be seen as vehicles, which carry enough information to recover a partial metric on the underlying domain. This point of view proves especially useful when we want to build partial metrics on product and function spaces: it is enough to perform the respective construction on corresponding measurements, which is a considerably simpler task (see next chapter). Finally, the interplay between the two quantitative notions offers a new perspective on both subjects. For example, we are now able to see continuous domains with Lebesgue measurement as metric spaces (recall Theorem 2.28, page 38), or, we can show that partial metrics exist on arbitrary countably based continuous dcpos (Theorem 3.71).

Having established the correspondence between measurements and partial semimetrics, we turn to a study of existence of certain partial metrics on domains. Our basic requirement is that the *partial metric topology agrees with the Scott topology* on the domain. In addition, we demand that *the induced metric is complete* and that the derived *measurement satisfies the kernel condition*. It appears that this ideal situation is rare but nevertheless we can establish it for the classes of ω-continuous lattices (Theorem 3.72) and Scott domains (Theorem 3.64).

Assuming just two of the three above mentioned requirements, we get some interesting existence results. Firstly, we prove that every countably based algebraic domain which satisfies the Lawson condition [Law97b] admits a partial metric for the Scott topology and a measurement with the kernel condition (Theorem 3.63). On the other hand, dropping the requirement on the measurement, we demonstrate that every countably based algebraic domain admits a partial metric such that the induced metric topology agrees with the Lawson topology (Theorem 3.69). In this case, Lawson-compactness of the domain implies completeness of the induced metric. Finally, generalizing the last construction from algebraic to the continuous case, we show that the induced metric topology may no longer agree with the Lawson topology but is complete whenever the domain is bounded-complete (Theorem 3.71).

3.1 Background

3.1.1 Domain Theory

We review some basic notions from domain theory, mainly to fix the language and notation. See [AJ94] for more information.

Posets

Let P be a poset. A pair of elements $x, y \in P$ is *consistent* (*bounded*), denoted $x{\uparrow}y$, if there exists an element $z \in P$ such that $z \sqsupseteq x, y$. The contrary case is written as $x\#y$. We say that a poset is *bounded-complete* if each finite, bounded set of elements has a supremum. In particular, a non-empty bounded-complete poset P has a least element, which arises as a supremum of the empty set. A subset $A \subseteq P$ of P is *directed* if it is nonempty and any pair of elements of A has an upper bound in A. If a directed set A has a supremum, it is denoted $\bigsqcup^{\uparrow} A$. A poset P in which every directed set has a supremum is called a *dcpo*. A dcpo with a least element is bounded-complete iff every nonempty subset has an infimum.

Approximation

Let x and y be elements of a poset P. We say that x *approximates* (*is way-below*) y if for all directed subsets A of P, $y \sqsubseteq \bigsqcup^{\uparrow} A$ implies $x \sqsubseteq a$ for some $a \in A$. We denote it as $x \ll y$. If $x \ll x$ then x is called a *compact* element. The subset of compact elements of a poset P is denoted $K(P)$. Now, $\mathord{\downarrow\!\!\!\downarrow} x$ is the set of all approximants of x below it. $\mathord{\uparrow\!\!\!\uparrow} x$ is defined dually. We say that a subset B of a poset P is a (*domain-theoretic*) *basis* for P if for every element x of P, the set $\mathord{\downarrow\!\!\!\downarrow} x \cap B$ is directed with supremum x. A poset is called *continuous* if it has a basis. One can show that a poset P is continuous iff $\mathord{\downarrow\!\!\!\downarrow} x$ is directed with supremum x, for all $x \in P$. A poset is called a *domain* if it is a continuous dcpo. Note that $K(P) \subseteq B$ for any basis B of P. If $K(P)$ is itself a basis, the poset P is called *algebraic.* If a poset admits a countable basis, we say that it is ω-continuous (or ω-algebraic providing that $K(P)$ is a countable basis for P). A *Scott domain* is a bounded-complete ω-algebraic dcpo with a least element.

The poset $[0, \infty)^{op}$ figures prominently in this thesis. It is a domain without a least element. We use $\sqsubseteq$ to refer to its order, which is dual to the natural one, $\leq$, but we usually prefer to work with the latter.

Abstract bases

Every basis in a continuous poset is an example of a so called abstract basis:

Definition 3.1. *An* abstract basis *is given by a set B together with a transitive relation $\prec$ on B, such that for all $a \in B$ and finite subsets M of B the following axiom holds:*

$$M \prec a \text{ implies } \exists b \in B.\ M \prec b \prec a. \qquad \textbf{(INT)}$$

For an abstract basis $\langle B, \prec \rangle$ and an element $b \in B$ set

$$b^* := \{a \in B \mid a \prec b\}.$$

In this thesis, an abstract basis such that the relation $\prec$ is reflexive is named a *reflexive abstract basis*. For any algebraic domain P, the set $K(P)$ is an example of a reflexive abstract basis.

For an abstract basis $\langle B, \prec \rangle$ let $\mathfrak{I}(B)$ be the set of all ideals (directed, lower subsets) ordered by inclusion. It is called the *rounded ideal completion* of B.

Theorem 3.2 ([AJ94, Law97a]). *Let $\langle B, \prec\rangle$ be an abstract basis.*

1. *$\mathcal{I}(B)$ is a continuous dcpo;*
2. *The set $\{a^* \mid a \in B\}$ is a domain-theoretic basis for $\mathcal{I}(B)$;*
3. *$x \ll z$ in $\mathcal{I}(B)$ iff $\exists b \in z.\ x \prec b$;*
4. *$\forall a, b \in B.\ a \prec b$ implies $a^* \ll b^*$;*
5. *$\forall x \in \mathcal{I}(B).\ \forall a \in B.\ a \in x$ implies $a^* \ll x$.*

Proof. We verify only the assertion (5), which does not appear in the quoted literature. Suppose $a \in x$ for some $a \in B$ and $x \in \mathcal{I}(B)$. Since $x = \bigcup^{\uparrow}_{c \in x} c^*$, there exists a $c \in x$ such that $a \prec c$. This implies $a^* \ll c^*$ by (4). Therefore, $a^* \ll x$. $\square$

For any algebraic dcpo P the rounded ideal completion of $\langle K(P), \sqsubseteq\rangle$ is isomorphic to P, in symbols: $\mathcal{I}(K(P)) \cong P$.

One of the consequences of the interpolation property **(INT)** is the fact that sets

$$b_* := \{a \in B \mid b \prec a\} \text{ for } b \in B$$

form a basis for a topology on B called the *pseudoScott* topology.

We will continue the study of abstract bases in Section 4.2.2, page 101.

Intrinsic topologies

A subset $U \subseteq P$ of a poset P is *upper* if $x \sqsupseteq y \in U$ implies $x \in U$. Upper sets inaccessible by directed suprema form a topology called the *Scott topology*; it is denoted $\sigma(P)$ (or σ for short). A continuous poset P admits a countable domain-theoretic basis iff its Scott topology is second countable. The Scott topology encodes the underlying order: $x \sqsubseteq y$ in P iff $\forall U \in \sigma.\ (x \in U \Rightarrow y \in U)$. This is the general definition of the so-called *specialisation order* for a topology. The collection $\{\twoheaduparrow x \mid x \in P\}$ forms a basis for the Scott topology on a continuous poset P. The topology satisfies only weak separation axioms: It is always T_0 on a poset but T_1 only if the order is trivial. For an introduction to T_0 spaces, see [Hec90]. An excellent general reference on Topology is [Eng89].

Two further intrinsic topologies on a continuous poset P are: The *weak topology* $\omega(P)$ generated by the collection $\{P \setminus \uparrow x \mid x \in P\}$ and the *Lawson topology* defined as $\lambda(P) := \sigma(P) \vee \omega(P)$, the join of the Scott topology and the weak topology in the lattice of topologies on P. It has a basis of the form $\{\twoheaduparrow x \setminus \uparrow F \mid x \in P, F \subseteq_{fin} P\}$ on any continuous poset P. The Lawson topology is Hausdorff and for ω-continuous posets it is separable metrizable [GHK⁺80].

The fourth intrinsic topology, which will play a role in this thesis is the *Martin topology* and will be defined in Section 3.1.3.

3.1.2 Partial metrics

Although we have implicitly learned much about partial metrics from Chapter 2 already, we will briefly review basic definitions and facts here. An introduction to partial metric theory can be also found in Heckmann [Hec99], Matthews [Mat92] and O'Neill [O'N95a, O'N95b, O'N98].

For ease of reference we recall that a *partial metric* on a set X is a map $p\colon X \times X \to [0, \infty)$ which satisfies for all $x, y, z \in X$,

1. $p(x, y) = p(y, x)$ **(symm)**.
2. $p(x, y) = p(x, x) = p(y, y)$ implies $x = y$ **($\mathbf{T_0}$)**.
3. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ $(\Delta^\sharp)$.
4. $p(x, x) \leq p(x, y)$ **(ssd)**.

If we abandon Axiom 4, the map p is called a *weak partial metric*. From the topological point of view, weak partial metrics and partial metrics are equivalent since for every weak partial metric p there is a corresponding one which satisfies **(ssd)** and induces the same topology. It is given by $p^{'}(x, y) := \max\{p(x, y), p(x, x), p(y, y)\}$.

Order

The following are equivalent for all pairs of elements x, y of a partial metric space X:

1. $x \sqsubseteq_{\tau_p(X)} y$.
2. $p(x, y) = p(x, x)$.
3. $\forall \varepsilon > 0.\ y \in B^p(x, \varepsilon)$.

We will say $x \sqsubseteq_p y$ if one of the above conditions holds.

Distance maps

A partial metric p induces various distance maps. A first comprehensive description of them appears in Matthews [Mat92], where the following two *correspondence theorems* are stated as well:

Theorem 3.3 (Matthews). *For every partial metric $p\colon X \times X \to [0,\infty)$ the map $d_p\colon X \times X \to [0,\infty)$ given by*

$$d_p(x,y) := 2p(x,y) - p(x,x) - p(y,y)$$

is a metric with $\tau_p(X) \subseteq \tau_d(X)$, which is weighted*, that is,*

$$\forall x, y \in X.\ \ d_p(x,y) \geq \mu_p x - \mu_p y,$$

where $\mu\colon X \to [0,\infty)$ is the self-distance map:

$$\mu_p x := p(x,x).$$

Conversely, every metric $d\colon X \times X \to [0,\infty)$ weighted by a map $\mu\colon X \to [0,\infty)^{op}$ gives rise to a partial metric $p_{d,\mu}\colon X \times X \to [0,\infty)$ by

$$p_{d,\mu}(x,y) := \frac{1}{2}(d(x,y) + \mu x + \mu y)$$

such that $d_{p_{d,\mu}} = d$ and $\mu_{p_{d,\mu}} = \mu$. $\square$

Theorem 3.4 (Matthews). *For every partial metric $p\colon X \times X \to [0,\infty)$ the mapping $q_p\colon X \times X \to [0,\infty)$ given by*

$$q_p(x,y) := p(x,y) - p(x,x)$$

is a quasi-metric such that $\tau_p(X) = \tau_{q_p}(X)$, which is weighted *by the self-distance of the map p in the sense that*

$$\forall x, y \in X.\ \ q_p(x,y) + \mu_p x = q_p(y,x) + \mu_p y.$$

Conversely, any quasi-metric $q\colon X \times X \to [0,\infty)$ weighted by a map $\mu\colon X \to [0,\infty)^{op}$ gives rise to a partial metric $p_{d,\mu}\colon X \times X \to [0,\infty)$ by

$$p_{q,\mu}(x,y) := q(x,y) + \mu x$$

with $q_{p_{q,\mu}} = q$ and $\mu_{p_{q,\mu}} = \mu$. $\square$

To summarize, there are three distance maps induced by a partial metric:

1. the *self-distance* map $\mu_p \colon X \to [0, \infty)$ given by

$$\mu_p x := p(x, x),$$

2. the *induced quasi-metric* $q_p \colon X \times X \to [0, \infty)$ given by

$$q_p(x, y) := p(x, y) - p(x, x),$$

3. the *induced metric* $d_p \colon X \times X \to [0, \infty)$ defined as

$$d_p(x, y) := 2p(x, y) - p(x, x) - p(y, y).$$

The following is proved in [Hec99]. The demonstration uses neither axiom **(ssd)** nor $(\mathbf{T_0})$.

Theorem 3.5 (Heckmann). *For every partial metric space $\langle X, p\rangle$, the function $p \colon X \times X \to [0, \infty)^{op}$ is continuous, where X is given the topology induced by p, and $[0, \infty)^{op}$ is endowed with the Scott topology.* $\square$

In later sections we will be mainly interested in the case when the space is a continuous poset equipped with the Scott topology. This special case deserves a separate reference:

Corollary 3.6. *Let P be a continuous poset equipped with a partial metric $p \colon P \times P \to [0, \infty)^{op}$ compatible with the Scott topology. Then the map p is Scott-continuous. Moreover, the self-distance mapping $\mu_p \colon P \to [0, \infty)^{op}$ is Scott-continuous and strictly monotone.*

Proof. Both claims about continuity are obvious. Strict monotonicity follows from Proposition 2.30, page 40. $\square$

3.1.3 Martin's theory

We will now give a brief summary of the main elements of Keye Martin's theory of measurements. Our main reference is [Mar00].

Quantitative approximation

Let P be a poset. For a monotone mapping $\mu \colon P \to Q$ between posets P and Q and any $x \in P$, $\varepsilon \in Q$ we define

$$\mu(x, \varepsilon) := \{y \in P \mid y \sqsubseteq x \wedge \varepsilon \ll \mu y\}.$$

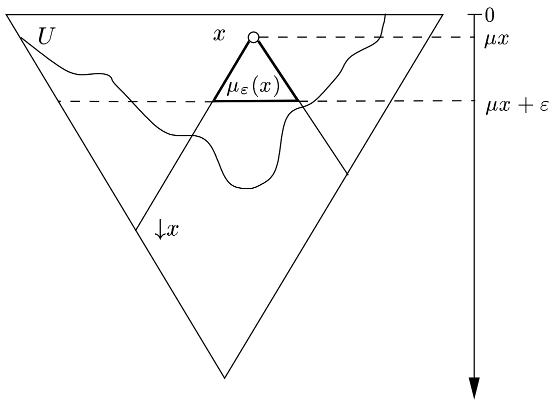


Figure 3.1: The set of elements ε-close to x.

One can readily see that in the case when $Q = [0, \infty)^{op}$ we can write

$$\mu(x, \varepsilon) = \{y \in P \mid y \sqsubseteq x \wedge \mu y < \mu x + \varepsilon\}.$$

We say that $\mu(x, \varepsilon)$ is the set of elements of P which are *ε-close to x*. The map μ can be thought of as a quantitative measure of a relative "distance" between elements in P.

Measurement

We say that a monotone mapping $\mu \colon P \to Q$ between posets P and Q *induces the Scott topology* on a subset X of P if

$$\forall U \in \sigma(P).\ \forall x \in X \cap U.\ \exists \varepsilon \in Q.\ x \in \mu(x, \varepsilon) \subseteq U.$$

We denote it as $\mu \longrightarrow \sigma(X)$.

Note that if $\mu \colon P \to [0, \infty)^{op}$, then the condition $\mu \longrightarrow \sigma(P)$ reduces to

$$\forall U \in \sigma(P).\ \forall x \in U.\ \exists \varepsilon > 0.\ \mu(x, \varepsilon) \subseteq U.$$

Definition 3.7. *Let P be a continuous poset and let $\mu \colon P \to [0, \infty)^{op}$ be a Scott-continuous map. If $\mu \longrightarrow \sigma(P)$, then we will say that μ* measures *P or that μ is a* measurement *on P. If $\mu \longrightarrow \sigma(\mu^{-1}\{0\})$, then we will say that μ is a* kernel measurement *on P.*[1]

We will often find it useful to employ the following characterization of measurements:

[1] Our definition of the measurement differs from the one given by Martin.

Proposition 3.8 (Martin). *A Scott-continuous mapping $\mu\colon P \to [0,\infty)^{op}$ on a continuous poset P is a measurement iff for all $x \in P$ and all sequences (x_n) in P with $(x_n) \ll x$ we have*

$$\lim_{n\to\infty} \mu x_n = \mu x \text{ implies } \bigsqcup x_n = x$$

and this supremum is directed. $\Box$

Define the *kernel of* μ by $\ker\mu := \{x \in P \mid \mu x = 0\}$. The kernel is always a G_δ subset of maximal elements of P and as such is a topologically important object of study. We often seek a measurement on a domain with $\ker\mu = \max P$; this is called the *kernel condition* for measurements.

The composition of a Scott-continuous mapping $\mu\colon P \to Q$ inducing the Scott topology on P and a measurement $\lambda\colon Q \to [0,\infty)^{op}$ is again a measurement. It satisfies the kernel condition whenever both μ and λ do.

Topology

The *Martin topology* arises naturally in the consideration of measurements. On a continuous poset P measured by $\mu\colon P \to [0,\infty)^{op}$, the collection $\{\mu(x,\varepsilon) \mid x \in P, \varepsilon > 0\}$ is a basis for a topology called the *Martin topology* on P denoted $\mu(P)$. One can show that the definition does not depend on the choice of a measurement. Moreover, it so happens that the collection

$$\{\twoheaduparrow x \cap \downarrow y \mid x, y \in P\}$$

is yet another basis for the topology, which shows that it is an intrinsic topology for a continuous poset P.

The Martin topology is always Hausdorff on a continuous poset P. If P is countably based with $\mathrm{card}(\max P) \geq 2^{\aleph_0}$, then the Martin topology is not normal. This fact will be useful later in Section 3.3.

Lastly, the Martin topology is tightly connected to the Scott topology. The single most important fact states that:

Proposition 3.9 (Martin). *For any continuous poset P measured by $\mu\colon P \to [0,\infty)^{op}$, the collection*

$$\{\uparrow\mu(x,\varepsilon) \mid x \in P, \varepsilon > 0\}$$

is a basis for the Scott topology. $\Box$

3.2 Correspondence between measurements and partial metrics

3.2.1 Measurements and partial semimetrics

Definition 3.10 (Martin). *Let P be a continuous poset with a measurement $\mu\colon P \to [0,\infty)^{op}$. The map $p_\mu\colon P \times P \to [0,\infty)^{op}$ defined by*

$$p_\mu(x,y) := \bigsqcup\{\mu z \mid z \ll x,y\} = \inf\{\mu z \mid z \ll x,y\}$$

is the partial semimetric associated with μ *(cf. Proposition 3.11 below).*

Note that the definition is well-formed if any two elements x, y of P are bounded from below. This condition, however, may be omitted: whenever x, y have no lower bound, we scale the measurement to $\mu^*\colon P \to [0,1)^{op}$ with $\mu^*(x) := \mu x/(1+\mu x)$ for any $x \in P$. Such a map is again a measurement with the same kernel (cf. Lemma 5.3.1 of [Mar00], page 135). Now, we define p^*_μ to be:

$$\forall x,y \in P. \quad p^*_\mu(x,y) = \begin{cases} \inf\{\mu^* z \mid z \ll x,y\} & \text{if } \exists z \in P.\ z \sqsubseteq x,y \\ 1 & \text{otherwise} \end{cases}$$

Proposition 3.11. *Let $\mu\colon P \to [0,\infty)^{op}$ be a measurement on a continuous poset P. Then:*

1. [Mar00] *p_μ is a Scott-continuous map from $P \times P$ to $[0,\infty)^{op}$.*
2. [Mar00] *$p_\mu(x,x) = \mu x$ for all $x \in P$.*
3. [Mar00] *$B^{p_\mu}(x,\varepsilon) = \uparrow\mu(x,\varepsilon)$ for all $x \in P$ and $\varepsilon > 0$. That is, p_μ is a stable partial semimetric, which induces the Scott topology.*
4. *A subset $A \subseteq P$ is Scott-closed iff for all $x \in P$, $p_\mu(x,A) = \mu x$ implies $x \in A$.*
5. *For a sequence (x_n) and any $x \in P$, $x_n \to x$ in the Scott topology on P iff $\lim p_\mu(x_n,x) = \mu x$.*

Proof. Note that (1) follows from the definition of p_μ and continuity of P. (2) holds, since μ is Scott-continuous. (3) follows from the definition of p_μ and Proposition 3.9. Note that (4) and (5) are consequences of Corollary 2.9 and Proposition 2.10, page 28, respectively. However, below we give alternative domain-theoretic proofs:

(4) The following is similar to Theorem 4.6 of [Fla97]. If A is Scott-closed and $d(x,A) = \mu x$, then for all $\varepsilon > 0$, for some $a \in A$, we have $p_\mu(x,a) < \mu x + \varepsilon$. Thus, for all $\varepsilon > 0$, $B^{p_\mu}(x,\varepsilon) \cap A \neq \emptyset$. This proves $x \in \mathbf{cl}_\sigma A = A$. Conversely, suppose $x \notin A \Rightarrow p_\mu(x,A) > \mu x$. Let $x \notin A$. Hence $p_\mu(x,A) > \mu x$ and there exists an $\varepsilon > 0$ such that $p_\mu(x,A) \not\leq \mu x + \varepsilon$. This means $B^{p_\mu}(x,\varepsilon) \subseteq X \setminus A$. Hence $X \setminus A$ is Scott-open.

(5) The proof is based on the idea from Corollary 5.3.1 of [Mar00]. Let $\varepsilon > 0$. Choose $z \ll x$ with $\mu z < \mu x + \varepsilon$. Then $\exists n_0.\ z \ll x_n$ for $n \geq n_0$. Thus $z \ll x_n, x$ and hence $p_\mu(x_n,x) \leq \mu z < \mu x + \varepsilon$. Since ε was arbitrary, $\lim p_\mu(x_n,x) = \mu x$. For the converse, let $x \in U \in \sigma(P)$. Then $x \in \mu(x,\varepsilon) \subseteq U$, for some $\varepsilon > 0$. By assumption, $\exists n_0.\ p_\mu(x_n,x) < \mu x + \varepsilon/2$ for $n \geq n_0$. By definition of p_μ, $\exists a_n \ll x_n, x$ such that $\mu a_n < p_\mu(x_n,x) + \varepsilon/2$. Hence $\mu a_n < \mu x + \varepsilon$ for all $n \geq n_0$ and so $a_n \in \mu(x,\varepsilon) \subseteq U$. But now $x_n \in \Uparrow a_n \subseteq U$ for $n \geq n_0$. This proves $x_n \to x$. $\Box$

By virtue of being a partial semimetric, the distance map associated with a measurement encodes order on the underlying domain (cf. Proposition 2.11, page 28). The following is a generalisation of Proposition 2.14 of [Hec99].

Proposition 3.12. *For a continuous poset P the following are equivalent:*

1. $x \sqsubseteq y$.

2. $p_\mu(x,y) = \mu x$.

3. $\forall \varepsilon > 0.\ y \in B_{p_\mu}(x,\varepsilon)$.

4. $\forall z \in P.\ p_\mu(x,z) \geq p_\mu(y,z)$.

Proof. (2)$\Rightarrow$(3) is obvious. (3)$\Rightarrow$(1), if $x \in U \in \sigma(P)$, then $x \in B^{p_\mu}(x,\varepsilon) \subseteq U$ for some $\varepsilon > 0$. But by assumption, also $y \in B_{p_\mu}(x,\varepsilon) \subseteq U$. Hence we have shown that $\forall U \in \sigma(P).\ x \in U \Rightarrow y \in U$, that is, $x \sqsubseteq y$. (1)$\Rightarrow$(4) holds by monotonicity of p_μ. (4)$\Rightarrow$(2) works if we put $z = x$. $\Box$

The following corollary collects some of the properties of p_μ.

Corollary 3.13. *If P is a continuous poset with a measurement $\mu\colon P \to [0,\infty)^{op}$, then:*

1. $p_\mu(x,y) = p_\mu(y,x)$.

2. $p_\mu(x,y) = \mu x = \mu y$ *iff* $x = y$.

3. $\mu x = p_\mu(x,x) \leq p_\mu(x,y)$.

4. $p_\mu(x, y) = 0$ *iff* $x = y \in \ker\mu$. $\square$

Define a poset P to have *consistent lower bounds* if every pair of consistent elements $a, b \in P$ has a lower bound in P. We are ready to state the main result of this section:

Theorem 3.14. *For a continuous poset P the following are equivalent:*

1. *P admits a stable partial semimetric compatible with the Scott topology;*

2. *P has consistent lower bounds and admits a partial semimetric compatible with the Scott topology, which is Scott-continuous as a map from $P \times P$ to $[0, \infty)^{op}$;*

3. *P has consistent lower bounds and admits a measurement.*

Proof. The implication (1)⇒(2) is trivial. For (2)⇒(3), the self-distance mapping of the partial semimetric has the defining measurement property $\mu \longrightarrow \sigma_P$ by Lemma 2.34, page 42. In addition, it is Scott-continuous since the partial semimetric is. Lastly, (3)⇒(1) is a consequence of Proposition 3.11.(3). $\square$

We state the following corollary for a future reference. In fact, this result will be slightly improved in Corollary 3.21.

Corollary 3.15. *Let P be a continuous poset with a partial metric $p\colon P \times P \to [0, \infty)$ compatible with the Scott topology. Then the self-distance mapping $\mu_p\colon P \to [0, \infty)^{op}$ is a measurement on P.* $\square$

3.2.2 Lebesgue measurements and partial semimetrics

For the purpose of the next definition we introduce the following notation.

$$\mu(A, \varepsilon) := \bigcup\{\mu(x, \varepsilon) \mid x \in A\}, \tag{3.1}$$

where A is a subset of a continuous poset P, the map $\mu\colon P \to [0, \infty)^{op}$ is monotone and $\varepsilon > 0$.

Definition 3.16. *Let P be a continuous poset. A Scott-continuous map $\mu\colon P \to [0, \infty)^{op}$ is a* Lebesgue measurement *on P if for all Scott-compact (we may assume saturated) subsets $K \subseteq P$ and for all Scott-open subsets $U \subseteq P$,*

$$K \subseteq U \quad \Rightarrow \quad \exists \varepsilon > 0.\ \mu(K, \varepsilon) \subseteq U.$$

One can immediately see from the definitions that Lebesgue measurements are measurements. The three propositions below give alternative descriptions of the defining property of Lebesgue measurements on domains.

Proposition 3.17. *Let P be a continuous poset equipped with a Scott-continuous mapping $\mu\colon P \to [0,\infty)^{op}$. The following are equivalent:*

1. *The map μ is a Lebesgue measurement.*

2. *If $\uparrow x \subseteq U$, for some x in P and a Scott-open subset U of P, then there exists $\varepsilon > 0$ such that $\mu(\uparrow x, \varepsilon) \subseteq U$.*

Proof. For the nontrivial direction, suppose that $K \subseteq U$, where K is a Scott-compact saturated subset of P and U is Scott-open in P. Then for all $k \in K$ choose an element $l \ll k$ with $l \in U$. The collection $\{\twoheaduparrow l \mid l \ll k\}$ is an open cover of K. Hence a finite subcollection $\twoheaduparrow l_1, \twoheaduparrow l_2, ..., \twoheaduparrow l_n$ covers K already. By assumption, for every l_i, $i = 1, ..., n$ there is $\varepsilon_i > 0$ with $\mu(\uparrow l_i, \varepsilon_i) \subseteq U$. Set $\varepsilon := \min\{\varepsilon_i \mid i = 1, 2, ..., n\}$. Then $K \subseteq \mu(K, \varepsilon) \subseteq U$. □

Next, we demonstrate that whenever μ is a Lebesgue measurement on a continuous poset, its induced partial semimetric p_μ satisfies condition **(L)**.

Proposition 3.18. *Let P be a continuous poset equipped with a Scott-continuous mapping $\mu\colon P \to [0,\infty)^{op}$. The following are equivalent:*

1. *μ is a Lebesgue measurement;*

2. *For all $x \in P$ and for all sequences $(x_n), (y_n)$ of P,*

$$\lim p_\mu(x, x_n) = \mu x \text{ and } \lim p_\mu(x_n, y_n) = \lim \mu x_n \quad \text{imply} \quad \lim p_\mu(x, y_n) = \mu x.$$

Proof. (1)⇒(2). Let $x \in P$ and take a Scott-open set U with $x \in U$. The assumption $\lim p_\mu(x, x_n) = \mu x$ is equivalent to saying that $(x_n) \to_\sigma x$ by Proposition 3.11.(5). That is,

$$\exists N_1 \in \omega.\, \forall n \geq N_1.\;\; x_n \in U.$$

By definition, $p_\mu(x_n, y_n) = \inf\{\mu z \mid z \ll x_n, y_n\}$, for each $n \in \omega$. Therefore, there exists a sequence (z_n) with $z_n \ll x_n, y_n$ and $\lim \mu z_n = \lim p_\mu(x_n, y_n)$, for all $n \in \omega$. Since by assumption, $\lim p_\mu(x_n, y_n) = \lim \mu x_n$, we have $\lim \mu z_n = \lim \mu x_n$. This means

$$\forall \varepsilon > 0.\, \exists N_2 \in \omega.\, \forall n \geq N_2.\;\; \mu z_n - \mu x_n < \varepsilon.$$

Note that the set $K := \{x_n \mid n \geq N_1\} \cup \{x\}$ is a compact subset of U, and since μ is a Lebesgue measurement, the condition specialises to:

$$\exists \lambda > 0.\ \forall x_n \in K.\ \forall z \in P.\ (z \sqsubseteq x_n \text{ and } \mu z < \mu x_n + \lambda) \text{ implies } z \in U.$$

Hence $z_n \in U$ for all $n \geq \max\{N_1, N_2\}$. Since $z_n \ll y_n$ for all $n \in \omega$, (y_n) is cofinally in U. That is, $(y_n) \to_\sigma x$, or equivalently, $\lim p_\mu(x, y_n) = \mu x$.

(2)$\Rightarrow$(1). For the converse, observe that by Proposition 2.19, page 33, for any Scott-open set $U \subseteq P$ and $x \in U$ we have $q_{p_\mu}(\uparrow x, P \setminus U) > 0$. This, however, is equivalent to saying that there exists $\varepsilon > 0$ such that

$$B_{p_\mu}(\uparrow x, \varepsilon) := \bigcup_{x \sqsubseteq y} B_{p_\mu}(y, \varepsilon) \subseteq U.$$

But the map p_μ is stable and hence $B_{p_\mu}(y, \varepsilon) = \uparrow\mu(y, \varepsilon)$ by the proof of Theorem 2.36, page 42. We conclude that $\mu(\uparrow x, \varepsilon) \subseteq U$. Therefore, the map μ is a Lebesgue measurement by Proposition 3.17. $\square$

Recall the condition **(U)** from Section 2.7, page 35, which in the light of Proposition 3.11(3) can be written for a Scott-continuous mapping $\mu \colon P \to [0, \infty)^{op}$ as:

$$\forall x \in P.\ \forall x \in U \in \sigma(P).\ \exists \varepsilon > 0.\ \exists V \in \sigma(P).\ x \in V \subseteq \uparrow\mu(V, \varepsilon) \subseteq U. \quad \textbf{(U)}$$

Proposition 3.19. *Let P be a continuous poset equipped with a Scott-continuous map $\mu \colon P \to [0, \infty)^{op}$. The following are equivalent:*

1. *μ is a Lebesgue measurement.*
2. *μ satisfies* **(U)**.

Proof. If $x \in U$ for $x \in P$ and $U \in \sigma(P)$, then $x \subseteq \twoheaduparrow y \subseteq U$ for some $y \in U$. By interpolation, there exists $z \in P$ such that $x \in \twoheaduparrow z \subseteq \uparrow z \subseteq \twoheaduparrow y \subseteq U$ and hence $x \in \twoheaduparrow z \subseteq \uparrow\mu(\twoheaduparrow z, \varepsilon) \subseteq \uparrow\mu(\uparrow z, \varepsilon) \subseteq U$, by **(L)**. Then $x \in V \subseteq \uparrow\mu(V, \varepsilon) \subseteq U$, where $V := \twoheaduparrow z$. Conversely, assuming $x \in U$, we have $x \in V \subseteq \uparrow\mu(V, \varepsilon) \subseteq U$ for some $V \in \sigma(P)$ by **(U)**. We immediately get $\mu(\uparrow x, \varepsilon) \subseteq U$. $\square$

Proposition 3.18 can be easily extended and stated in a form analogous to Theorem 3.14.

Theorem 3.20. *For a continuous poset P the following are equivalent:*

1. *P admits a stable partial semimetric with* **(L)** *for the Scott topology;*

2. *P admits a partial semimetric with* **(L)** *for the Scott topology, which is Scott-continuous as a mapping from $P \times P$ to $[0, \infty)^{op}$;*

3. *P admits a Lebesgue measurement.*

Proof. It is enough to show (2)⇒(3). Let $p\colon P \times P \to [0, \infty)$ be a partial semimetric for the Scott topology, which satisfies **(L)**. Then for any Scott-compact subset K and for any Scott-open subset U of P with $K \subseteq U$, we have $q_p(K, P \setminus U) > 0$, by Proposition 2.19, page 33. This is equivalent to saying that for some $\varepsilon > 0$ we have $K \subseteq B_p(K, \varepsilon) \subseteq U$. Denote the self-distance map for p by μ_p. Now, by Lemma 2.31, page 41, we have $\mu_p(K, \varepsilon) \subseteq B_p(K, \varepsilon) \subseteq U$ and so the mapping μ_p is a Lebesgue measurement on P. □

The following result improves Corollary 3.15.

Corollary 3.21. *Let P be a continuous poset with a partial metric $p\colon P \times P \to [0, \infty)$ compatible with the Scott topology. Then the self-distance mapping $\mu_p\colon P \to [0, \infty)^{op}$ is a Lebesgue measurement on P.* □

3.2.3 Weakly modular measurements and stable partial metrics

Definition 3.22. *Let P be a continuous poset with a measurement $\mu\colon P \to [0, \infty)^{op}$. If for all consistent pairs $a, b \in P$, for all upper bounds r of a and b and for all $\varepsilon > 0$, there exists an $s \sqsubseteq a, b$ such that*

$$\mu r + \mu s \leq \mu a + \mu b + \varepsilon,$$

then we say that μ is a weakly modular *measurement on P.*

Proposition 3.23. *Let P be a continuous poset with a weakly modular measurement $\mu\colon P \to [0, \infty)^{op}$. Then $p_\mu\colon P \times P \to [0, \infty)$ is a partial metric on P such that the partial metric topology $\tau_p(P)$ agrees with the Scott topology $\sigma(P)$.*

The converse also holds: Whenever p_μ is a partial metric for some measurement $\mu\colon P \to [0, \infty)^{op}$, then μ is weakly modular.

Proof. The first part of the proof is an extension of the argument in Corollary 5.4.1 of [Mar00]. By Corollary 3.13, it is enough to prove that p_μ satisfies $(\Delta^\sharp)$. Take any $x, y, z \in P$. By definition of p_μ, for any $\varepsilon > 0$ there exist $a \sqsubseteq x, z$ and $b \sqsubseteq y, z$ such that $p_\mu(x, z) + \varepsilon \geq \mu a$ and $p_\mu(y, z) + \varepsilon \geq \mu b$. Since a, b are consistent, there is $s \sqsubseteq a, b$ such that

$$p_\mu(x, y) \leq p_\mu(a, b) \leq \mu s \leq \mu a + \mu b - \mu z + \varepsilon.$$

Hence,

$$p_\mu(x,y) \leq p_\mu(x,z) + p_\mu(z,y) - \mu z + 3\varepsilon,$$

for all $\varepsilon > 0$. This proves that p_μ satisfies $(\Delta^\sharp)$.

For the converse, let a, b be a consistent pair of elements of P, say, $a, b \sqsubseteq r$ and $\varepsilon > 0$. Then there exists $z \ll a, b$ such that

$$\mu z \leq p_\mu(a,b) + \varepsilon \leq p_\mu(a,r) + p_\mu(r,b) - \mu r + \varepsilon = \mu a + \mu b - \mu r + \varepsilon,$$

which means that μ is weakly modular. $\square$

Notice that if P is bounded-complete and μ is modular, that is, for all consistent pairs $x, y \in P$ we have $\mu(x \sqcup y) + \mu(x \sqcap y) = \mu x + \mu y$, then the conditions of the proposition hold and $p_\mu = \mu(x \sqcap y)$ is a partial metric on P. Hence we complemented the result by O'Neill [O'N95a] who gave a construction of a partial metric from a valuation on a so called *valuation space*, i.e. on a bounded-complete dcpo with a least element. As our last result shows, the existence of suprema and infima is not necessary.

Without weak modularity, p_μ is no longer a partial metric. An example can be found in Section 3.3.3. Sometimes, however, p_μ still satisfies the classical triangle inequality for metrics.

Proposition 3.24. *Let P be a continuous poset with a measurement $\mu\colon P \to [0,\infty)^{op}$ such that*

$$\forall x, y \in P.\ \exists z \sqsubseteq x, y.\ \mu z \leq \mu x + \mu y.$$

Then the associated partial semimetric $p_\mu\colon P \times P \to [0,\infty)$ satisfies the triangle inequality and induces the Scott topology on P.

The reasoning is essentially the same as in the proof of the preceding proposition. Further generalisation is still possible, but this involves applying a construction due to Frink [Fri37] to the map p_μ, and is beyond our present concern.

Proposition 3.23 yields:

Theorem 3.25. *Let P be a continuous poset. The following are equivalent:*

1. *P admits a stable partial metric $p\colon P \times P \to [0,\infty)$ with $\tau_p(P) = \sigma(P)$.*

2. *P admits a weakly modular measurement $\mu\colon P \to [0,\infty)^{op}$.* $\square$

This result establishes an equivalence between weakly modular measurements and stable partial metrics on continuous posets.

Note that by the last theorem and Corollary 3.21 a weakly modular measurement always satisfies the local axiom of triangle **(L)**.

We give plenty of concrete examples of weakly modular measurements in Section 3.3.3. Here we show how to construct one on an arbitrary countably based continuous semilattice.

Proposition 3.26. *Let D, E be continuous dcpos and $\sqcap$-semilattices. Assume that the map $f\colon D \to E$ is Scott-continuous and $\sqcap$-preserving and the mapping $\mu\colon E \to [0,\infty)^{op}$ is a weakly modular measurement. Then the composition*

$$D \xrightarrow{f} E \xrightarrow{\mu} [0,\infty)^{op}$$

is a weakly modular measurement on D.

Proof. Denote the composition $\mu \circ f$ by g. Let x, y be a consistent pair of elements of D bounded by some $z \in D$. The map f is monotone and hence $fx, fy \sqsubseteq fz$. Therefore, for any $\varepsilon > 0$, there exists $w \in E$ with $w \sqsubseteq fx, fy$ and

$$\mu w + gz \le gx + gy + \varepsilon.$$

Since f preserves infima and μ is monotone, we have $g(x \sqcap y) \le \mu w$. We conclude that

$$g(x \sqcap y) + gz \le gx + gy,$$

as required. $\square$

Remark 3.27. *If D is a countably based domain and an $\sqcap$-semilattice, then it admits a weakly modular measurement. It is the composition of the map $f\colon D \to \mathcal{P}\omega$ given by*

$$fx = \{n \mid x \in U_n,\ U_n \text{ is a Scott-open filter}\}$$

and the weakly modular measurement $|\cdot|\colon \mathcal{P}\omega \to [0,\infty)^{op}$

$$|x| := \sum_{n \notin x} 2^{-(n+1)}.$$

3.2.4 Extended example: the interval domain

Recall that the interval domain $\mathbf{I}\mathbb{R}$ is a set of all compact, connected intervals of the real line ordered by the inverse inclusion

$$\mathbf{I}\mathbb{R} = \{[x,y] \mid x, y \in \mathbb{R}, x \le y\}.$$

We are going to describe $\mathbf{I}\mathbb{R}$ in more detail in Section 3.3.3, page 76, however at the moment it is enough to note that the map $\mu\colon \mathbf{I}\mathbb{R} \to [0,\infty)^{op}$ given by

$$\forall [x,y] \in \mathbf{I}\mathbb{R}.\ \mu[x,y] := y - x$$

is a weakly modular measurement on $\mathbf{I}\mathbb{R}$.

It is clear that the weak modularity property of the natural measurement on the interval domain reduces to the usual modularity law: if $[x,y],[u,w] \in \mathbf{I}\mathbb{R}$ are consistent, then

$$\mu([x,y] \sqcup [u,w]) + \mu([x,y] \sqcap [u,w]) = \mu[x,y] + \mu[u,w].$$

Our goal is to characterize all modular measurements on the interval domain $\mathbf{I}\mathbb{R}$.

Theorem 3.28. *For a mapping $\lambda\colon \mathbf{I}\mathbb{R} \to [0,\infty)^{op}$ the following are equivalent:*

1. *λ is modular measurement with* $\ker\lambda = \max\mathbf{I}\mathbb{R}$.
2. *There exists a Lawson-continuous, strictly increasing function $f\colon \mathbb{R} \to \mathbb{R}$ with*
$$\lambda[x,y] = f(y) - f(x).$$

Proof. For a mapping $\lambda\colon \mathbf{I}\mathbb{R} \to [0,\infty)^{op}$, we define a function $f_\lambda\colon \mathbb{R} \to \mathbb{R}$ by

$$f_\lambda(x) = \begin{cases} \lambda[0,x], & \text{if } x > 0 \\ -\lambda[x,0], & \text{otherwise.} \end{cases}$$

The claim now follows from Lemmata 3.29-3.32 below. □

To simplify notation, we will often abbreviate f_λ to f.

Lemma 3.29. *For a mapping $\lambda\colon \mathbf{I}\mathbb{R} \to [0,\infty)^{op}$ the following are equivalent:*

1. *$\lambda[x,y] = fy - fx$,*
2. *λ is modular and $\max \mathbf{I}\mathbb{R} \subseteq \ker \lambda$.*

In addition, if either holds, f is increasing and λ is monotone.

Proof. For (1)⇒(2), suppose that $[x,y]$ and $[u,w]$ in $\mathbf{I}\mathbb{R}$ are consistent. Then,

$$\begin{aligned}&\lambda([x,y] \sqcap [u,w]) + \lambda([x,y] \sqcup [u,w]) = \\ &\lambda[\min\{x,u\},\max\{y,w\}] + \lambda[\max\{x,u\},\min\{y,w\}] = \\ &f(\max\{y,w\}) - f(\min\{x,u\}) + f(\min\{y,w\}) - f(\max\{x,u\}) = \\ &fy - fx + fw - fu = \lambda[x,y] + \lambda[u,w].\end{aligned}$$

We will show (2)⇒(1) by cases:

- Case $0 < x \leq y$. $\lambda[0,y] = \lambda[0,x] + \lambda[x,y]$, by modularity and $\max \mathbf{I}\mathbb{R} \subseteq \ker\lambda$. Hence $\lambda[x,y] = \lambda[0,y] - \lambda[0,x] = fy - fx$.
- Case $x < 0 \leq y$ is similar.
- Case $x < y \leq 0$. $\lambda[x,y] + \lambda[y,0] = \lambda[x,0]$, by modularity and $\max \mathbf{I}\mathbb{R} \subseteq \ker\lambda$. Hence $\lambda[x,y] = \lambda[x,0] - \lambda[y,0] = fy - fx$.

Next, for any $x, y \in \mathbb{R}$, $x \leq y$ iff $\lambda[x,y] \geq 0$ iff $fy - fx \geq 0$ iff $fx \leq fy$. Hence the map f is increasing.

Let $[u,w] \sqsubseteq [x,y]$. We have $\lambda[u,w] = fw - fu \geq fy - fx = \lambda[x,y]$, that is $\lambda[u,w] \sqsubseteq \lambda[x,y]$ in $[0,\infty)^{op}$. Hence the map λ is monotone. □

Lemma 3.30. *For a modular mapping* $\lambda\colon \mathbf{I}\mathbb{R} \to [0,\infty)^{op}$ *with* $\max\mathbf{I}\mathbb{R} \subseteq \ker\lambda$ *the following are equivalent:*

1. *λ is strictly monotone,*
2. $\ker\lambda = \max\mathbf{I}\mathbb{R}$*,*
3. *f is strictly increasing.*

Proof. For (1)⇒(2), let $[x,y] \in \ker\lambda$. Let $[u,w] \sqsupseteq [x,y]$. By monotonicity,

$$0 \leq \lambda[u,w] \leq \lambda[x,y] = 0.$$

By strict monotonicity, $[x,y] = [u,w]$. Hence $[x,y] \in \max\mathbf{I}\mathbb{R}$. Conversely, if $[x] := [x,x] \in \max\mathbf{I}\mathbb{R}$, then $\lambda[x] = fx - fx = 0$. That is, $[x] \in \ker\lambda$.

To show (2)⇒(3), let $x < y$ in $\mathbb{R}$. Since $\ker\lambda = \max\mathbf{I}\mathbb{R}$, $\lambda[x,y] = fy - fx > 0$. Hence $fx < fy$.

For (3)⇒(1), $[u,w] \sqsubset [x,y]$ means $u < x$ or $y < w$. In either case, $\lambda[u,w] = fw - fu > fy - fx = \lambda[x,y]$. □

Lemma 3.31. *For a modular mapping* $\lambda\colon \mathbf{I}\mathbb{R} \to [0,\infty)^{op}$ *with* $\max \mathbf{I}\mathbb{R} \subseteq \ker\lambda$*, the following are equivalent:*

1. *The map λ preserves directed suprema,*
2. *The map f is Lawson continuous.*

Proof. In order to show (1)⇒(2), firstly, we will prove that f is continuous at $x > 0$. Let $y_n \to x$ from the left for a sequence (y_n) of real numbers and a real number x. Then

$$\lim_{n\to\infty}(fx - fy_n) = \lim_{n\to\infty}\lambda[y_n, x] = \lambda[\bigsqcup{}^{\uparrow} y_n, x] = \lambda[x,x] = 0.$$

Hence $fy_n \to fx$. The proof for $y_n \to x$ from the right is almost identical. For (2)$\Rightarrow$(1), let $[x_n, y_n]$ be a chain of intervals with $\bigsqcup^{\uparrow}[x_n, y_n] = [x, y]$. Hence,

$$\begin{aligned}
\lambda[x,y] &= \lambda[\sup x_n, \inf y_n] \\
&= f(\inf\, y_n) - f(\sup\, x_n) \\
&= \inf(fy_n - fx_n) \\
&= \bigsqcup^{\uparrow}\lambda[x_n, y_n].
\end{aligned}$$

□

Lemma 3.32. *Let $\lambda\colon \mathbf{I}\mathbb{R} \to [0,\infty)^{op}$ be a modular, Scott-continuous map. Then it measures $\mathbf{I}\mathbb{R}$ iff it is strictly monotone.*

Proof. Define $\phi\colon \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ by $[a,b] \mapsto [fa, fb]$. By properties of f, the map ϕ is injective and order reflecting. Therefore, it is an order isomorphism to its image, $\mathbf{I}\mathbb{R} \cong \phi[\mathbf{I}\mathbb{R}]$. To show Scott-continuity, note that for any sequence of intervals (x_n), we have $\phi(\bigsqcup^{\uparrow}x_n) = \bigsqcup^{\uparrow}\phi x_n$. But since f_λ is continuous, $\bigsqcup^{\uparrow}\phi x_n \in \phi[\mathbf{I}\mathbb{R}]$.

Let $x \in \mathbf{I}\mathbb{R}$, $U \in \sigma(\mathbf{I}\mathbb{R})$. Suppose that $x \in U$. Choose $y \in U$ such that $y \ll x$. If $a \sqsubseteq x$ and $\phi y \ll \phi a$, then since ϕ reflects the way-below relation, $y \ll a \sqsubseteq x$. Hence $a \in U$, as required. We have proved that $\phi \longrightarrow \sigma(\mathbf{I}\mathbb{R})$.

Finally, we have that $\lambda = \mu \circ \phi$, where μ is the natural measurement on $\mathbf{I}\mathbb{R}$. Therefore λ measures $\mathbf{I}\mathbb{R}$ as a composition of two maps that induce the Scott topology. □

As an immediate application of Theorem 3.28 we observe that the classical mean value theorem (as stated below) for continuous self-maps on the real line can be extended to the interval domain.

Theorem 3.33 (Lagrange). *If $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous on the closed interval $[x,y]$ and for all $w \in (x,y)$ the derivative $f'(w) < \infty$ exists, then there is a $z \in (x,y)$ such that*

$$\frac{f(y) - f(x)}{y - x} = f'(z).$$

□

First of all, we need to define a derivative on the interval domain. An appropriate definition, which works on any domain is given in [Mar00], Chapter 7.

Definition 3.34. *Let D be a continuous dcpo equipped with a Scott-continuous mapping $\mu\colon D \to [0,\infty)^{op}$ and let X be a subset of D. If*

1. $\mu \longrightarrow \sigma(X)$,

2. $f\colon D \to \mathbb{R}$ *is a partial map,*

3. $p \in dom(f) \cap X$ *is not isolated in* $dom(f)$,

then

$$\frac{df}{d\mu}(p) := \lim_{x \to p} \frac{f(x) - f(p)}{\mu(x) - \mu(p)}$$

is called a derivative *of* f *at* p *with respect to* μ*, whenever the limit* $x \to p$ *exists in the relative Martin topology on* $dom(f)$ *and the limit*

$$\lim_{x \to p} \frac{f(x) - f(p)}{\mu(x) - \mu(p)}$$

exists the Euclidean topology on the reals.

When a mapping has a derivative, it is unique. Moreover,

Lemma 3.35 (Martin). *If* $f\colon \mathbb{R} \to \mathbb{R}$ *is a function on the real line and we define*

$$\Delta f\colon \mathbf{I}\mathbb{R} \to \mathbb{R}$$

$$\Delta f[x, y] = f(y) - f(x),$$

then for all $p \in \mathbb{R}$,

$$\frac{\Delta f}{d\mu}(p) \text{ exists} \iff f'(p) \text{ exists}$$

and in either case these quantities coincide. $\Box$

The preceding lemma and Theorem 3.28 yield:

Theorem 3.36 (Mean Value Theorem for IR). *Let* $\lambda\colon \mathbf{I}\mathbb{R} \to [0, \infty)^{op}$ *be a modular, Scott-continuous map with* $\ker\lambda = \max\mathbf{I}\mathbb{R}$*. If the derivative exists for all* $p \in (x, y)$*, then*

$$\lambda[x, y] = \frac{d\lambda}{d\mu}(z) \cdot \mu[x, y],$$

for some $z \in (x, y)$. $\Box$

3.2.5 Characterization of partial metrizability of algebraic domains

In this section we apply the knowledge about Lebesgue measurements to show a complete characterization of partial metrizability of the Scott topology on an algebraic domain. We start from a similar result obtained by Künzi and Vajner in [KV92], Proposition 3, page 73.

Künzi and Vajner's theorem

Theorem 3.37 (Künzi and Vajner). *A poset P admits a partial metric for its Alexandroff topology iff there is a function $|\cdot|\colon P \to [0,\infty)$ such that*

$$\forall x \in P.\ \exists \varepsilon > 0.\ \forall y \in \uparrow x.\ \forall z \in \downarrow y \setminus \uparrow x. \quad |z| - |y| \geq \varepsilon. \tag{*}$$

□

Since Künzi and Vajner's theorem is of crucial importance to our characterization, in the rest of this section we sketch its proof. Let us start with introducing the necessary terminology.

For any relation R on a set X and $x \in X$ we define:

$$[x]R := \{y \in X \mid xRy\}.$$

Definition 3.38. *Let X be a set equipped with a binary relation R. A path $\mathcal{W}$ from an element x to an element y in X is a finite sequence of elements $\mathcal{W} = (x_0, x_1, ..., x_n)$ such that $n \in \omega$, $x = x_0$, $y = x_n$ and $x_{i+1} \in [x_i](R \cup R^{-1})$, whenever $i \in \{0, 1, 2, ..., n-1\}$.*

We observe that if $\mathcal{W} = (x_0, x_1, ..., x_{n-1}, x_n)$ is a path from x to y in X then $\bar{\mathcal{W}} = (x_n, x_{n-1}, ..., x_1, x_0)$ is a path from y to x in X.

For any map $\mu\colon X \to [0,1]$ and a path $\mathcal{W}$ from x to y define the *length of the path $\mathcal{W}$ with respect to μ* to be the number

$$l(\mathcal{W}) := \sum_{i=0}^{n-1} \max\{0, \mu x_{i+1} - \mu x_i\}.$$

Note that

$$\mu y - \mu x = l(\mathcal{W}) - l(\bar{\mathcal{W}}).$$

We show that there exists a function $q\colon X \times X \to [0,\infty)$ such that

1. $\forall x \in X.\ q(x,x) = 0.$

2. $\forall x, y, z \in x.\ q(x,z) \leq q(x,y) + q(y,z)$.

3. $\forall x, y \in X.\ q(x,y) + \mu x = q(y,x) + \mu y$.

For arbitrary $x \neq y$ in X connected by a path define

$$d(x,y) := \inf\{l(\mathcal{W}) \mid W \text{ is a path from } x \text{ to } y\}$$

and

$$q(x,y) := \min\{d(x,y), 2 - \mu x\}.$$

If there is no path between x and y we set $q(x,y) := 2 - \mu x$. Finally, for all x in X define $q(x,x) := 0$.

Then (1) holds by definition. To check (2) let $x, y, z \in X$. It is enough to assume that $x \neq y$ and $y \neq z$. We consider three cases:

- If $q(x,y) = 2 - \mu x$, then $q(x,z) \leq 2 - \mu x \leq q(x,y) + q(y,z)$.
- If $q(x,y) \neq 2 - \mu x$ and $q(y,z) = 2 - \mu y$. Then for any $\varepsilon > 0$ we have $q(x,y) + \varepsilon \geq l(\mathcal{W}^\varepsilon)$ for some path $\mathcal{W}^\varepsilon$ from x to y. Hence, $q(x,z) \leq 2 - \mu x = l(\mathcal{W}) - l(\bar{\mathcal{W}}^\varepsilon) + 2 - \mu y \leq q(x,y) + q(y,z) + \varepsilon$ for any $\varepsilon > 0$.
- Suppose that $q(x,y) \neq 2 - \mu x$ and $q(y,z) \neq 2 - \mu y$. Then for arbitrary $\varepsilon > 0$ we have $q(x,y) + \varepsilon \geq l(\mathcal{W}_1^\varepsilon)$ for some path $\mathcal{W}_1^\varepsilon$ from x to y and $q(y,z) + \varepsilon \geq l(\mathcal{W}_2^\varepsilon)$ for some path $\mathcal{W}_2^\varepsilon$ from y to z. Thus, $q(x,z) \leq l(\mathcal{W}_1^\varepsilon \circ \mathcal{W}_2^\varepsilon) = l(\mathcal{W}_1^\varepsilon) + l(\mathcal{W}_2^\varepsilon) \leq q(x,y) + q(y,z) + 2\varepsilon$ for any $\varepsilon > 0$, where $\mathcal{W}_1^\varepsilon \circ \mathcal{W}_2^\varepsilon$ denoted the path from x to z by first traveling along $\mathcal{W}_1^\varepsilon$ and then along $\mathcal{W}_2^\varepsilon$.

For (3), let $x, y \in X$. It suffices to consider the case when x and y are connected by some path and, let us say, $q(x,y) = d(x,y) < 2 - \mu x$. Set $\varepsilon := 2 - \mu x - d(x,y)$. Let $\mathcal{W}$ be a path from x to y with $l(\mathcal{W}) < d(x,y) + \varepsilon$. Then $l(\bar{\mathcal{W}}) = \mu x - \mu y + l(\mathcal{W}) < 2 - \mu y$. Hence $q(y,x) = d(y,x)$. Since for any path $\mathcal{A}$ from y to x we have $l(\mathcal{A}) = \mu x - \mu y + l(\bar{\mathcal{A}}) \geq \mu x - \mu y + d(x,y)$ we conclude that $d(y,x) \geq \mu x - \mu y + d(x,y)$. By symmetry, (3) is satisfied for d and hence for q.

Therefore, the map $p \colon X \times X \to [0, \infty)$ defined for any $x, y \in X$ by

$$p(x,y) := q(x,y) + \mu x \tag{3.2}$$

is a partial metric on X.

In the particular case when P is a poset, then Künzi and Vajner show that the condition $(*)$ of Theorem 3.37 implies that

$$\forall x \in P.\ \exists \varepsilon_x > 0.\ {\uparrow}x = \{y \mid p(x,y) = p(x,x)\} = \{y \mid p(x,y) < p(x,x) + \varepsilon_x\} \tag{+}$$

and so the partial metric p induces the Alexandroff topology on P.

The characterization

For the following crucial lemma, recall the notation from Section 3.1.1.

Lemma 3.39. *Let $(B, \prec)$ be a reflexive abstract basis equipped with a mapping $\mu\colon B \to [0,\infty)^{op}$, which satisfies the condition (*) from Theorem 3.37. Then there exists a partial metric compatible with the Scott topology on the rounded ideal completion $\mathcal{I}(B)$ of B.*

Proof. Note that by the discussion in [KV92] p.74 we can always assume that μ is bounded by 1. Therefore, applying Theorem 3.37 we conclude that there exists a partial metric $p\colon B \times B \to [0,\infty)$ which is bounded by 3 and such that $\mu_p = \mu$. The partial metric p satisfies the formula $(+)$ above.

Now, extend the function p to $\mathcal{I}(B)$ in the following way: for each $I, J \in \mathcal{I}(B)$ define $\hat{p}\colon \mathcal{I}(B) \times \mathcal{I}(B) \to [0,\infty)$ by

$$\hat{p}(I,J) := \inf\{p(x,y) \mid x \in I, y \in J\}.$$

Since p is continuous as a map from B equipped with the pseudoScott topology to $[0,\infty)^{op}$ in its Scott topology, the mapping $\hat{p}\colon \mathcal{I}(B) \times \mathcal{I}(B) \to [0,\infty)^{op}$ is Scott continuous. Note that for every $x, y \in B$ we have $\hat{p}(x^*, y^*) = p(x,y)$. This means that p induces the subspace Scott topology on the image of B in $\mathcal{I}(B)$ under the canonical embedding.
Step 1: First, we will prove that $\hat{p}$ is a partial pseudometric. For the sharp triangle inequality, note that for all $x \in I$, $y \in J$ and $z \in K$, where $I, J, K \in \mathcal{I}(B)$, we have $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$ and this inequality extends to infima. Similarly we prove the **(ssd)** axiom. Symmetry is trivial.
Step 2: If $I \subseteq J$ in $\mathcal{I}(B)$, then $\{p(x,y) \mid x \in I, y \in I\} \subseteq \{p(x,y) \mid x \in I, y \in J\}$. Therefore, $\hat{p}(I,I) \geq \hat{p}(I,J)$. Hence $\hat{p}(I,I) = \hat{p}(I,J)$ by **(ssd)**.
Step 3: To prove that $\hat{p}$ induces the Scott topology on $\mathcal{I}(B)$ we will show that $\twoheaduparrow x^* = B^{\hat{p}}(x^*, \varepsilon)$ for some $x \in B$ and $\varepsilon > 0$. Suppose $x^* \ll K$ for some $K \in \mathcal{I}(B)$. Then $x^* \subseteq K$ and so $\hat{p}(x^*, K) = \hat{p}(x^*, x^*)$ by step 2. Hence, $\twoheaduparrow x^* \subseteq B^{\hat{p}}(x^*, \varepsilon)$ for any $\varepsilon > 0$. Conversely, if for some $L \in \mathcal{I}(B)$ we have that $\hat{p}(x^*, L) < \hat{p}(x^*, x^*) + \varepsilon_x$, then $\hat{p}(x^*, w^*) < \hat{p}(x^*, x^*) + \varepsilon_x$ already for some $w \in L$. This is however equivalent to $p(x,w) < p(x,x) + \varepsilon_x$ and so $w \in B^p(x, \varepsilon_x) = \uparrow x$ by $(+)$. That is, $x \prec w$. This means that $x \in L$ and so $x^* \ll L$. We have shown that $B^{\hat{p}}(x^*, \varepsilon_x) \subseteq \twoheaduparrow x^*$.
Step 4: By step 3, the order induced by $\hat{p}$ agrees with the subset inclusion, which is the specialisation order for the Scott-topology on $\mathcal{I}(B)$. Hence if $\hat{p}(I,J) = \hat{p}(I,I) = \hat{p}(J,J)$ for some $I, J \in \mathcal{I}(B)$, then $I \subseteq J$ and $J \subseteq I$. Therefore, $I = J$ and so $\hat{p}$ is a partial metric on $\mathcal{I}(B)$. $\square$

Lemma 3.40. *Let P be an algebraic poset with a monotone map $\mu\colon P \to [0,\infty)^{op}$. Then the local triangle inequality of μ implies the condition (*) of Theorem 3.37 for the restriction of μ to $K(P)$.*

Proof. Take any $x \in K(P)$. The set $\uparrow x$ is Scott open and Scott-compact in P. By the local triangle inequality,

$$q_\mu(\uparrow x, P \setminus \uparrow x) > 0.$$

Therefore, there exists an $\varepsilon > 0$ such that

$$q_\mu(\uparrow x, P \setminus \uparrow x) > \varepsilon.$$

This implies in particular that for every $y \in \uparrow x$ and $z \in \downarrow y \setminus \uparrow x$ we have $p_\mu(y,z) \geq \mu y + \varepsilon$. Hence $\mu z = p_\mu(y,z) \geq \mu y + \varepsilon$. This shows (*). $\square$

Therefore, we have obtained a complete characterization of partial metrizability on algebraic domains:

Theorem 3.41. *Let P be an algebraic dcpo. The following are equivalent:*

(1) *P admits a Lebesgue measurement $\mu\colon P \to [0,\infty)^{op}$.*

(2) *P admits a partial metric compatible with the Scott topology on P.*

Proof. By Lemmata 3.40 and 3.39, every Lebesgue measurement on P extends to a partial metric for the Scott topology on $\mathcal{I}(P) \cong P$. Conversely, by Corollary 3.21, the self-distance map for a partial metric compatible with the Scott topology is a Lebesgue measurement on P. $\square$

In fact, the method of constructing partial metrics employed to obtain the characterization above is the most general procedure of this sort known to us.

Example 3.42. *Let P be a continuous poset with a weakly modular measurement $\mu\colon P \to [0,\infty)^{op}$. Künzi and Vajner's construction yields the following mapping:*

$$\forall x,y \in P. \quad p(x,y) = \begin{cases} \inf\{\mu z \mid z \ll x,y\} & \textit{if } \exists z \in P.\ z \sqsubseteq x,y \\ 2 & \textit{otherwise} \end{cases}$$

which coincides with the distance p_μ induced by the measurement μ. However, by virtue of Proposition 3.23, the mapping $p = p_\mu$ is a partial metric on P. That is, Proposition 3.23 is a special case of Theorem 3.41.

Proof. A formal proof employs simple induction. We only give an informal argument here. For any path $\mathcal{W}$ from x to y, where $x, y \in P$, we will make explicit the relations between the elements of the path. For instance, if $\mathcal{W} = (x, x_1, x_2, ..., x_{n-1}, y)$ and $x \sqsubseteq x_1, x_1 \sqsubseteq x_2, ..., x_{n-1} \sqsupseteq y$, then we simply write

$$\mathcal{W} = (x \sqsubseteq x_1 \sqsubseteq x_2 \sqsupseteq x_3 ... x_{n-1} \sqsupseteq y).$$

Observe that any path of the form

$$\mathcal{W} = (x ... x_{i-1} \sqsubseteq x_i \sqsubseteq x_{i+1} ... y)$$

can be replaced by

$$\widehat{\mathcal{W}} = (x ... x_{i-1} \sqsubseteq x_{i+1} ... y)$$

with $l(\widehat{\mathcal{W}}) = l(\mathcal{W})$. Similarly, for the opposite order $\sqsupseteq$. Moreover, by weak modularity of the map μ, every path of the form

$$\mathcal{W} = (x ... x_{i-1} \sqsubseteq x_i \sqsupseteq x_{i+1} ... y)$$

can be replaced by

$$\widehat{\mathcal{W}} = (x ... x_{i-1} \sqsupseteq x_i \sqsubseteq x_{i+1} ... y)$$

with $l(\widehat{\mathcal{W}}) \leq l(\mathcal{W})$. Since paths are finite, a finite number of changes described above transform any path $\mathcal{V}$ from x to y to the one of the form

$$\widehat{\mathcal{V}} = (x \sqsupseteq z \sqsubseteq y)$$

and we have $l(\widehat{\mathcal{V}}) \leq l(\mathcal{V})$. The claim now follows immediately from the definition of the partial metric p (cf. formula 3.2). $\square$

3.3 The induced metric

Let (P, p) be a partial metric space. According to Theorem 3.3, the map $d \colon P \times P \to [0, \infty)$ defined for all $x, y \in P$ by

$$d(x, y) := 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on P.

3.3.1 Basic properties of the induced metric

Proposition 3.43. 1. *For all $x, y \in P$, $x \sqsubseteq_p y$ iff $d(x,y) = p(x,x) - p(y,y)$.*

2. *d is* order convex*, that is, for all $x, y, z \in P$, $x \sqsubseteq_p y \sqsubseteq_p z \Rightarrow d(x,z) = d(x,y) + d(y,z)$.*

Proof. For (1), by the definition of the order induced by p, $x \sqsubseteq_p y$ iff $p(x,y) = p(x,x)$. Multiplying both sides of the equation by 2 and subtracting $p(x,x)$ and $p(y,y)$ we immediately obtain $d(x,y) = p(x,x) - p(y,y)$, as required. (2) is now clear. □

The main objective of this section is to locate the metric topology in the lattice of topologies on P and to outline some of the consequences of this analysis. Primarily, we ask how the partial metric topology $\tau_p(P)$, its dual $\tau_p^{op}(P)$, and the induced metric topology $\tau_d(P)$ relate to the intrinsic topologies on P.

We start with a comparison of the dual topology $\tau_p^{op}(P)$ and the weak topology $\omega(P)$.

Lemma 3.44. *The dual topology $\tau_p^{op}(P)$ is always finer than the weak topology $\omega(P)$ on $(P, \sqsubseteq_p)$.*

Proof. It is enough to show that $\uparrow x$ is closed in $\tau_p^{op}(P)$. Let $x \not\sqsubseteq_p y$. Then $p(x,y) > p(x,x)$ and so for some $\varepsilon > 0$ we have $p(x,y) > p(x,x) + \varepsilon$. Consider $B^{op}(y,\varepsilon) = \{z \mid p(y,z) < p(z,z) + \varepsilon\}$. Assume that $z \in B^{op}(y,\varepsilon)$ and $x \sqsubseteq_p z$. Then $p(x,y) \leq p(x,z) + p(z,y) - p(z,z) < p(x,x) + p(z,z) - p(z,z) + \varepsilon = p(x,x) + \varepsilon$, a contradiction. Therefore, $y \in B^{op}(y,\varepsilon) \subseteq P \setminus \uparrow x$, which proves that $\uparrow x$ is closed in $\tau_p^{op}(P)$. □

Denote by $\mathbf{c}(\tau_p)$ the topology of τ_p-closed sets. Recall that the collection of dual c-balls $\{N^{op}(x,\varepsilon) \mid x \in P, \varepsilon > 0\}$ forms a neighbourhood basis for the dual topology $\tau_p^{op}(P)$.

Lemma 3.45. *The dual topology $\tau_p^{op}(P)$ is always coarser than $\mathbf{c}(\tau_p)$.*

Proof. It is enough to show that the dual c-balls are closed in $\tau_p(P)$. The proof is adapted from [FK97]. Suppose $y \notin N^{op}(x,\delta)$, that is $p(x,y) > p(y,y) + \delta$. Hence also $p(x,y) > p(y,y) + \delta + \varepsilon$, for some $\varepsilon > 0$. If now $z \in B^p(y,\varepsilon)$, then $p(y,z) < p(y,y) + \varepsilon$ and therefore we may conclude $p(x,y) > p(y,z) + \delta$. Using the sharp triangle inequality we have $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$ and hence $\delta < p(x,z) - p(z,z)$. That is, $z \notin N^{op}(x,\delta)$. We have shown $B_p(y,\varepsilon) \subseteq D \setminus N^{op}(x,\delta)$. Hence $N^{op}(x,\delta)$ is closed in $\tau_p(P)$. □

To gain further insight, we specialise to the case when the partial metric topology agrees with the Scott topology $\sigma(P)$. In this case we are able to compare the induced metric topology $\tau_d(P)$ to the Lawson and Martin topologies on P. First, observe the following characterization of the Martin topology.

Lemma 3.46. *The Martin topology on a continuous poset P is the join of the Scott topology $\sigma(P)$ and the topology of Scott-closed sets $\mathbf{c}(\sigma(P))$.*

Proof. The collection $\{\twoheaduparrow x \cap \downarrow y \mid x, y \in P\}$ is a basis for the Martin topology on P. Thus, the Martin topology is always coarser than $\sigma(P) \vee \mathbf{c}(\sigma(P))$. To prove the converse, let $U := O \cap C$, where O is a Scott-open set and C is Scott-closed and let $x \in U$. Then choose an element $y \in U$ way-below x such that $y \in O$. We claim that the set $A := \twoheaduparrow y \cap \downarrow x$ is a subset of U. Indeed, if $z \in A$, then $z \in \twoheaduparrow y \subseteq O$. Also, $z \in \downarrow x \subseteq C$. Therefore, $z \in U$. Since A is Martin-open, we are done. $\square$

Theorem 3.47. *For a partial metric on a continuous poset P that is compatible with the Scott topology on P, the induced metric topology $\tau_d(P)$ is finer than the Lawson topology $\lambda(P)$ and coarser than the Martin topology $\mu(P)$.*

Proof. In the lattice of topologies on P, $\lambda(P) = \sigma(P) \vee \omega(P) \subseteq \tau(P) \vee \tau_p^{op}(P) = \tau_d(P)$, while the inclusion holds by Lemma 3.44. On the other hand, by Lemmata 3.45 and 3.46, $\mu(P) = \sigma(P) \vee \mathbf{c}(\sigma(P)) \supseteq \tau_p(P) \vee \tau_p^{op}(P) = \tau_d(P)$. $\square$

Recall from Section 3.1.2 that if P is an ω-continuous poset and $\mathrm{card}(\max P) \geq 2^{\aleph_0}$, then the Martin topology on P is not normal and hence not metrizable, and so the induced metric topology must be strictly coarser than the Martin topology on P.

Example 3.68, page 81, shows that $\tau_d(P)$ is strictly finer than the Lawson topology in general. However, whenever $\tau_d(P)$ is compact, the two topologies agree.

Corollary 3.48. *Let D be a continuous dcpo equipped with a partial metric $p\colon D \times D \to [0, \infty)$ for the Scott topology $\sigma(D)$ on D. Then the following are equivalent:*

1. *D is ω-continuous.*
2. *$\sigma(D)$ is second-countable.*
3. *$\tau_{d_p}(D)$ is second-countable.*

4. $\tau_{d_p}(D)$ *is separable.*

5. $\tau_{d_p}(D)$ *is Lindelöf.*

Proof. Since $\tau_{d_p}(D)$ is metrizable, the equivalence of (3),(4) and (5) follows from General Topology (cf. [Wil70], Thm. 16.11, page 112). The equivalence of (1) and (2) is noted in [AJ94]. For (2)⇒(4), note that the Martin topology is separable on D (cf. [Mar00], Corollary 3.1.1, page 51), and so is $\tau_{d_p}(D)$. Finally, (5)⇒(1) follows immediately from the fact that $\sigma(D)$ is coarser than $\tau_{d_p}(D)$. □

Corollary 3.49. *Let D be a continuous dcpo with a partial metric $p\colon D \times D \to [0, \infty)$ for the Scott topology $\sigma(D)$ on D. If the induced metric topology is compact, then D is countably based and Lawson-compact.*

Proof. If $\tau_{d_p}(D)$ is compact, then it agrees with the Lawson topology and hence D is Lawson-compact. Since every compact space is Lindelöf, D has a countable basis by Corollary 3.48. □

For the converse to the above result on algebraic domains, see Corollary 3.70, page 88.

Proposition 3.50. *Let P be a continuous poset equipped with a partial metric $p\colon P \times P \to [0, \infty)$ for the Scott topology $\sigma(P)$ on P. The self-distance map $\mu_p\colon P \to [0, \infty)^{op}$ is uniformly continuous as a map from $\tau_d(P)$ to the Euclidean topology on $[0, \infty)$. If $\tau_d(P)$ is compact, μ_p is uniformly Lawson-continuous.*

Proof. It is easy to see that for all x, y in P, $|\mu_p x - \mu_p y| \leq d(x, y)$, which means that μ is a non-expansive as a map from $\tau_d(P)$ to the Euclidean topology on $[0, \infty)$. Therefore, it is continuous. The second assertion is clear by the discussion above. □

Corollary 3.51. *Let P be a continuous poset equipped with a partial metric $p\colon P \times P \to [0, \infty)$ for the Scott topology $\sigma(P)$ on P. The set* $\ker\mu_p$ *is closed in $\tau_d(P)$.*

Corollary 3.52. *Let P be a continuous poset equipped with a partial metric $p\colon P \times P \to [0, \infty)$ for the Scott topology $\sigma(P)$ on P. The relative induced metric topology on* $\ker\mu_p$ *agrees with the subspace Scott topology.*

Also, local compactness, compactness and completeness are hereditary properties with respect to closed subspaces. Hence the kernel of a self-distance map for some partial metric on a domain P in its relative Scott topology inherits much of the structure of the metric topology on the whole domain P. In this light, the kernel condition, $\ker\mu_p = \max P$, gains much significance.

3.3.2 Completeness

As remarked in Section 2.7, partial metric spaces are examples of syntopological spaces and as such are *completable* in the sense of M. B. Smyth who gives an appropriate construction in [Smy94]. This completion is nowadays known as *S-completion* or *Smyth-completion*. In [Kün93] Künzi noted that in fact each partial metric induces a Smyth-completable quasi-uniform space. Consequently, Sünderhauf [Sün95] showed that for S-completable spaces, Smyth-completion reduces to a so called *bicompletion*. In order to avoid introducing unnecessary terminology, we just note that in our setting this means precisely that the Smyth-completion of a partial metric space is the usual completion of the induced metric.

Definition 3.53. *A partial metric space $\langle X, p\rangle$ is complete if and only if every Cauchy sequence with respect to the induced metric $d_p\colon X \times X \to [0,\infty)$ converges in X.*

It follows from metric space theory that such a completion always exists, see for example [Wil70], Theorem 24.4, page 176.

In this section we switch from the problem of completion to the one of *completeness*. In particular, we are interested in the relationship between completeness of the induced metric and the structure of the underlying domain. A main result is that bounded-completeness of the domain implies completeness of the metric. The "converse" dependence will be sketched in the last paragraph of Section 3.4.1: in algebraic domains, compactness of the induced metric space implies Lawson-compactness of the domain and its second-countability (cf. Corollary 3.70).

Let us start with an auxiliary characterization of completeness by appropriate continuity of the self-distance mapping.

Theorem 3.54. *Let P be a bounded-complete continuous dcpo equipped with a stable partial metric $p\colon P \times P \to [0,\infty)$ compatible with the Scott topology on P. The following are equivalent:*

1. *The induced metric d_p is complete.*
2. *The self-distance map μ_p is cocontinuous, i.e. it preserves infima of filtered subsets of P.*

Proof. (2)$\Rightarrow$(1). Let (x_n) be a Cauchy sequence in P with respect to the induced metric d_p. Define $y_n := \bigsqcap_{i=n+1}^{\infty} x_i$. Since P is a dcpo, the chain (y_n) has a lub, say, $z := \bigsqcup^{\uparrow} y_n$. We will show that z is the metric limit of (x_n). Since the sequence (x_n) is Cauchy, for all $\varepsilon > 0$ there is $n_0 \in \omega$

such that for all $n \geq n_0$ we have $d_p(x_n, x_{n+1}) < \varepsilon$. Stability of p yields $\mu_p(x_n \sqcap x_{n+1}) - \mu_p x_n < \varepsilon$. Therefore $\inf_n \mu_p(x_n, x_{n+1}) = \inf_n \mu_p x_n$. It follows easily by induction that for a fixed $k \in \omega$ we have

$$\inf_n \mu_p(\prod_{i=n+1}^{n+1+k} x_i) = \mu_p x_n. \tag{3.3}$$

Now, we claim that

$$\lim_n p(z, x_n) = \mu_p z \tag{3.4}$$

and the limit on the left-hand side exists. For, $\mu_p z \leq p(z, x_{n+1}) \leq p(z, y_n) + p(y_n, x_{n+1}) - \mu_p y_n = p(y_n, x_{n+1}) = \mu_p y_n$ and since $\lim_n \mu_p y_n = \mu_p z$, we have $\lim_n p(z, x_n) = \mu_p z$.

Next, we calculate

$$\begin{aligned}
\lim_n \mu_p y_n &= \inf_n \mu_p(\prod_{i=n+1}^{\infty} x_i) = \inf_n \mu_p(\prod_{k\in\omega} \prod_{i=n+1}^{n+1+k} x_i) \\
&= \inf_n \sup_k \ \mu_p(\prod_{i=n+1}^{n+1+k} x_i) = \sup_k \inf_n \mu_p(\prod_{i=n+1}^{n+1+k} x_i) \\
&\overset{(3.3)}{=} \sup_k \inf_n \mu_p x_n = \inf_n \mu_p x_n.
\end{aligned}$$

Therefore, $\lim(\mu_p y_n - \mu_p x_n)$ exists and equals 0. Combining this observation with (3.3) we get

$$\lim(p(z, x_n) - \mu_p z) + \lim(p(z, x_n) - \mu_p y_n) + \lim(\mu_p y_n - \mu_p x_n) = 0,$$

which is equivalent to saying that $\lim_n d_p(z, x_n) = 0$.

The proof of (1)$\Rightarrow$(2) follows from Lemma 3.4 in [O'N95a]. We sketch the construction: let F be a filtered subset of P. By bounded-completeness of P, F has infimum $x \in P$. Since $F \sqsupseteq x$, we have $\mu_p(F) \leq \mu_p x$ and so $\alpha := \lim \mu_p(F)$ exists in $[0, \infty)$. Next, we find a descending chain (z_n) in F with $\lim z_n = \alpha$. Since (z_n) is Cauchy, it has a metric limit $c \in P$. By showing that $c = x$ (using the sharp triangle inequality of p), we complete the proof. $\square$

In order to prove that bounded-completeness of the domain implies metric completeness, we employ a topological characterization of complete metric spaces.

Theorem 3.55 (Engelking [Eng89]). *A Tychonoff space X is Čech-complete if and only if there is a countable collection $\{\mathcal{R}_n \mid n \in \omega\}$ of open covers of X, which has the following property: for any collection $\mathcal{F}$ of closed sets of X such that*

1. The family $\mathcal{F}$ has the finite intersection property,

2. $\forall k \in \omega.\ \exists F_k \in \mathcal{F}.\ \exists U_k \in \mathcal{R}_k.\ F_k \subseteq U_k$,

we have $\bigcap \mathcal{F} \neq \emptyset$.

The following result uses the idea of Theorem 4.1 in [Marb], which again goes back to [Eng89].

Theorem 3.56. *Let $\langle D, \rho\rangle$ be a bounded-complete continuous dcpo equipped with a partial symmetric $\rho\colon D\times D \to [0,\infty)$ compatible with the Scott topology on D. If*

1. The map ρ satisfies ($\mathbf{L}$),

2. The dual topology has a neighbourhood basis consisting of Scott-closed sets,

then the induced symmetric topology is completely metrizable.

Proof. By assumption (1) the topology of the induced symmetric d_ρ, denoted for convenience as τ_d, is metrizable (cf. Theorem 2.28, page 38) and hence Tychonoff. To show complete metrizability we will prove Čech-completeness of τ_d and then the desired conclusion will follow from Theorem 3.55 above.

Let $\{\mathcal{G}_n \mid n \in \omega\}$ be a development for τ_d. Let $\{A_i \mid i \in I\}$ be a basis for the dual topology τ_ρ^{op} consisting of Scott-closed sets. Define a countable collection of τ_d-open covers $\{\mathcal{R}_n \mid n \in \omega\}$ of D by

$$\mathcal{R}_n := \{\mathbf{int}_{\tau_d}(\twoheaduparrow b \cap A_i) \mid \exists U \in \mathcal{G}_n.\ \uparrow b \cap A_i \subseteq U \text{ for some } b \in D \textit{ and } i \in I\}.$$

The family $\{\mathcal{R}_n\}$ is well-defined thanks to interpolation in D and due to the fact that the sets of the form $\twoheaduparrow b \cap A_i$ form a neighbourhood basis for the induced topology τ_d.

Let $\mathcal{F}$ be a collection of τ_d-closed sets with the finite intersection property such that

$$\forall k \in \omega.\ \exists F_k \in \mathcal{F}.\ \exists R \in \mathcal{R}_k.\ F_k \subseteq R. \tag{3.5}$$

We must demonstrate that $\bigcap \mathcal{F} \neq \emptyset$. Define for every $n \in \omega$ a set $C_n := \bigcap_{k=1}^n F_k$ and choose $x_n \in C_n$. By formula (3.5) and definition of $\{R_n\}$, we have

$$\forall n \in \omega.\ \ x_n \in C_n \subseteq \uparrow b_n \cap A_n \subseteq U_n, \tag{3.6}$$

where $U_n \in \mathcal{G}_n$. Next, define $y_n := \bigsqcap_{i=n}^{\infty} x_i$. Since the sequence (C_n) is descending,

$$\forall n.\ \forall k \geq n.\ x_k \in \uparrow b_n \cap A_n$$

and so $b_n \sqsubseteq y_k \sqsubseteq x_k \in A_n$. Since each A_n is downward-closed,

$$\forall n.\ \forall k \geq n.\ y_k \in \uparrow b_n \cap A_n. \tag{3.7}$$

Now, since D is a dcpo, the lub of the chain (y_n) exists. Put $z := \bigsqcup^{\uparrow} y_n$. By formula (3.7) and the fact that each A_n is closed under directed suprema,

$$\forall n.\ \forall k \geq n.\ z \in \uparrow b_n \cap A_n$$

and hence by formula (3.6) we have

$$\forall n \in \omega.\ \ z \in U_n \in \mathcal{G}_n. \tag{3.8}$$

We claim that $(x_n) \to_{\tau_d} z$. Let V be any τ_d-open set around z. By developability, there exists $m \in \omega$ such that $z \in \mathbf{St}(z, \mathcal{G}_m) \subseteq V$. Consequently, by (3.6),(3.7) and (3.8),

$$\forall k \geq m.\ x_k \in C_m \subseteq U_m \subseteq \mathbf{St}(z, \mathcal{G}_m) \subseteq V,$$

and this proves that $(x_n) \to_{\tau_d} z$.

Since each set C_n is τ_d-closed and the sequence (x_n) is cofinally in each of C_n, we infer that $z \in C_n$ and so $z \in F_n$ for all $n \in \omega$. Since $\{\mathbf{St}(z, \mathcal{G}_n)\}_{n \in \omega}$ is a basis at z and τ_d is Hausdorff,

$$\{z\} \subseteq \bigcap_n F_n \subseteq \bigcap_n \mathbf{St}(z, \mathcal{G}_n) = \{z\}.$$

Therefore, $\bigcap_n F_n = \{z\}$.

Now, for any $F \in \mathcal{F}$ we have that $\{F \cap F_n \mid n \in \omega\}$ is again a nonempty sequence of τ_d-closed sets satisfying equation (3.5). Hence, the same argument proves that its intersection is nonempty. We have

$$\emptyset \neq \bigcap_n (F \cap F_n) = F \cap \bigcap_n F_n = F \cap \{z\}$$

and then $z \in F$. We conclude that $z \in \bigcap \mathcal{F}$. That is, $\bigcap \mathcal{F} \neq \emptyset$, as required. □

3.3.3 Examples

The Cantor set model Σ^∞

Let Σ^∞ denote the set of all finite and infinite words over a finite alphabet Σ, with the prefix ordering. This is an ω-algebraic domain. For all $x, y \in \Sigma^\infty$,

$x \ll y$ holds iff $x \sqsubseteq y$ and x is finite. The mapping $2^{-|\cdot|}\colon \Sigma^\infty \to [0,\infty)^{op}$, where $|\cdot|\colon \Sigma^\infty \to \omega \cup \{\infty\}$ takes a string to its length is a measurement on Σ^∞ [Mar00]. The measurement is modular so the partial semimetric $p_\mu(x,y) = 2^{-|r|}$, where r is the largest common prefix of x and y, is a partial metric on Σ^∞. The induced metric is Polish and compact [O'N95b]. In fact, it gives the Lawson topology on Σ^∞.

The powerset of naturals $\mathcal{P}\omega$

The collection of all subsets of ω ordered by inclusion is an ω-algebraic domain. The supremum of a directed set $S \subseteq \mathcal{P}\omega$ is $\bigcup S$ and for all elements x, y of $\mathcal{P}\omega$ the approximation relation is given by $x \ll y$ iff $x \subseteq y$ and x finite. The mapping $|\cdot|\colon \mathcal{P}\omega \to [0,\infty)^{op}$ given by $|x| := \sum_{n \notin x} 2^{-(n+1)}$ is a measurement on $\mathcal{P}\omega$ [Mar00], which is modular. The partial semimetric associated with μ is given by $p_\mu(x,y) = \sum_{n \notin x \cap y} 2^{-(n+1)}$ and is a partial metric on $\mathcal{P}\omega$. The induced metric $d(x,y) = \sum_{n \in x \dot{-} y} 2^{-(n+1)}$, where $x \dot{-} y = (x \setminus y) \cup (y \setminus x)$ makes $\mathcal{P}\omega$ a Polish, compact space [O'N95b]. The metric is compatible with the Lawson topology on $\mathcal{P}\omega$.

The interval domain $\mathbf{I}\mathbb{R}$

The collection of compact, connected intervals of the real line ordered under reverse inclusion is an ω-continuous dcpo. The supremum of a directed set $S \subseteq \mathbf{I}\mathbb{R}$ is $\bigcap S$ and for all intervals $x, y \in \mathbf{I}\mathbb{R}$ we have $x \ll y$ iff x is contained in the interior of y. The length function $|\cdot|\colon \mathbf{I}\mathbb{R} \to [0,\infty)^{op}$ given by $|x| = b - a$, where $x = [a,b] \in \mathbf{I}\mathbb{R}$, is a weakly modular measurement on $\mathbf{I}\mathbb{R}$ [Mar00]. The partial semimetric given by $p_\mu([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$, where $[a,b],[c,d] \in \mathbf{I}\mathbb{R}$, is a partial metric on $\mathbf{I}\mathbb{R}$. The induced metric is given by $d(x,y) = |a-c| + |b-d|$ for any intervals $[a,b]$ and $[c,d]$ of $\mathbf{I}\mathbb{R}$ and hence it is compatible with the Lawson topology on $\mathbf{I}\mathbb{R}$. Indeed,

$$\begin{aligned}
d([a,b],[c,d]) &= 2\max\{b,d\} - 2\min\{a,c\} - (b-a) - (d-c) \\
&= (2\max\{b,d\} - b - d) - (2\min\{a,c\} - a - c) \\
&= (\max\{b,d\} - \min\{b,d\}) + (\max\{a,c\} - \min\{a,c\}) \\
&= |b-d| + |a-c|.
\end{aligned}$$

Therefore, $(\mathbf{I}\mathbb{R}, d)$ is a Polish space.

Each of the domains above admits a weakly modular measurement and hence the associated partial semimetric is a partial metric by Proposition 3.23. In the example that follows, this is no longer the case.

The formal ball model $\mathbf{B}X$

Introduced in [EH98]. Let (X, d_X) be a metric space.

$$\mathbf{B}X := \{(x, r) \mid x \in X, r \in [0, \infty)\}$$

is a continuous poset ordered by $(x, r) \sqsubseteq (y, s)$ iff $d_X(x, y) \leq r - s$. The way-below relation is characterised by $(x, r) \ll (y, s)$ iff $d_X(x, y) < r - s$. One can show that $\mathbf{B}X$ is a dcpo iff the metric d_X is complete. The mapping $\mu\colon \mathbf{B}X \to [0, \infty)^{op}$ given by $\mu(x, r) = r$ is a measurement on $\mathbf{B}X$ [Mar00]. The associated partial semimetric p_μ is not a partial metric in general. For example, take the four-element subset $A := \{x = 0, y = 100, w = 101, z = 102\}$ of the real line with the natural metric inherited from $\mathbb{R}$. Then $(x, 100), (y, 1), (z, 1) \in \mathbf{B}A$. We calculate:

$$\begin{aligned}
&p_\mu[(x, 100), (y, 1)] = \mu(x, 101) = 101.\\
&p_\mu[(y, 1), (z, 1)] = \mu(w, 2) = 2.\\
&p_\mu[(x, 100), (z, 1)] = \mu(x, 103) = 103.
\end{aligned}$$

Therefore,

$$p_\mu[(x, 100), (z, 1)] + \mu(y, 1) = 104 > 101 + 2 = p_\mu[(x, 100), (y, 1)] + p_\mu[(y, 1), (z, 1)],$$

and so p_μ on $\mathbf{B}A$ does not satisfy $\Delta^\sharp$. In general, however, the formal ball model $\mathbf{B}X$ admits a weak partial metric $p\colon \mathbf{B}X \times \mathbf{B}X \to [0, \infty)$ compatible with the Scott topology [Hec99]. It is defined as $p((x, r), (y, s)) := d_X(x, y) + r + s$. A partial metric for the Scott topology on $\mathbf{B}X$ with **(ssd)** is then obtained by a standard construction. One can readily see that its induced metric topology is compact iff (X, d_X) is. In this case the induced metric gives the Lawson topology on $\mathbf{B}X$.

The domain $\mathbb{T}^\omega$

The domain $\mathbb{T}^\omega$ was first described in [Plo78]. It was shown to be universal for the class of all coherent Scott domains. $\mathbb{T}^\omega$ can be viewed as the class of pairs of disjoint subsets of ω. Thus,

$$\mathbb{T}^\omega = \{\langle u_0, u_1\rangle \mid u_0, u_1 \subseteq \omega, u_0 \cap u_1 = \emptyset\}.$$

If $t = \langle u_0, u_1\rangle \in \mathbb{T}^\omega$, then we put $(t)_0 := u_0$ and $(t)_1 := u_1$. The order is characterised by $t \sqsubseteq s$ iff $(t)_0 \subseteq (s)_0$ and $(t)_1 \subseteq (s)_1$. The maximal elements in $\mathbb{T}^\omega$ are elements t, such that $(t)_0 \cup (t)_1 = \omega$. $\mathbb{T}^\omega$ is ω-algebraic. The compact elements are of the form $\langle (b)_0, (b)_1\rangle$, where both $(b)_0$ and $(b)_1$ are finite subsets of ω. $\mathbb{T}^\omega$ is a bounded-complete dcpo with a least element. A subset X of a dcpo D is *pairwise-consistent* if every pair of elements of X is bounded. A dcpo D is *coherent* iff every pairwise-consistent subset of D has a supremum. $\mathbb{T}^\omega$ is coherent. For a bounded-complete dcpo, coherence amounts to:

$$\text{(coh)} \qquad \forall t \in \mathbb{T}^\omega.\, \forall S \subseteq \mathbb{T}^\omega. \quad (\bigsqcup S \text{ exists } \wedge\ t\#\bigsqcup S) \Rightarrow \exists s \in S.\ t\#s.$$

Proposition 3.57. *The map $\mu\colon \mathbb{T}^\omega \to [0,1)^{op}$ given by*

$$\mu(t) := \sum_{n \notin (t)_0 \cup (t)_1} \frac{1}{2^{n+1}}$$

is a modular measurement on $\mathbb{T}^\omega$ with $\ker\mu = \max \mathbb{T}^\omega$.

Proof. The map μ is easily seen to be monotone. If $D := (t_i)_{i\in I}$ is a directed, lower set (ideal) of elements with supremum t, then $\mu(\bigcup t_i) \le \bigsqcup^\uparrow \mu t_i$, by monotonicity. On the other hand, for all $m \in \omega$, we have

$$\begin{aligned}
\mu(\bigcup t_i) &= \sum_{n \notin \bigcup_i (t_i)_0 \cup (t_i)_1} \frac{1}{2^{n+1}} \ge \sum_{\substack{n \notin \bigcup_i (t_i)_0 \cup (t_i)_1 \\ n<m}} \frac{1}{2^{n+1}} = \sum_{n \notin (t_j)_0 \cup (t_j)_1} \frac{1}{2^{n+1}} \\
&\ge \mu(t_j) - \frac{1}{2^m} \ge \bigsqcup_{i\in I}{}^{\uparrow} \mu(t_i) - \frac{1}{2^m},
\end{aligned}$$

where

$$t_j := \langle \{(t_i)_0 \mid i \in I\} \cap \{0,1,...,m-1\}, \{(t_i)_1 \mid i \in I\} \cap \{0,1,...,m-1\} \rangle.$$

(Note that $(t_j)_0 \cup (t_j)_1$ is a finite subset of $\bigcup_i (t_i)_0 \cup (t_i)_1$ and D is a lower set. Thus $t_j \in D$.) Hence μ is Scott-continuous. Suppose $t \in \Uparrow z$ for some $z \in \mathbb{T}^\omega$. Then $(z)_0$ and $(z)_1$ are finite sets. Set $m := \max\{n \mid n \in (z)_0 \cup (z)_1\} + 1$. We want to show that $\mu(t, 1/2^m) \subseteq \uparrow z$. If $y \in \mu(t, 1/2^m)$, then

$$\sum_{n \notin (y)_0 \cup (y)_1} \frac{1}{2^{n+1}} < \sum_{n \notin (t)_0 \cup (t)_1} \frac{1}{2^{n+1}} + \frac{1}{2^m}.$$

For the sake of contradiction, suppose that $a \in (z)_0 \setminus (y)_0$ or $b \in (z)_1 \setminus (y)_1$. Then we have that either $y \sqsubseteq \langle (t)_0 \setminus \{a\}, (t)_1 \rangle$ or $y \sqsubseteq \langle (t)_0, (t)_1 \setminus \{b\} \rangle$. Therefore,

$$\sum_{n \notin (y)_0 \cup (y)_1} \frac{1}{2^{n+1}} \ge \sum_{n \notin ((t)_0 \setminus \{a\}) \cup ((t)_1 \setminus \{b\})} \frac{1}{2^{n+1}} \ge \sum_{n \notin (t)_0 \cup (t)_1} \frac{1}{2^{n+1}} + \frac{1}{2^m},$$

which is a contradiction. We conclude that $\mu(t, 1/2^m) \subseteq \uparrow z$, which means that the map μ measures $\mathbb{T}^\omega$. Now, the modularity of μ follows straight from the fact that infima and suprema of elements in $\mathbb{T}^\omega$ are given respectively by intersections and unions of the underlying sets $(\cdot)_0$ and $(\cdot)_1$. Finally, for every $t \in \mathbb{T}^\omega$ we have $\mu z = 0$ iff $(z)_0 \cup (z)_1 = \omega$ iff z is maximal in $\mathbb{T}^\omega$, what is equivalent to say that $\ker\mu = \max \mathbb{T}^\omega$. $\square$

Since $\mathbb{T}^\omega$ admits a modular measurement, the partial semimetric associated with μ is a partial metric.

$$p_\mu(x, y) = \mu(x \sqcap y) = \sum_{n \notin [(x)_0 \cap (y)_0] \cup [(x)_1 \cap (y)_1]} \frac{1}{2^{n+1}}.$$

Now, the metric induced by p_μ is given by

$$\begin{aligned}
d(x,y) &= 2(1 - \sum_{n \in [(x)_0 \cap (y)_0] \cup [(x)_1 \cap (y)_1]} \frac{1}{2^{n+1}}) \\
&- (1 - \sum_{n \in (x)_0 \cup (x)_1} \frac{1}{2^{n+1}}) - (1 - \sum_{n \in (y)_0 \cup (y)_1} \frac{1}{2^{n+1}}) \\
&= \sum_{n \in (x)_0 \cup (x)_1} \frac{1}{2^{n+1}} + \sum_{n \in (y)_0 \cup (y)_1} \frac{1}{2^{n+1}} - 2 \cdot \sum_{n \in [(x)_0 \cap (y)_0] \cup [(x)_1 \cap (y)_1]} \frac{1}{2^{n+1}} \\
&= \sum_{n \in [(x)_0 \setminus (y)_0] \cup [(y)_0 \setminus (x)_0]} \frac{1}{2^{n+1}} + \sum_{n \in [(x)_1 \setminus (y)_1] \cup [(y)_1 \setminus (x)_1])} \frac{1}{2^{n+1}} \\
&= \sum_{n \in (x)_0 \dot{-} (y)_0} \frac{1}{2^{n+1}} + \sum_{n \in (x)_1 \dot{-} (y)_1} \frac{1}{2^{n+1}}.
\end{aligned}$$

Proposition 3.58. *$(\mathbb{T}^\omega, d)$ is Polish.*

Proof. Let (x_n) be a Cauchy sequence in $(\mathbb{T}^\omega, d)$. Then for any $k \geq 0$, there exists an $N_k \geq 0$ such that for any $n, m \geq N_k$,

$$\sum_{i \in (x_n)_0 \dot{-} (x_m)_0} \frac{1}{2^{i+1}} + \sum_{i \in (x_n)_1 \dot{-} (x_m)_1} \frac{1}{2^{i+1}} < \frac{1}{2^{k+1}}.$$

Hence,

$$\forall n, m \geq N_k.\ \ k \in (x_n)_0 \text{ iff } k \in (x_m)_0 \tag{3.9}$$

and the same holds for the $(\cdot)_1$ components. We will show that the element $z \in \mathbb{T}^\omega$ given by

$$z = \langle \{a \mid a \in (x_{N_a})_0\}, \{b \mid b \in (x_{N_b})_1\} \rangle$$

is the limit of (x_n) in $\mathbb{T}^\omega$. Let $\varepsilon > 0$. Then there exists $k \geq 0$ such that $\sum_{j=k+1}^{\infty} 2^{-(j+1)} < \varepsilon$. Take any $n \geq \max\{N_j \mid j = 0, ..., k\}$. Then

$$d(x_n, z) = \sum_{i \in (x_n)_0 \dot{-} (z)_0} \frac{1}{2^{i+1}} + \sum_{i \in (x_n)_1 \dot{-} (z)_1} \frac{1}{2^{i+1}} < \sum_{j=k+1}^{\infty} \frac{1}{2^{j+1}} < \varepsilon.$$

The first inequality above is indeed correct since for each $j = 0, 1, ..., k$ we have $j \in (x_n)_0$ iff $j \in (x_{N_j})_0$ by equation (3.9), and $j \in (x_{N_j})_0$ iff $j \in (z)_0$ by definition of z. Note that the same is true for the $(\cdot)_1$ components. Therefore, $(x_n) \to z$ in $\mathbb{T}^\omega$, as required. Now, by Theorem 3.7 of [O'N95b], $\tau_d(\mathbb{T}^\omega)$ is second-countable since the Scott topology is second-countable. Equivalently, $\tau_d(\mathbb{T}^\omega)$ is separable and hence Polish. □

Proposition 3.59. $(\mathbb{T}^\omega, d)$ *is totally bounded.*

Proof. Let $\varepsilon > 0$ and $k \in \omega$ be such that $0 < 2^{-(k+1)} < \varepsilon$. Define $A := \{x \in \mathbb{T}^\omega \mid (x)_0 \cup (x)_1 \subseteq \{0, ..., k\}\}$. Then A is finite. For $y \in \mathbb{T}^\omega$ define $x \in \mathbb{T}^\omega$ by $(x)_0 := (y)_0 \cap \{1, ..., k\}$ and $(x)_1 := (y)_1 \cap \{1, ..., k\}$. Then $x \in A$ and $d(x, y) \leq \sum_{i=k+2}^{\infty} 2^{-i} = 2^{-(k+1)} < \varepsilon$. □

Proposition 3.60. $(\mathbb{T}^\omega, d)$ *is compact. The topology induced by d is the Lawson topology on $\mathbb{T}^\omega$.*

Proof. Since $(\mathbb{T}^\omega, d)$ is completely metrizable and totally bounded, it must be compact. Since the Lawson topology is always contained in the metric topology, they must be equal. □

3.4 Existence of measurements and partial metrics on domains

We seek a construction of a partial metric p on a domain D such that the following three criteria are fulfilled:

1. The partial metric topology coincides with the Scott topology, $\tau_p(D) = \sigma(D)$.

2. The measurement induced by p assigns zero precisely to the set of the maximal elements of the domain, $\ker \mu_p = \max D$ (the kernel condition).

3. The induced metric d_p is complete.

3.4.1 Algebraic domains

Let P be an algebraic poset and $K(P)$ its basis. First, we characterise the space of maximal elements of P.

Lemma 3.61. *For any algebraic poset the following are equivalent:*

1. *x is maximal in P,*

2. *$\forall b \in K(P).\ (b \ll x$ or $b \# x)$.*

Proof. Suppose $x \notin \max P$. Then there exists an $y \in P$ such that $x \sqsubset y$. This implies that y is not below x. Therefore, there exists b in $K(P)$ such that $b \ll y$ and b is not way-below x. Hence $y \sqsupseteq x, b$, that is, $b \uparrow x$. Together with the fact that b is not way-below x, we have the negation of the statement (2). Conversely, if $x \in \max P$, then for each b in $K(P)$ either $b \ll x$ or not. In the former case, we are done. In the latter case, suppose $b \uparrow x$. But since x is maximal, and b compact, $b \ll x$, a contradiction. Therefore, $b \# x$, as required. $\square$

Note that the statement of the lemma remains valid for any continuous poset P. Indeed, let B be a basis of P. One can show that an x in P is maximal iff either $b \ll x$ or $\hat{b} \# x$ holds for all pairs $b \ll \hat{b}$ from B.

The Lawson condition

By the Lawson condition we understand the requirement that the Scott and the Lawson topology agree at the top of D, written as

$$\forall x \in D. \quad \uparrow x \cap \max D = A \cap \max D \text{ for some Scott}-\text{closed subset } A \text{ of } D. \tag{Law}$$

In fact, (Law) is equivalent to the following:

$$\forall x \in D. \quad \uparrow x \cap \max D = \mathbf{cl}_\sigma(\uparrow x) \cap \max D. \tag{Law'}$$

Whenever (Law) holds, Lemma 3.61 has an alternative formulation.

Lemma 3.62. *Let P be an algebraic poset with* (Law)*. Then the following are equivalent:*

1. *x is maximal in P.*

2. *$\forall b \in K(P).\ b \ll x$ or $\exists c \in K(P).\ c \ll x$ and $b \# c$.*

Proof. Let $x \in \max P$. Suppose that b is not way-below x. Equivalently, $x \notin \uparrow b$. By (Law), there exists a Scott-closed set A such that $x \notin A$ and $\uparrow b \subseteq A$. Hence $P \setminus A$ is a Scott-open neighbourhood of x. Therefore, there exists $c \in K(P)$ such that $c \ll x$ and $c \notin A$. This implies $c \# b$. Conversely, suppose that x is not maximal in P. Then $x \sqsubset y$ for some $y \in P$. Hence there exists $b \in K(P)$ with $b \ll y$ and $b \not\sqsubseteq x$. Obviously, $y \sqsupseteq x, b$ and so $b \uparrow x$. Therefore, for all $c \ll x$, b is not way-below x and $b \uparrow c$, which is the negation of the statement (2). □

Theorem 3.63. *Every ω-algebraic dcpo D with* (Law) *admits a partial metric $p\colon D \times D \to [0, \infty)$ for the Scott topology on D such that $\tau_p(D) = \sigma(D)$ and such that the induced measurement μ_p satisfies the kernel condition.*

Proof. Our result is a direct consequence of Theorem 20 of Plotkin [Plo78]. Define $\eta\colon D \to \mathbb{T}^\omega$ by

$$\forall x \in D. \quad \eta(x) := \langle \{n \mid b_n \ll x\}, \{m \mid \exists n \in \omega.\ b_n \ll x \text{ and } b_n \# b_m\} \rangle,$$

where $K(D) = \{b_1, b_2, ...\}$. The function η is a well-defined Scott-continuous order-embedding. If U is a Scott-open set of D, then since $x \in U$ iff $\exists i \in \omega.\ x \in \uparrow b_i \subseteq U$,

$$\eta[U] = \{\eta(x) \mid x \in U\} = \eta[D] \cap \{t \in \mathbb{T}^\omega \mid \exists i \in \omega.\ \uparrow b_i \subseteq U \wedge i \in (t)_0\}.$$

Therefore, $\eta[U]$ is a Scott-open subset of $\eta[D]$ in the subspace topology, proving that η is a homeomorphism onto $\eta[D]$. Hence the restriction of the partial metric from $\mathbb{T}^\omega$ to the subspace $\eta[D]$ is a partial metric for the Scott topology on D. Since $\tau_p(D) = \sigma(D)$, by Corollary 3.15 the self-distance map of the partial metric on D is a measurement on D. Finally, Lemma 3.62 assures that $\ker \mu_p = \max D$. □

The last result and Theorem 3.56 together imply that

Theorem 3.64. *Every Scott domain admits a partial metric for the Scott topology, which satisfies the kernel condition and such that the induced metric is complete.* □

More on the Lawson condition

Let P be an algebraic poset and $K(P)$ its basis. We examine three other requirements in the spirit of the Lawson condition, namely:

(cp) D is Lawson-compact,

(†) $\forall x \in P. \quad \downarrow\uparrow x = \mathbf{cl}_\sigma(\uparrow x)$,

(‡) $\forall c \in K(P). \quad \downarrow\uparrow c = \mathbf{cl}_\sigma(\uparrow c)$.

Proposition 3.65. *For an algebraic dcpo P the conditions above form the following strict hierarchy:*

$$\text{(cp)} \Rightarrow (\dagger) \Rightarrow (\ddagger) \Rightarrow \text{(Law)}.$$

Proof. Assume (cp). To show (†) it is enough to demonstrate that for every $z \in D$, the set $\downarrow\uparrow z$ is closed with respect to existing suprema of directed sets. Let $(x_i)_{i\in I}$ be a directed set of elements from $\downarrow\uparrow z$ and let $x = \bigsqcup^{\uparrow} x_i$. For every $i \in I$, there is a $z_i \sqsupseteq z, x_i$. Hence $\uparrow x_i \cap \uparrow z \neq \emptyset$. In fact, by Lawson-compactness, the collection of subsets $\{C_i\}_{i\in I}$, where $C_i := \uparrow x_i \cap \uparrow z$, is a filtered family of nonempty compact saturated subsets of $\downarrow\uparrow z$. Therefore, by the Hofmann-Mislove Theorem, their intersection $\bigcap_{i\in I} C_i$ is nonempty as well. Obviously, any element $x \in \bigcap_{i\in I} C_i$ is an upper bound of all (x_i) and z. Hence $x = \bigsqcup^{\uparrow} x_i \subseteq \downarrow x \subseteq \downarrow\uparrow z$. That is, $x \in \downarrow\uparrow z$, as required.

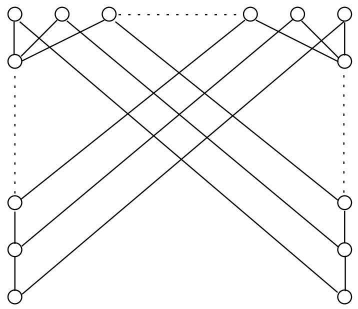
(†) implies (‡) trivially. Hence assume (‡) and let $x \in P$. Suppose that $y \in \max P$ and $x \not\sqsubseteq y$. Hence there exists $c_y \in K(P)$ with $c_y \ll x$ and $c_y \not\sqsubseteq y$. By (‡), $y \in P \setminus \mathbf{cl}_\sigma(\uparrow c_y)$. Therefore, there exists $b_y \in K(P)$ with $b_y \ll y$ and such $b_y \# c$. Now, define $A := P \setminus \bigcup_{y \in \max P \setminus \uparrow x} \uparrow b_y$. A is a Scott-closed set and $\uparrow x \cap \max P = A \cap \max P$. This shows (Law). The examples in Figure 3.2 show that all implications are strict. □

We will now use (‡) to get a variation of Lemma 3.62, which as a result allows us to obtain a simple description of the partial metric constructed in Theorem 3.63 and its induced metric. For any $c \in K(P)$ define an auxiliary function $\delta_c \colon P \to \mathbb{B}_\perp$ by

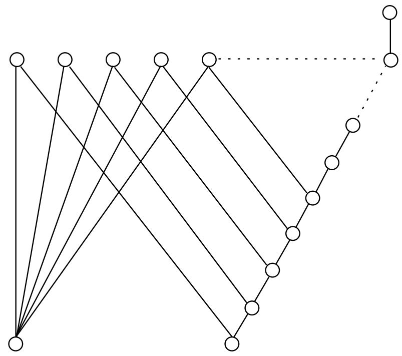
$$\delta_c(y) := \begin{cases} tt & \text{if } c \sqsubseteq y \\ f\!f & \text{if } c \# y \\ \perp & \text{otherwise} \end{cases}$$

Proposition 3.66. *Let P be an algebraic poset. The following are equivalent:*

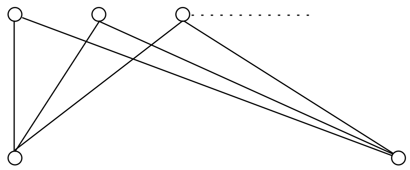
1. *P satisfies (‡).*
2. $\forall x \in P$. $\{b \in K(P) \mid \exists c \in K(P).\ c \ll x$ and $b \# c\} = \{b \in K(P) \mid b \# x\}$.
3. $\forall c \in K(P)$. *δ_c is Scott-continuous.*



(‡) but not (†)



(L) but not (‡)



(†) but not (cp)

Figure 3.2: Separating examples

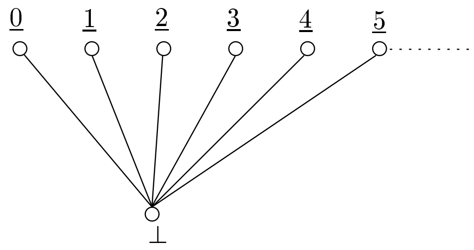


Figure 3.3: Flat natural numbers $\omega_\perp$

Proof. We will show the equivalence of (1) and (2). Assume (‡). Let $x \in P$. Suppose that there exists $c \in K(P)$ such that $c \ll x$ and $b\#c$ for some $b \in K(P)$. But this immediately implies $b\#x$. Conversely, if $b\#x$ for some $b \in K(P)$, then $\uparrow x \cap \downarrow\uparrow b = \emptyset$. Now, by (‡), $D \setminus \downarrow\uparrow b$ is a Scott-open set around x. Therefore, there exists $c \in K(P)$ with $x \in \uparrow c \subseteq D \setminus \downarrow\uparrow b$ The last inclusion is equivalent to $\uparrow c \cap \downarrow\uparrow b = \emptyset$, which allows us to conclude $b\#c$. $\square$

Corollary 3.67. *The partial metric on an ω-algebraic dcpo with (‡) constructed in Theorem 3.63 is given by*

$$p(x, y) = 1 - \sum_{\{n \mid (x,y \sqsupseteq b_n) \,\vee\, (x,y\#b_n)\}} \frac{1}{2^{n+1}}$$

and its induced metric is

$$d_p(x, y) = \sum_{\{n \mid (x \sqsupseteq b_n \,\wedge\, y \not\sqsupseteq b_n) \,\vee\, (x \not\sqsupseteq b_n \,\wedge\, y \sqsupseteq b_n)\}} \frac{1}{2^{n+1}} + \sum_{\{n \mid (x\#b_n \,\wedge\, y\uparrow b_n) \,\vee\, (x\uparrow b_n \,\wedge\, y\#b_n)\}} \frac{1}{2^{n+1}}.$$

$\square$

The metric given above in not complete in general:

Example 3.68. The domain of flat natural numbers $\omega_\perp$, see Figure 3.3, is a Scott-domain where every element is compact. Theorem 3.63 guarantees the existence of a partial metric for the Scott topology on $\omega_\perp$. Its induced metric, given by the formula in Corollary 3.67, reduces to:

$$d(\underline{n}, \underline{m}) := \begin{cases} 0 & \text{if } \underline{n} = \underline{m} \\ \frac{1}{2^{n+1}} + \frac{1}{2^{m+1}} & \text{if } \underline{n} \neq \underline{m} \text{ and } \underline{n}, \underline{m} \neq \perp \\ 1 & \text{otherwise} \end{cases}$$

The metric is not complete: the sequence $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \ldots$ is Cauchy but not convergent. The topology induced by the metric is discrete and hence different from the Lawson topology, in which $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \ldots$ converges to $\perp$.

Nevertheless, the Lawson topology on an arbitrary ω-algebraic poset P always comes from a metric induced by a partial metric on P.

Theorem 3.69. *Every ω-algebraic poset P admits a partial metric $p\colon D \times D \to [0, \infty)$ with $\tau_p(P) = \sigma(P)$ and $\tau_d(P) = \lambda(P)$.*

Proof. Let $\{\twoheaduparrow b_n \mid n \in \omega\}$ be a countable basis for the Scott topology on P. First, consider a map $q\colon P \times P \to [0, \infty)$ given by

$$q(x, y) := \sum_{\{n\in\omega \mid b_n \in \twoheaddownarrow x \setminus \twoheaddownarrow y\}} \frac{1}{2^{n+1}}.$$

Let $x, y, z \in P$. Surely, $q(x, x) = 0$. Moreover, since P is a T_0 space in its Scott topology, $q(x, y) = q(y, x) = 0$ implies that $x = y$. For the triangle inequality, suppose that $b_n \in \twoheaddownarrow x \setminus \twoheaddownarrow y$ for some $n \in \omega$. We have either $b_n \notin \twoheaddownarrow z$ or $b_n \in \twoheaddownarrow z$. Hence, $q(x, y) \le q(x, z) + q(z, y)$. Now, it is easy to check that $q(x, y) + wx = q(y, x) + wy$, where the map $w\colon P \to [0, \infty)^{op}$ is defined as $wx := 1 - \sum_{\{n \mid b_n \ll x\}} 2^{-(n+1)}$ Therefore, the mapping $p\colon P \times P \to [0, \infty)$ given by

$$p(x, y) := q(x, y) + wx = 1 - \sum_{\{n\in\omega \mid b_n \ll x,y\}} \frac{1}{2^{n+1}}$$

is a partial metric on P [Mat92]. We will show that it induces the Scott topology on P. Since, for any $x \in P$ and $k \in \omega$,

$$\bigcap\{\twoheaduparrow b_i \mid b_i \ll x \text{ and } i \le k+2\} \subseteq B^p(x, \frac{1}{2^{k+1}}),$$

we have $\tau_p(P) \subseteq \sigma(P)$. Conversely, suppose that $x \in \twoheaduparrow b_{n_0}$ for some $n_0 \in \omega$. If $y \in B^p(x, 1/2^{n_0+1})$, then for all $i \in \{0, ..., n_0+2\}$ we have that $b_i \ll x$ implies that $b_i \ll y$. Hence $b_{n_0} \ll y$. We have shown that $x \in B^p(x, 1/2^{n_0+1}) \subseteq \twoheaduparrow b_{n_0}$. Therefore, $\tau_p(P) \supseteq \sigma(P)$.

To prove $\tau_d(P) = \lambda(P)$, in the light of Lemma 3.44, it is enough to prove that the weak topology $\omega(P)$ is finer than the dual topology $\tau_p^{op}(P)$. Indeed, let $k \in \omega$. Then

$$\begin{aligned}
B^{op}(x, \frac{1}{2^{k+1}}) &= \{y \mid p(y,x) < p(y,y) + \frac{1}{2^{k+1}}\} = \{y \mid q(y,x) < \frac{1}{2^{k+1}}\} \\
&= \{y \mid \sum_{\{n \mid b_n \in \mathord{\downarrow\!\!\downarrow} y \setminus \mathord{\downarrow\!\!\downarrow} x\}} \frac{1}{2^{n+1}} < \frac{1}{2^{k+1}}\} \\
&\supseteq \{y \mid b_i \ll y \text{ implies } b_i \ll x, \text{ for all } i = 0, ..., k+2\} \\
&= P \setminus \{y \mid b_i \ll y \text{ and } b_i \not\ll x, \text{ for some } i = 0, 1, ..., k+2\} \\
&= P \setminus \bigcup \{\mathord{\uparrow\!\!\uparrow} b_i \mid b_i \not\ll x \text{ and } 0 \leq i \leq k+2\} \\
&= P \setminus \bigcup \{\uparrow b_i \mid b_i \not\ll x \text{ and } 0 \leq i \leq k+2\} \in \omega(P).
\end{aligned}$$

□

Note that whenever P is Lawson-compact, the induced metric d_p must be complete.

Corollary 3.70. *Let D be an algebraic dcpo. The following are equivalent:*

1. *D admits a partial metric for the Scott topology such that the induced metric topology is compact.*
2. *D is countably based and Lawson-compact.*

Proof. (1)⇒(2) holds by Corollary 3.49, page 72. The converse follows immediately from Theorem 3.69. □

3.4.2 Continuous domains

We demonstrate that every ω-continuous poset can be stably partially metrized. In addition, directed and bounded completeness of the underlying poset implies the completeness of the induced metric. Unfortunately, in general the kernel of the induced measurement does not agree with the set of maximal elements of the domain.

Theorem 3.71. *Let D be an ω-continuous poset. There exists a stable partial metric $p\colon D \times D \to [0, \infty)$ with $\tau_p(D) = \sigma(D)$. If D is a bounded-complete dcpo, the induced metric d_p is complete.*

Proof. Let $\{U_n \mid n \in \omega\}$ be a countable base for the Scott topology on D, consisting of Scott-open filters [AJ94]. The map

$$p(x,y) := 1 - \sum_{\{n \in \omega \mid x, y \in U_n\}} \frac{1}{2^{n+1}},$$

is a Scott-continuous partial metric on D. Indeed,

$$\begin{aligned} p(x,y) &= 1 - \sum_{\{n \,|x,y\in U_n\}} \frac{1}{2^{n+1}} = \bigsqcup\{1 - \sum_{\{n \,|z\in U_n\}} \frac{1}{2^{n+1}} \mid z \ll x,y\} \\ &= \bigsqcup\{\mu z \mid z \ll x,y\} = p_\mu(x,y), \end{aligned}$$

where μ is a measurement on D given by $\mu x := 1 - \sum_{\{n \,|x\in U_n\}} \frac{1}{2^{n+1}}$ described for a first time in [Mar00], Example 2.5.3, page 38, and p_μ is the associated partial semimetric. Note that because every U_n for $n \in \omega$ is a filter, the condition

$$x, y \in U_n \Rightarrow \exists z \in U_n.\ z \ll x, y$$

holds and the second equality above is indeed correct. Note that the sharp triangle inequality $\Delta^\sharp$ for p_μ is equivalent to:

$$\sum_{\{n \,|x,z\in U_n\}} \frac{1}{2^{n+1}} + \sum_{\{n \,|y,z\in U_n\}} \frac{1}{2^{n+1}} \leq \sum_{\{n \,|x,y\in U_n\}} \frac{1}{2^{n+1}} + \sum_{\{n \,|z\in U_n\}} \frac{1}{2^{n+1}}$$

and is readily checked to hold. Hence p_μ is a stable partial metric with $\tau_p(D) = \sigma(D)$. The induced metric is given by:

$$d_p(x,y) = \sum_{\{n\in\omega|(x\in U_n \,\wedge\, y\notin U_n) \,\vee\, (y\in U_n \,\wedge\, x\notin U_n)\}} \frac{1}{2^{n+1}}.$$

Assume D is a bounded-complete dcpo. The proof of completeness of the induced metric d_p can be immediately deduced from Theorem 3.56 but we prefer to give a separate proof that is a generalisation of the one given in [O'N95a], Lemma 4.4. Let (x_n) be a Cauchy sequence in D. Define $y_n := \sqcap\{x_k \mid k > n\}$. Then (y_n) is a chain in D. Let $z = \bigsqcup^\uparrow y_n$. We will show that z is the metric limit of (x_n). Take any $\varepsilon > 0$. Then there exists $K \in \omega$ with $2^{-K} < \varepsilon$. Since (x_n) is a Cauchy sequence, there exists $N_1 \in \omega$ such that for all $n, m \geq N_1$ we have $d_p(x_n, x_m) < 2^{-(K+1)}$. Therefore, for all $i = 0, ..., K$, for all $m, n \geq N_1$, $x_n \in U_i$ iff $x_m \in U_i$. Let $i \in \{0, ..., K\}$ and $n \geq N_1$. Suppose $x_n \in U_i$. Then $y_n \in U_i$ and consequently $z \in U_i$. Now, since z is the lub of (y_n), if $z \in U_i$ then there exists $N_2 \in \omega$ such that $y_n \in U_i$ for all $n \geq N_2$ and so $x_n \in U_i$ for all $n \geq N_2$. We conclude that for all $n \geq \max\{N_1, N_2\}$, $x_n \in U_i$ iff $z \in U_i$. Therefore, for $n \geq \max\{N_1, N_2\}$ we have $d_p(x_n, z) \leq \sum_{k=K+1}^{\infty} \frac{1}{2^{k+1}} < 2^{-K} < \varepsilon$. That is, $(x_n) \to z$, which means that the induced metric is complete. □

Corollary 3.72. *Every ω-continuous lattice L admits a partial metric $p \colon L \times L \to [0, \infty)$ such that $\tau_p(L) = \sigma(L)$, its induced metric d_p is complete and the induced measurement μ_p satisfies* $\ker\mu_p = \max L$.

Proof. Let μ_p be the induced measurement of the partial metric defined in the preceding theorem. One can readily see that $\top$ is the only element in L with measure 0, that is, $\ker\mu_p = \max L$. $\square$

Künzi and Vajner [KV92] prove that each topological T_0 space with a σ-point-bounded base admits a partial metric compatible with the topology. (A family $\mathcal{B}$ of subsets of a set X is point-bounded if there is an $m \in \omega$ such that

$$\mathrm{card}\{A \in \mathcal{B} \mid x \in A\} \leq m,$$

whenever $x \in X$. A base for a topological space is σ-point-bounded if it can be represented as a countable union of point-bounded families.) Since any countable base for the Scott-topology on a continuous poset P is a σ-point-bounded base, P must be partially metrizable. One can see that Theorems 3.69 and 3.71 are both special cases of Künzi and Vajner's construction.

It should be remarked that the two mentioned theorems were also demonstrated independently by O'Neill [O'N98] (Lemmata 6.2.5 and 6.2.4, pages 123–124) using the language of so called *generalized valuation spaces*. The latter result also appears in [Scha] with yet a different proof.

Chapter 4

Applications

In this chapter we apply the theory developed so far to two kinds of problems. Firstly, we perform constructions of distance maps (mainly focusing on partial metrics) on the product, a function space and an inverse limit of domains. Secondly, we investigate the problem of modelling first-countable topological spaces by domains.

4.1 Constructions

Our study of distance on continuous domains is motivated by Denotational Semantics. We wish to be able to use our quantitative domains as models of programming languages such as PCF. In fact, any cartesian closed subcategory of countably based continuous domains would be suitable for this purpose, since we know that any ω-continuous dcpo carries a natural partial metric (Theorem 3.71). In this light, the category of Scott domains seems to be best-suited for the modelling task, since any object admits a complete partial metric, which satisfies the kernel condition (Theorem 3.64). However, the situation is far from being satisfactory since the partial metric on the constructs (products, function spaces) is *not* formed from the distances on the components but comes from a general construction. In this section we attempt to improve our understanding of constructions on partially metrized domains. Our approach is simple to follow: we use measurements on the components to build a measurement on the more complex domain and then we recover the distance from it. This way of constructing stable partial metrics from weakly modular measurements was first described by Dieter Spreen in [Spr01][1]. We complement his results by showing that arbitrary

[1]Parts 1,3 and 5 in Theorems 4.1, 4.4 and 4.6 have been announced by Spreen. We nevertheless give their proofs, which have been found independently.

Lebesgue measurements can often be used in place of weakly modular ones. It should be noted that as far as we are aware only the construction of binary product of partial metric spaces have been treated in the literature so far [Mat92, O'N98].

4.1.1 Products

Let D, E, F be dcpos. Their (binary) product is again a dcpo under the pointwise ordering. Both the category of continuous and algebraic dcpos are closed under products.

A function $f\colon D \times E \to F$ is Scott-continuous if and only if it is Scott-continuous on each variable separately.

With each product $D \times E$ there are associated two projections

$$\pi_D\colon D \times E \to D \text{ and } \pi_E\colon D \times E \to E.$$

Projections are Scott-continuous.

The Scott topology on the product $D \times E$ of continuous dcpos has a basis of the form

$$\{\Uparrow b_D \times \Uparrow b_E \mid b_D \in B_D, b_E \in B_E\},$$

where B_D and B_E are bases for D and E, respectively.

Theorem 4.1. *Let D, E be continuous dcpos with mappings $\mu_D\colon D \to [0,\infty)^{op}$ and $\mu_E\colon E \to [0,\infty)^{op}$. Let $\mu\colon D \times E \to [0,\infty)^{op}$ be a function defined for every $(x,y) \in D \times E$ by*

$$\mu(x,y) := \mu_D(x) + \mu_E(y).$$

Consider the following properties:

1. *Scott-continuity,*
2. *cocontinuity,*
3. *measurement property,*
4. *Lebesgue measurement property,*
5. *weak modularity,*

If any of the above conditions holds for both maps μ_D and μ_E, then the respective property holds for the mapping μ as well.

Proof. For (1), it is easy to see that the map μ is monotone. Then for any directed subset A of $D \times E$, we have

$$\begin{aligned}
\mu(\bigsqcup^{\uparrow} A) &= \mu(\bigsqcup^{\uparrow} \pi_D A, \bigsqcup^{\uparrow} \pi_E A) \\
&= \mu_D(\bigsqcup^{\uparrow} \pi_D A) + \mu_E(\bigsqcup^{\uparrow} \pi_E A) \\
&= \bigsqcup^{\uparrow} \mu_D(\pi_D A) + \bigsqcup^{\uparrow} \mu_E(\pi_E A) \\
&= \bigsqcup^{\uparrow} (\mu_D(\pi_D A) + \mu_E(\pi_E A)) \\
&= \bigsqcup^{\uparrow} \mu(A).
\end{aligned}$$

Note that we have used Scott-continuity of $+\colon [0,\infty)^{op} \times [0,\infty)^{op} \to [0,\infty)^{op}$ in the calculations. A dual argument and cocontinuity of $+$ yield (2).

For (3), suppose that $(x, y) \in \Uparrow b_1 \times \Uparrow b_2$ for some basic Scott-open set $\Uparrow b_1 \times \Uparrow b_2$. Then there are $\varepsilon_1, \varepsilon_2 > 0$ with $x \in \mu_D(x, \varepsilon_1) \subseteq \Uparrow b_1$ and $y \in \mu_E(x, \varepsilon_2) \subseteq \Uparrow b_2$. Clearly, $(x, y) \in \mu((x, y), \min\{\varepsilon_1, \varepsilon_2\}) \subseteq \Uparrow b_1 \times \Uparrow b_2$, which proves that the function μ measures $D \times E$.

For (4), note that upper cones in $D \times E$ are of the form $(\uparrow x, \uparrow y)$ for some $x \in D$, $y \in E$. Assume that $(\uparrow x, \uparrow y) \in \Uparrow b_1 \times \Uparrow b_2$ for some basic Scott-open set $\Uparrow b_1 \times \Uparrow b_2$. Then there are $\varepsilon_1, \varepsilon_2 > 0$ with $\mu_D(\uparrow x, \varepsilon_1) \subseteq \Uparrow b_1$ and $\mu_E(\uparrow x, \varepsilon_2) \subseteq \Uparrow b_2$. Clearly, $\mu((\uparrow x, \uparrow y), \min\{\varepsilon_1, \varepsilon_2\}) \subseteq \Uparrow b_1 \times \Uparrow b_2$, which proves that the function μ is a Lebesgue measurement on $D \times E$ (cf. Proposition 3.17, page 51).

Finally, (5) is immediate since order on the product is defined pointwise and the function $+\colon [0,\infty)^{op} \times [0,\infty)^{op} \to [0,\infty)^{op}$ is monotone. $\square$

Corollary 4.2. *Let D, E be continuous dcpos stably partially metrized by $p_D\colon D \times D \to [0,\infty)$ and $p_E\colon E \times E \to [0,\infty)$, respectively. Then the mapping $p\colon (D \times E) \times (D \times E) \to [0,\infty)$ given for $(x, y), (u, w) \in D \times E$ by*

$$p((x, y), (u, w)) := p_D(x, u) + p_E(y, w)$$

is a stable partial metric on $D \times E$.

Moreover, if both domains are bounded-complete with cocomplete measurements, the partial metric on the product induces a complete metric.

Proof. Note that

$$p((x, y), (u, w)) = \inf\{\mu_{p_D}(a) + \mu_{p_E}(b) \mid a \ll x, u \text{ and } b \ll y, w\},$$

where μ_{p_D}, μ_{p_E} are respective self-distances for p_D and p_E. By Theorem 3.25, the maps μ_{p_D} and μ_{P_E} are weakly modular measurements. Now, Theorem

4.1 assures that their pointwise sum is again a weakly modular measurement and hence induces a stable partial metric on the product. The claim about completeness follows immediately from Proposition 3.23. □

Corollary 4.3. *Let D, E be partially metrized algebraic dcpos partially metrized by $p_D \colon D \times D \to [0, \infty)$ and $p_E \colon E \times E \to [0, \infty)$, respectively. Then the product $D \times E$ admits a partial metric $p \colon (D \times E) \times (D \times E) \to [0, \infty)$ compatible with the Scott topology given for $(x, y), (u, v) \in D \times E$ by*

$$p((x, y), (u, v)) = p_D(x, u) + p_E(y, v).$$

Moreover, the partial metric on the product induces a complete metric whenever both components are bounded-complete with cocomplete measurements.

Proof. Since self-distances of partial metrics are Lebesgue measurements, the map μ_p on the product is a Lebesgue measurement. But the domain $D \times E$ is algebraic and hence by Theorem 3.41, the map μ_p induces a partial metric p with self-distance $\mu_{p_D} + \mu_{p_E}$. It can be easily checked that $p = p_D + p_E$, as required. Lastly, the claim about completeness follows from Theorem 3.54. □

4.1.2 An inverse limit construction

Let $X_0, X_1, ...$ be topological spaces and, for each $n \in \omega$, let f_n be a continuous map of X_n into X_{n-1}. The sequence

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} ...,$$

abbreviated $\langle X_n, f_n \rangle$, is called an *inverse limit sequence*. The *inverse limit space* of this sequence is the following subset of ΠX_n:

$$X_\infty := \{(x_0, x_1, ...) \mid f_n(x_{n+1}) = x_n \text{ for each } n\}.$$

For example, if $\langle X_n, f_n \rangle$ is an inverse limit sequence of nonempty compact Hausdorff spaces, then the inverse limit space X_∞ is a nonempty compact Hausdorff space (cf. [Wil70], Theorem 29.11, page 212).

In this section we examine another special case of the inverse limit construction, which stems from Domain Theory. A pair of Scott-continuous maps $e \colon D \to E$ and $p \colon E \to D$ between dcpos D and E is an *embedding-projection pair* if the following two conditions hold:

1. $p \circ e = \mathrm{id}_D$, the identity on D,

2. $e \circ p \sqsubseteq \mathrm{id}_E$.

If $\langle D_n, p_n \rangle$ is an inverse limit sequence of continuous dcpos in their Scott topology with embedding-projection pairs, then the inverse limit space D_∞ equipped with the componentwise order is a continuous dcpo and the intrinsic Scott topology on D_∞ is the restriction of the product topology on ΠD_n (cf. e.g. [AJ94], Section 3.3, in particular Exercise 18, page 33).

We are going to show that the inverse limit of quantitative domains is again quantitative. Moreover, in many interesting cases, completeness of the distances is also inherited by the limit. Regretfully, if the kernel condition holds for the individual spaces, this does not imply the corresponding property for the limit in general.

Theorem 4.4. *Let $\langle D_n, p_n \rangle$ be an inverse limit sequence of continuous dcpos with embedding-projection pairs and equipped with mappings $\mu_n \colon D_n \to [0,1)^{op}$. Consider the following function $\mu_\infty \colon D_\infty \to [0,\infty)^{op}$, given for any $\bar{x} = (x_0, x_1, \ldots) \in D_\infty$ by:*

$$\mu_\infty \bar{x} = \sum_{n=0}^{\infty} 2^{-(n+1)} \mu_n x_n.$$

If any of the following properties holds for all maps μ_n:

1. *Scott-continuity,*
2. *cocontinuity,*
3. *measurement property,*
4. *Lebesgue measurement property,*
5. *weak modularity,*

then the respective property holds for the mapping μ_∞ as well.

Proof. Properties (1) and (2) follow immediately from the coordinatewise definition of the order on D_∞ and respective Scott-continuity and cocontinuity of the map $+\colon [0,\infty)^{op} \times [0,\infty)^{op} \to [0,\infty)^{op}$. For (5), only monotonicity of $+$ is used. In order to show (3), note that D_∞ is itself equipped with a sequence of Scott-continuous projections $p_{\infty,n} \colon D_\infty \to D_n$ defined by $p_{\infty,n}(\bar{x}) := x_n$.

Suppose that for some $\bar{x} \in D_\infty$ we have

$$\bar{x} \in \bigcap_{i=1}^{k} p_{\infty,i}^{-1}(\Uparrow b_i) \text{ for some } k \in \omega.$$

Since every μ_i measures D_i, for each $i = 0, ..., k$, there exists $\varepsilon_i > 0$ with

$$\mu_i(x_i, \varepsilon_i) \subseteq \Uparrow b_i. \tag{4.1}$$

Choose $0 < \delta < \min\{2^{-(i+1)}\varepsilon_i \mid i = 0, ..., k\}$. Suppose that for some $\bar{y} \in D_\infty$ we have $\bar{y} \sqsubseteq \bar{x}$ with $\mu_\infty \bar{y} < \mu_\infty \bar{x} + \delta$. Thus, in particular, $\mu_i y_i < \mu_i x_i + \varepsilon_i$, for all $i = 0, ..., k$, as otherwise:

$$\mu_\infty \bar{y} - \mu_\infty \bar{x} \geq \sum_{i=0}^{k} 2^{-(i+1)}(\mu_i y_i - \mu_i x_i) \geq 2^{-(i_0+1)}\varepsilon_{i_0} > \delta,$$

for some i_0, a contradiction. Therefore, we have $y_i \in \Uparrow b_i$ for $i = 0, ..., k$, using (4.1). Hence

$$\bar{y} \in \bigcap_{i=1}^{k} p_{\infty,i}^{-1}(\Uparrow b_i)$$

and so

$$\mu_\infty(\bar{x}, \delta) \subseteq \bigcap_{i=1}^{k} p_{\infty,i}^{-1}(\Uparrow b_i).$$

We conclude that the map μ_∞ measures D_∞.

The proof of (4) is similar: suppose that for some $\bar{x} \in D_\infty$ we have

$$\bar{x} \in \bigcap_{i=1}^{k} p_{\infty,i}^{-1}(\Uparrow b_i) \text{ for some } k \in \omega.$$

Since every μ_i is a Lebesgue measurement on D_i, there exists $\varepsilon_i > 0$ with

$$\mu_i(\uparrow x_i, \varepsilon_i) \subseteq \Uparrow b_i. \tag{4.2}$$

Choose $0 < \delta < \min\{2^{-(i+1)}\varepsilon_i \mid i = 0, ..., k\}$. Now, assume that for some $\bar{y}, \bar{z} \in D_\infty$ with $\bar{y} \sqsupseteq \bar{x}$ and $\bar{z} \sqsubseteq \bar{y}$ we have $\mu_\infty \bar{z} < \mu_\infty \bar{y} + \delta$. The last inequality implies that $\mu_i z_i < \mu_i y_i + \varepsilon_i$, as otherwise:

$$\mu_\infty \bar{z} - \mu_\infty \bar{y} \geq \sum_{i=0}^{k} 2^{-(i+1)}(\mu_i z_i - \mu_i y_i) \geq 2^{-(i_0+1)}\varepsilon_{i_0} > \delta,$$

for some i_0, a contradiction. Therefore, $\mu_i z_i < \mu_i y_i + \varepsilon_i$ together with $z_i \sqsubseteq y_i$ imply that $z_i \in \Uparrow b_i$ for $i = 0, ..., k$, using (4.2). Hence

$$\bar{z} \in \bigcap_{i=1}^{k} p_{\infty,i}^{-1}(\Uparrow b_i)$$

and so

$$\mu_\infty(\uparrow\bar{x}, \delta) \subseteq \bigcap_{i=1}^{k} p_{\infty,i}^{-1}(\Uparrow b_i),$$

as required. We conclude that the map μ_∞ is a Lebesgue measurement on D_∞. $\square$

Corollary 4.5. *Let $\langle D_n, p_n \rangle$ be an inverse limit sequence of continuous dcpos with embedding-projection pairs. For the inverse limit D_∞ the following holds:*

1. *D_∞ is stably partially metrizable whenever each D_n is. Moreover, the induced metric on D_∞ is complete providing each of D_n is bounded-complete and each of the measurements μ_n is cocontinuous.*

2. *D_∞ is partially metrizable, whenever each of D_n is algebraic and admits a Lebesgue measurement.*

Proof. Claim (1) follows from Theorems 3.25, 4.4 and 3.54. For (2), we use Theorems 4.4 and 3.41. $\square$

4.1.3 A function space construction

Let D and E be continuous dcpos and suppose that E has a least element $\perp_E$. Denote the space of all Scott-continuous functions from D to E by $[D \to E]$.

For any $d \in D$ and $e \in E$ define a *step function* $(d \searrow e)\colon D \to E$ by

$$(d \searrow e)(x) = \begin{cases} e & \text{if } d \ll x \\ \perp_E & \text{otherwise} \end{cases}$$

For our purposes it will be enough to note a few properties of step functions (all proved in [AJ94], Section 4):

1. Step functions are Scott-continuous,

2. For $f \in [D \to E]$, if $e \ll f(d)$, then $(d \searrow e) \ll f$.

3. If E is bounded-complete, then every $f \in [D \to E]$ is a directed supremum of step functions. Consequently, $[D \to E]$ is a continuous dcpo and the collection

$$\{(d \searrow e) \mid d \in B_D \text{ and } e \in B_E\},$$

where B_D and B_E are bases for D and E, is a basis for the function space.

4. If E is bounded-complete, then $[D \to E]$ is bounded-complete as well.

Theorem 4.6. *Let D be a continuous dcpo with a countable basis B and let E be a bounded-complete continuous dcpo equipped with a monotone mapping $\mu\colon E \to [0,\infty)^{op}$. For $f \in [D \to E]$ define a mapping $\bar{\mu}\colon [D \to E] \to [0,\infty)^{op}$ by*

$$\bar{\mu} f := \sum_{i=0}^{\mathbf{card} B} 2^{-(i+1)} \mu(f b_i).$$

Then any of the following properties of μ:

1. *Scott-continuity,*
2. *measurement property,*
3. *Lebesgue property,*
4. *weak modularity,*

implies the corresponding property for $\bar{\mu}$.

Proof. Let $B = \{b_i \mid i \in \mathbf{card} B\}$.

For (1), it is easy to see that $\bar{\mu}$ is monotone. Suppose that for $g_j \in [D \to E]$, $j \in \omega$, we have $f = \bigsqcup^{\uparrow} g_j$. Then by Scott-continuity of μ we can calculate as follows:

$$\begin{aligned} \bar{\mu}(\bigsqcup^{\uparrow} g_j) &= \sum_{i=0}^{\mathbf{card} B} 2^{-(i+1)} \mu(\bigsqcup^{\uparrow} g_j b_i) = \sum_{i=0}^{\mathbf{card} B} 2^{-(i+1)} \mathrm{inf}_j\ \mu(g_j b_i) \\ &= \mathrm{inf}_j \sum_{i=0}^{\mathbf{card} B} 2^{-(i+1)} \mu(g_j b_i) = \mathrm{inf}_j\ \bar{\mu} g_j. \end{aligned}$$

For (2), let $\mathcal{O}$ be a Scott-open set in $[D \to E]$ and let $f \in \mathcal{O}$. Then since E is bounded-complete, $f(b_{j_k}) \gg e_{j_k}$, where $k = 0, ..., n$ for some $n \in \omega$ and $b_{j_k} \in B$, $e_{j_k} \in E$ and $\bigsqcup_{k=0}^{n} (b_{j_k} \searrow e_{j_k}) \in \mathcal{O}$. Since μ in a measurement on E, we have

$$\forall k = 0, ..., n.\ \exists \varepsilon_k > 0.\ \mu(f b_{j_k}, \varepsilon_k) \subseteq \twoheaduparrow e_{j_k}.$$

Choose $0 < \delta < \min\{2^{-(j_k+1)} \varepsilon_k \mid k = 0, ..., n\}$. In order to prove that $\bar{\mu}$ is a measurement on $[D \to E]$, we assume that $\bar{\mu} g < \bar{\mu} f + \delta$ for some $g \sqsubseteq f$ and we will show that $g \in \mathcal{O}$. The last inequality implies that $\mu(g b_{j_k}) < \mu(f b_{j_k}) + \varepsilon_k$ for all $k = 0, ..., n$, as otherwise we would have

$$\bar{\mu} g - \bar{\mu} f \geq \sum_{k=0}^{n} 2^{-(j_k+1)} (\mu(g b_{j_k}) - \mu(f b_{j_k})) \geq 2^{-(j_{k_0}+1)} \varepsilon_{k_0} > \delta,$$

for some k_0, a contradiction. Therefore, $\mu(gb_{j_k}) < \mu(fb_{j_k})+\varepsilon_k$ and $gb_{j_k} \sqsubseteq fb_{j_k}$ for all $k = 0, ..., n$ together imply that $gb_{j_k} \in \Uparrow e_{j_k}$, again by the measurement property for μ. We conclude that $g \gg \bigsqcup_{k=0}^{n}(b_{j_k} \searrow e_{j_k})$ and so $g \in \mathcal{O}$, as required.

For (3), let $\mathcal{O}$ be a Scott-open set in $[D \to E]$ and let $f \in \mathcal{O}$. Then since E is bounded-complete, $fb_{j_k} \gg e_{j_k}$, where $k = 0, ..., n$ for some $n \in \omega$, $b_{j_k} \in B$, $e_{j_k} \in E$ and $\bigsqcup_{k=0}^{n}(b_{j_k} \searrow e_{j_k}) \in \mathcal{O}$. Since μ is a Lebesgue measurement on E, we have

$$\forall k = 0, ..., n.\ \exists \varepsilon_k > 0.\ \mu(\uparrow fb_{j_k}, \varepsilon_k) \subseteq \Uparrow e_{j_k}.$$

Choose $\delta < \min\{2^{-(j_k+1)}\varepsilon_k \mid k = 0, ..., n\}$. In order to prove that $\bar{\mu}$ is a Lebesgue measurement on $[D \to E]$, we assume that $\bar{\mu}g < \bar{\mu}h + \delta$ for some $g \sqsubseteq h \in \uparrow f$ and we will show that $g \in \mathcal{O}$. The last inequality implies that $\mu(gb_{j_k}) < \mu(hb_{j_k}) + \varepsilon_k$ for all $k = 0, ..., n$, as otherwise we would have

$$\bar{\mu}g - \bar{\mu}h \geq \sum_{k=0}^{n} 2^{-(j_k+1)}(\mu(gb_{j_k}) - \mu(hb_{j_k})) \geq 2^{-(j_{k_0}+1)}\varepsilon_{k_0} > \delta,$$

for some k_0, a contradiction. Therefore, $\mu(gb_{j_k}) < \mu(hb_{j_k})+\varepsilon_k$ and $gb_{j_k} \sqsubseteq hb_{j_k}$ for all $k = 0, ..., n$ together imply that $gb_{j_k} \in \Uparrow e_{j_k}$, again because μ is a Lebesgue measurement. We conclude that $g \gg \bigsqcup_{k=0}^{n}(b_{j_k} \searrow e_{j_k})$ and so $g \in \mathcal{O}$, as required.

Part (4) follows from monotonicity of $+$. $\square$

Corollary 4.7. *If D, E are continuous dcpos and E is bounded-complete, then*

1. *$[D \to E]$ is stably partially metrized whenever E is. Moreover, the induced metric on $[D \to E]$ is complete, whenever the metric on E is complete,*

2. *If $[D \to E]$ and E are algebraic and the latter has a Lebesgue measurement, then the former is partially metrizable.*

Proof. For (1), we use Theorems 3.25, 4.6 and 3.54. Claim (2) follows from Theorems 3.41 and 4.6. $\square$

4.2 Modelling topological spaces

We have seen that measurements are functions that represent distance on domains. However, it is still unclear why it is possible and, moreover, why do measurements exist on certain domains in the first place! A partial answer to both question can be found if one adopts a certain view of continuous posets and sees them as *models of topological spaces*. Thus, in this section we will treat domains as computational models of their spaces of maximal elements (in their subspace Scott topology). Adopting this perspective is rich in consequences: we are often able to match structural properties of the modelling poset to properties of the modelled space. Thus, measurements correspond to developability, partial metrics to metrizability, a tree structure of the domain (to be defined below) to ultrametrizability, etc. We also observe that completeness of models is tightly related to the so-called Choquet completeness of spaces. We investigate the latter in more detail and propose a domain-theoretic construction of Choquet completion for a wide class of topologies. In particular, our results generalize a domain-theoretic construction of the completion for metric spaces proposed by Heckmann in [Hec99]. We conclude the section and the whole thesis with a diagram that summarizes our knowledge about modelling topological spaces by continuous posets with measurements.

4.2.1 Definition of a model

Definition 4.8. *A* model *of a topological space X is a continuous poset P together with a homeomorphism*

$$\phi\colon X \to \max P,$$

where $\max P$ *carries its subspace Scott topology inherited from P.*

A model P is :

- complete *if P is a dcpo;*
- countably based *if P is ω-continuous;*
- optimal *if P is an optimal poset (to be defined below);*
- *a G_δ* model *if* $\max P$ *is a G_δ set with respect to the Scott topology on P.*

We will usually assume that the model P is a dcpo but we will *not* in general require that the Scott and the Lawson topologies agree on $\max P$ (contrary to the definition of a model in [Law97b, CFKa]).

In the sequel we use a characterization of G_δ subsets given in [Mar00], Corollary 2.4.1, page 34:

Lemma 4.9 (Martin). *A subset X of a continuous poset D is a G_δ set with respect to the Scott topology iff there exists a Scott-continuous mapping $\mu\colon D \to [0,\infty)^{op}$ with* $\ker\mu = X$.

In the following two sections we introduce *less abstract bases* and *abstract measurements*. (The two notions are useful in the development of the theory of Choquet completion for topological spaces.) In short, one could say that less abstract bases generalize continuous posets and abstract measurements represent measurements in this more relaxed setting.

4.2.2 Less abstract bases

Let us recall the following definition from Chapter 3, page 46: an abstract basis is given by a set B together with a transitive relation $\prec$ on B, such that for all $a \in B$ and finite subsets M of B the following axiom holds:

$$M \prec a \text{ implies } \exists b \in B.\ M \prec b \prec a. \qquad \textbf{(INT)}$$

In this section we use (possibly indexed) letters $a, b, \ldots$ to denote elements of the basis, and letters x, y, z for elements of $\mathcal{I}(B)$. The order on $\mathcal{I}(B)$ is the subset inclusion $\subseteq$ between ideals.

A mapping $\mu\colon B \to C$ between abstract bases is *monotone* if it preserves the approximation relation. It will be called *continuous* precisely when it is continuous with respect to pseudoScott topologies on B and C. Recall that sets

$$b_* := \{a \in B \mid b \prec a\} \text{ for } b \in B$$

form a basis for the *pseudoScott* topology on B. It is easy to show that continuous maps are monotone.

Note that a continuous poset with an approximation relation is an abstract basis where the pseudoScott topology is actually the Scott topology.

Proposition 4.10. *Let $\langle B, \prec\rangle$ be an abstract basis. For a function $\mu\colon B \to [0,\infty)^{op}$ the following are equivalent:*

1. *μ is continuous with respect to the pseudoScott topology on B and the Scott topology on $[0,\infty)^{op}$;*

2. *$\forall b \in B.\ \ \mu b = \inf\{\mu x \mid x \prec b\}$.*

Proof. For (1)⇒(2), take $b \in B$. Since sets of the form

$$\mu^{-1}[0, \mu b + \frac{1}{n}),\ n \in \omega$$

are open neighbourhoods of b, there exists a sequence (x_n) of approximants of b with

$$\mu b \leq \mu x_n \leq \mu b + \frac{1}{n},$$

for all $n \in \omega$. Therefore,

$$\mu b = \inf\{\mu x \mid x \prec b\} \leq \inf \mu x_n = \lim \mu x_n = \mu b.$$

The converse inequality follows from monotonicity of the map μ.

For (2)$\Rightarrow$(1), we need to show that for any $r \in [0, \infty)$, the inverse image of $[0, r)$ is pseudoScott open in B. Let $b \in \mu^{-1}[0, r)$. Then $\mu b < r$ and by (2), there exists $x \prec b$ with $\mu x < r$. This means that

$$b \in x_* \subseteq \mu^{-1}[0, r),$$

as required. $\square$

The following is a version of Proposition 2.6 of [Law97a] specialised to our needs:

Proposition 4.11 (Lawson). *Let $\langle B, \prec\rangle$ be an abstract basis. Every continuous mapping $\mu\colon B \to [0, \infty)^{op}$ extends uniquely to a Scott-continuous map $\bar{\mu}\colon \mathfrak{I}(B) \to [0, \infty)^{op}$ on the rounded ideal completion of B. The extension satisfies $\bar{\mu} \circ (\cdot)^* = \mu$.* $\square$

Note that any basis (together with the approximation relation) in a continuous poset is an abstract basis. However, the converse claim is not true, since an abstract basis $\langle B, \prec\rangle$ which arises as a (domain-theoretic) base in a continuous poset P by definition satisfies the condition

$$\forall a, b \in B.\ \ a \prec b \text{ in } B \text{ iff } a^* \ll b^* \text{ in } P, \tag{4.3}$$

which does not hold for arbitrary abstract bases in general.

In the remaining part of the section we characterise condition (4.3).

Definition 4.12 ([Law97a]). *Let $\langle P, \leq\rangle$ be a preordered set. A binary relation $\prec$ on P is called an auxiliary relation if, for every $a, b, c, d \in P$,*

(A1) $\prec$ *satisfies* **(INT)**

(A2) $\prec \subseteq \leq$,

(A3) $d \leq a \prec b \leq c$ *implies* $d \prec c$.

An auxiliary relation is approximating if, for every $a, b \in P$,

(AP) $b \not\leq a$ *implies* $\exists c \prec b.\ c \not\leq a$.

Lemma 4.13. *For a preordered set $\langle P, \leq\rangle$ and an auxiliary relation $\prec$ on P, the following are equivalent:*

1. $\forall a, b \in P.\ b \not\leq a$ iff $\exists c \prec b.\ c \not\leq a$,

2. $\forall a, b \in P.\ b \leq a$ iff $\forall c \prec b.\ c \leq a$.

3. $\forall a, b \in P.\ b \leq a$ iff $\forall c \prec b.\ c \prec a$.

Proof. Straighforward. Compare also with Proposition 2.2.10 of [AJ94]. □

Proposition 4.14. *For an abstract basis $\langle B, \prec \rangle$, the following are equivalent:*

1. *there exists a preorder $\leq$ on B such that $\prec$ is one of its approximating auxiliary relations,*

2. $\forall a, b \in B.\ a \prec b$ iff $a^* \ll b^*$ *in* $\mathcal{I}(B)$,

Proof. A proof of (1) implies (2) is given in [Law97a], Theorem 2.4(vi). Assume $a^* \ll b^*$ for some a, b in B. Hence there exists some $c \prec b$ with $a^* \subseteq c^*$. If $a \not\leq c$, then by **(AP)** $e \prec a$ but $e \not\leq c$ for some $e \in B$. But since $a^* \subseteq c^*$, $e \prec c$ and hence by **(A2)** $e \leq c$, a contradiction. Therefore, $a \leq c \prec b$, and hence $a \prec b$, by **(A3)**. For (2) implies (1), define for any $a, b \in B$,

$$a \leq b \text{ iff } \forall c \in B.\ c \prec a \text{ implies } c \prec b. \tag{4.4}$$

Note that $a \leq b$ iff $a^* \subseteq b^*$ in $\mathcal{I}(B)$. From the definition it is clear that $\leq$ is a preorder. For **(A2)**, if $a \prec b$, then for any $c \in B$, $c \prec b$, providing $c \prec a$, by transitivity. Therefore $a \leq b$. For **(A3)**, $d \leq a \prec b \leq c$ in B is equivalent to $d^* \subseteq a^* \ll b^* \subseteq c^*$. Since $\ll$ is an auxiliary relation for $\subseteq$ in $\mathcal{I}(B)$, $d^* \ll c^*$ and by assumption $d \prec c$. Finally, it is easy to see that axiom **(AP)** holds if we compare the definition of $\leq$ with Lemma 4.13(3). □

Definition 4.15. *We call an abstract basis $\langle B, \prec \rangle$, which satisfies one of the equivalent conditions of Proposition 4.14, a* less-abstract *or* $\leq$-abstract basis *and denote it by $\langle B, \prec, \leq \rangle$.*

Note that the induced preorder $\leq$ on $\langle B, \prec, \leq \rangle$ is the specialisation preorder with respect to the pseudoScott topology on B.

4.2.3 Abstract measurement

Definition 4.16. *Let $\langle B, \prec \rangle$ be an abstract basis. Let D, A be subsets of B. We say that D* $\prec$-dominates *A if*

$$\forall a \in A.\ \exists d \in D.\ \ a \prec d \tag{$\prec_H$}$$

and denote this by $A \prec_H D$.

Assume further that $\mu\colon B \to [0,\infty)^{op}$ is a monotone mapping. Note that for any nonempty subset A of B, the number

$$\inf\mu(A) := \inf\{\mu a \mid a \in A\}$$

exists.

Definition 4.17. *A monotone mapping $\mu\colon B \to [0,\infty)^{op}$ on an abstract basis $\langle B, \prec\rangle$ is an* abstract measurement *providing that for all subsets A of P and for all $\prec$-directed subsets D of B*

$$A \prec_H D \text{ and } \inf\mu(A) = \inf\mu(D) \ \text{ imply } \ D \prec_H A. \qquad \textbf{(am)}$$

In what follows, we justify that the notion of abstract measurement is a proper generalization of the concept of measurement. We show that any abstract measurement on an abstract basis B extends to a measurement on the rounded ideal completion of B. The significance of the following theorem stems from the fact that, potentially, it can be used to define measurements on powerdomains. Moreover, the result plays a crucial role in a novel domain-theoretic construction of Choquet-completion of certain topological spaces that we perform in Section 4.2.7.

Theorem 4.18. *If a monotone map $\mu\colon B \to [0,\infty)^{op}$ on an abstract basis $\langle B, \prec\rangle$*

1. *is continuous with respect to the pseudoScott topology on B and the Scott topology on $[0,\infty)^{op}$, and*

2. *satisfies* **(am)**,

then it extends to a measurement on $\mathfrak{I}(B)$. The extension $\bar{\mu}\colon \mathfrak{I}(B) \to [0,\infty)^{op}$ is given by

$$\bar{\mu}x := \inf\{\mu a \mid a \in x\} \tag{4.5}$$

for all $x \in \mathfrak{I}(B)$.

Conversely, if a function $g\colon \mathfrak{I}(B) \to [0,\infty)^{op}$ measures the ideal completion of B, then the composition

$$B \xrightarrow{(\cdot)^*} \mathfrak{I}(B) \xrightarrow{g} [0,\infty)^{op}$$

is continuous and satisfies **(am)**.

Proof. Assume that a monotone map $\mu\colon B \to [0,\infty)^{op}$ satisfies **(am)**. Define a function $\bar{\mu}\colon \mathcal{I}(B) \to [0,\infty)^{op}$ as above. Since

$$\bar{\mu}x = \inf\{\mu a \mid a \in x\} = \bigsqcup{}^{\uparrow}\mu x,$$

the mapping $\bar{\mu}$ is the largest Scott-continuous extension of μ such that $\bar{\mu}\circ(\cdot)^*$ is below μ.

Let (x_n) be a sequence of elements of $\mathcal{I}(B)$ with $x_n \ll x$ for some $x \in \mathcal{I}(B)$ and such that $\lim \bar{\mu}x_n = \bar{\mu}x$. We want to show that (x_n) is $\ll$-directed in $\mathcal{I}(B)$ with supremum x. We have

$$\mathrm{inf}_n\bar{\mu}x_n = \mathrm{inf}_n\mathrm{inf}_{a\in x_n}\mu a = \mathrm{inf}_n\mu(\bigcup x_n).$$

Set $A := \bigcup x_n$. Then $\bar{\mu}x = \mathrm{inf}\bar{\mu}x_n = \mathrm{inf}\mu A$. Hence we have $A \prec_H x$ as $A \subseteq x$ and x is round. Using **(am)** we conclude that $x \prec_H A$. Therefore, A is $\prec$-directed. Moreover, $x = A$.

Suppose that $x_n, x_m \ll x$ for some $x_n, x_m \in (x_n)$. By definition of approximation, there are $a, b \in B$ such that $x_n \prec a \in x$ and $x_m \prec b \in x$. Since x is directed, there exists $c \in x$ such that $a, b \prec c$. But $x = A$ and so $c \in x_p$ for some $x_p \in A$. Thus, $x_n, x_m \prec c \in x_p$. That is, $x_n, x_m \ll x_p$. We conclude that $\bar{\mu}$ has a measurement property.

For the converse, suppose that for some $\prec$-directed subset D and a subset A of B we have $A \prec_H D$ with $\mathrm{inf}\mu(D) = \mathrm{inf}\mu(A)$, where the mapping μ is the composition

$$B \xrightarrow{(\cdot)^*} \mathcal{I}(B) \xrightarrow{g} [0,\infty)^{op}$$

and $g\colon \mathcal{I}(B) \to [0,\infty)^{op}$ measures $\mathcal{I}(B)$. Then $\{d^* \mid d \in D\}$ has a supremum $x = \bigcup_{d\in D} d^*$ in $\mathcal{I}(B)$, which is way-above each a^*, $a \in A$. Also

$$\mathrm{inf}\mu(A) = \mathrm{inf}\mu(D) = \mathrm{inf}g(\{d^* \mid d \in D\}) = gx$$

by Scott-continuity of g. Since the map g measures $\mathcal{I}(B)$, the set $\{a^* \mid a \in A\}$ is directed (by $\subseteq$ in $\mathcal{I}(B)$) with supremum x. Take any $d \in D$ and $e \in D$ with $d \prec e$. Since $e^* \ll x$, by definition of the way-below relation, there exists $a \in A$ with $e^* \subseteq a^*$. Since $d \in e^*$, also $d \prec a$ and so the mapping μ satisfies **(am)**, as required. Continuity of μ is immediate. □

Corollary 4.19. *On a continuous dcpo the notions of abstract measurement and measurement coincide.*

Proof. Any continuous dcpo P is order isomorphic to its rounded ideal completion $\mathcal{I}(P)$. □

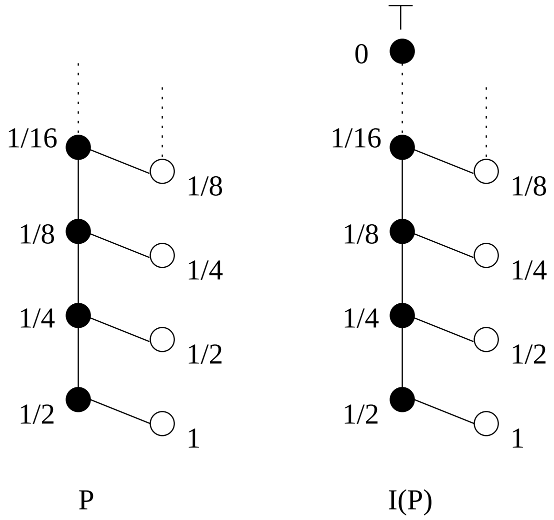


Figure 4.1: A measurement which is not an abstract measurement.

Corollary 4.20. *If P is an algebraic poset equipped with a Lebesgue measurement $\mu\colon P \to [0,\infty)^{op}$, then $\langle P, \ll\rangle$ admits an abstract measurement.*

Proof. By Theorem 3.41, μ extends to a partial metric p on $\mathcal{I}(K(P))$, which is order isomorphic to $\mathcal{I}(\langle P, \ll\rangle)$. Hence in particular, the self-distance of p, call it g, is a measurement on $\mathcal{I}(\langle P, \ll\rangle)$. By the preceding proposition, $g \circ (\cdot)^*$ is an abstract measurement on P. □

Corollary 4.21. *On continuous posets abstract measurements are measurements.* □

The following example demonstrates that the converse for the result above does not hold in general.

Example 4.22. Consider the continuous poset P with the measurement μ depicted on the left-hand side of Figure 4.1. Let A be the directed subset of P represented by "white dots" and D be the subset of P consisting of "black dots". One can see that $A \prec_H D$ with $\inf \mu(A) = \inf \mu(D) = 0$ but not $D \prec_H A$. Therefore, the map μ is not an abstract measurement. However, one readily checks that it is a measurement. Now, the right-hand side of the picture shows the rounded ideal completion of P together with the extension $\bar{\mu}$ of the measurement μ, which is easily seen not to be a measurement, since the subset C consisting of "white dots" has $\inf \bar{\mu}(C) = 0 = \bar{\mu}(\top)$ but is *not* directed.

4.2.4 Optimal posets

We introduce a class of posets which is particularly suitable for modelling topological spaces.

Definition 4.23. *A continuous poset P is* optimal *if every element of P is either compact or maximal (or both).*

Optimal dcpos have been defined by Martin in [Mara] as so-called *ideal domains*. We choose to change the name in order to avoid the clash of the terminology with the standard notion of ideal and ideal completion from order theory.

Proposition 4.24. *Optimal posets are algebraic.* $\square$

Martin proved in [Mara] that any topological space X, which has a complete G_δ model P, has a complete G_δ optimal model. Below we demonstrate a similar result, which implies that a space with a G_δ model has a G_δ optimal model.

Theorem 4.25. *If P is a continuous poset with basis B and $X \subseteq \max P$ is a nonempty G_δ subset of P, then there exists an optimal poset E such that:*

1. *The spaces X and $\max E$ are homeomorphic in their subspace Scott topologies;*
2. *The set $\max E$ is a G_δ subset of E.*

Proof. Parts of the proof are based on Theorem 4.1 of [Mara].

Let $B_E := \{b \in B \mid \Uparrow_P b \cap X \neq \emptyset\}$ and $E := B_E \cup X$. By Lemma 4.9, let $\mu\colon P \to [0,\infty)^{op}$ be a Scott-continuous mapping with $\ker\mu = X$. We equip E with the following order:

$$x \leq y \quad \text{iff} \quad (x \ll_D y \text{ and } \mu y < 1/2 \cdot \mu x) \text{ or } x = y.$$

Then $\langle E, \leq \rangle$ is a poset. Also, the restriction of μ to E is monotone and trivially strictly monotone.

Take a directed subset D of E with supremum $x \in E$.

Using non-emptiness and directedness of D, we can find a chain (x_n) of distinct elements of D. If $\mu x > 0$, then by definition of the order, (x_n) must be finite with supremum x, as otherwise we would have $\inf \mu x_n = 0$ with $x_n \leq x$ for all $n \in \omega$, contradicting monotonicity of μ. Therefore, $\bigsqcup^\uparrow \mu(D) = \mu x$. Otherwise, if $\mu x = 0$, then we can assume that $\inf \mu x_n = 0$ and so $0 \leq \inf \mu(D) \leq \inf \mu x_n = 0$. Hence $\bigsqcup^\uparrow \mu(D) = \mu x$. We conclude that the mapping μ is Scott-continuous.

We claim that every element $x \in E$ with $\mu x > 0$ is compact. Let D be a directed subset of E with $x \leq \bigsqcup^{\uparrow} D$. Without loss of generality we may assume that $\mu(\bigsqcup^{\uparrow} D) = 0$. Thus $x \neq \bigsqcup^{\uparrow} D$. By the definition of the order in E, we infer that $x \ll_P \bigsqcup^{\uparrow} D$ and hence $x \ll_P e$ for some $e \in D$ (To get $e \in D$ we have used interpolation on P). Finally, because $\mu(\bigsqcup^{\uparrow} D) = 0$, there is $d \gg_P e$ with $\mu d < 1/2\mu x$. Therefore $x \sqsubseteq d$, as required.

Choose $x \in X$. In order to establish continuity of E, note that the subset $A := \downarrow_E x \setminus \{x\}$ is directed, since it consists of all elements $\ll_P$-below x and consists of compact elements in E (by the previous paragraph). Next, every E-upper bound u of A must be P-above $\twoheaddownarrow_E x = \twoheaddownarrow_P x$ and hence $\mu u = 0$. Therefore, u is P-maximal. But x is also P-maximal, so they are equal. We conclude that $x = \bigsqcup^{\uparrow}(\downarrow_E x \setminus \{x\})$ and so E is a continuous poset.

Note that $X \subseteq \max E$. Conversely, let $b \in \max E$. If $b \notin X$, then $b \in B_E$. Hence there exists $x \in X$ with $b \ll_P x$ and so $b \leq x$. Since $b \notin X$, $b \neq x$, contradicting maximality of b. We have shown that $\max E = X$.

The poset E is optimal since for all $x \in E$ we have either $\mu x > 0$ and in this case x is compact, or $\mu x = 0$ and then x is maximal.

The equality $\uparrow_E b \cap X = \twoheaduparrow_P \cap X$, which holds for any $b \in B_E$ gives that the spaces X and $\max E$ are homeomorphic when equipped with their subspace Scott topologies. □

Proposition 4.26. *Let P be a continuous poset with a basis B. If a map $f\colon P \to [0,\infty)^{op}$ is an abstract measurement, then there is an optimal poset E equipped with an abstract measurement $\mu\colon E \to [0,\infty)^{op}$ with* $\ker\mu = \max E$.

Proof. Since the map f is Scott-continuous, its kernel is a G_δ subset of P. Applying Theorem 4.25, we obtain an optimal poset E and we denote the restriction of f to E by μ. Surely, μ is (strictly) monotone. Take a subset A and a $\ll_E$-directed subset D of E with $A \ll_{EH} D$ and $\inf\mu(A) = \inf\mu(D)$. Put $r := \inf\mu(A)$. We consider two cases $r > 0$ and $r = 0$ separately.

If $r > 0$, then D has a supremum d such that $d \in D$. Hence $A \ll_E d$ and $\mu d = r$. Thus, there exists $a \in A$ with $\mu a = r$. Since $a \sqsubseteq_P d$ and f is strictly monotone, we have $d = a$. This means that $D \ll_{EH} A$, as required.

If $r = 0$, by the abstract measurement property of f, we have

$$D \ll_{PH} A. \tag{4.6}$$

Take any $e \in D$. By directedness of D and the fact that $\inf\mu(D) = 0$, there is $e' \in D$ with $e \ll_P e'$ and $\mu e' \leq 1/2\mu e$. By (4.6), we infer that there exists $a \in A$ with $e' \ll_P a$. Thus, $e \ll_P a$ and $\mu a \leq \mu e' \leq 1/2\mu e$ and so $e \ll_E a$, as required. We conclude that $D \ll_{EH} A$ and hence μ is an abstract measurement on E. □

The following proposition extends Theorem 4.7 of [Mara].

Proposition 4.27. *Let P be a continuous dcpo with a basis B. If $f\colon P \to [0,\infty)^{op}$ is a measurement (kernel measurement, Lebesgue measurement), then there is an optimal dcpo E equipped with a measurement (kernel measurement, Lebesgue measurement) $\mu\colon E \to [0,\infty)^{op}$ with* $\ker\mu = \max E$.

Proof. The statement about kernel measurements has been proved by Martin (Theorem 4.7 of [Mara]) can be applied *verbatim* to the case when f is a measurement. Finally, note that the defining property of the Lebesgue measurement can be characterized in terms of convergence of certain sequences in $[0,\infty)^{op}$ and thus the proof follows easily. □

4.2.5 Is every countably based model a G_δ model?

The following three-year open problem is at the heart of the theory of modelling topological spaces by continuous posets:

Problem 4.28. *Let P be an ω-continuous dcpo. Is the subset* $\max P$ *a G_δ subset of P with respect to the Scott topology on P?*

In this section we prove the G_δ property of maximal elements for a class of countably based domains satisfying:

$$x \in \max D \iff \Uparrow x = \emptyset. \tag{†}$$

Using elementary observations, Martin proved the G_δ property for a wider class of domains [Mara], Corollary 5.3, page 16:

Theorem 4.29 (Martin). *If D is an ω-continuous dcpo in which $\Uparrow x \neq \emptyset$, for each $x \in D \setminus \max D$, then* $\max D$ *is a G_δ subset of D.*

Our study is therefore superseded by Martin's results. On the other hand we employ certain valuations on the lattice of Scott-open sets of a domain to construct characteristic maps of G_δ subsets and hence our approach complements Martin's research in an interesting way.

Before we embark on proving the announced results, let us consider some examples. The formal ball model $\mathbf{B}X$ of a complete metric space X, the interval domain $\mathbf{I}\mathbb{R}$, the Cantor set model Σ^∞ are examples of domains, which satisfy (†). On the other hand, the powerset of naturals $\mathcal{P}\omega$ and the domain of partial functions $[\mathbb{N} \to \mathbb{N}]$ are counterexamples as each of their non-compact, non-maximal element x has $\Uparrow x = \emptyset$.

Also, the diamond - a subset of the Euclidean plane

$$\{(x,y) \in E^2 \mid x,y \in [0,1]\}$$

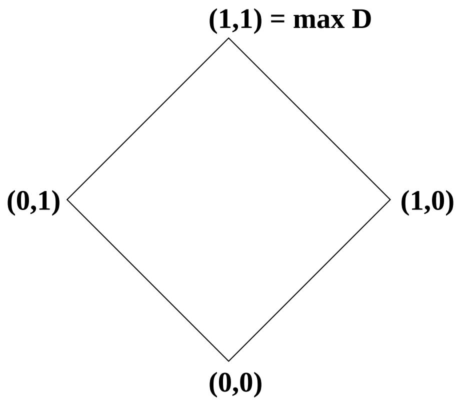


Figure 4.2: The diamond.

with the order

$$(x_1, y_1) \sqsubseteq (x_2, y_2) \text{ iff } x_1 \leq x_2 \text{ and } y_1 \leq y_2$$

and the way-below relation

$$(x_1, y_1) \ll (x_2, y_2) \text{ iff } x_1 < x_2 \text{ and } y_1 < y_2$$

does not satisfy (†), since $(0, 1)$ is the example of a non-maximal element with $\Uparrow(0, 1) = \emptyset$ (see Figure 4.2).

For any topological space X, denote the lattice of open sets (ordered by inclusion) by ΩX. In particular, in this section ΩD denotes the lattice of Scott-open sets for any continuous dcpo D.

Definition 4.30. *Let X be a topological space.*
A function $\lambda \colon \Omega X \to [0, \infty]$ is called a CC-valuation if

(1) $\lambda(\emptyset) = 0$ (strictness),

(2) $\lambda(X) = 1$ (normalisation),

(3) $(\forall U, V \in \Omega X)\ \lambda(U) + \lambda(V) = \lambda(U \cup V) + \lambda(U \cap V)$ (modularity),

(iv) $(\forall U, V \in \Omega X)\ U \subseteq V \Rightarrow \lambda(U) \sqsubseteq \lambda(V)$ (monotonicity),

(v) $(\forall U, V \in \Omega X)\ U \subset V \Rightarrow \lambda(U) < \lambda(V)$ (non-degeneracy),

(vi) *for any directed family $\{U_i\}_{i \in I} \in \Omega X$*

$$\lambda(\bigcup U_i) = \bigsqcup{}^{\uparrow} \lambda(U_i) \quad \text{(continuity)},$$

(vii) *for any filtered family* $\{U_i\}_{i\in I} \in \Omega X$

$$\lambda(\mathbf{int}(\bigcap U_i)) = \bigsqcap \lambda(U_i) \quad \text{(cocontinuity)}.$$

Theorem 4.31 (Bukatin). *For any ω-continuous dcpo D, there exists a CC-valuation $\lambda\colon \Omega D \to [0,1]$.*

Proof. A direct construction of the valuation is given in [BS98]. A transparent and elegant proof of the theorem exploiting properties of the lattice of open sets can be found in [Kei98]. □

For any CC-valuation $\lambda\colon \Omega D \to [0,1]$ on a ω-continuous dcpo D with (†), we define a map $\mu_\lambda\colon D \to [0,\infty)^{op}$ associated with the CC-valuation by

$$(\forall x \in D)\ \mu_\lambda x := \lambda(\twoheaduparrow x).$$

Lemma 4.32. *Let D be a continuous dcpo. For any directed set $S \subseteq D$ with $x = \bigsqcup^{\uparrow} S$ we have*

$$\twoheaduparrow x = \mathrm{Int}(\bigcap_{d\in S} \twoheaduparrow d).$$

Proof. Let S be a directed set with supremum x. The family $\{\twoheaduparrow d\}_{d\in S}$ of Scott-open sets is filtered. Indeed, let $d_1, d_2 \in S$. Since S is directed, there exists a $d_3 \in S$ such that $d_3 \sqsupseteq d_1, d_2$. Hence for any $z \in \twoheaduparrow d_3$ we have that $d_1, d_2 \sqsubseteq d_3 \ll z$, what implies $d_1, d_2 \ll z$. That is, $\twoheaduparrow d_3 \subseteq \twoheaduparrow d_1, \twoheaduparrow d_2$. Now, notice that since $\twoheaduparrow x \subseteq \bigcap \twoheaduparrow d$, also $\twoheaduparrow x \subseteq \mathrm{Int}(\bigcap \twoheaduparrow d)$.

Conversely, let $a \in \mathrm{Int}(\bigcap \twoheaduparrow d)$. By continuity of D, there exists a $b \in \mathrm{Int}(\bigcap \twoheaduparrow d)$ such that $b \ll a$. Since $b \in \mathrm{Int}(\bigcap \twoheaduparrow d) \subseteq \bigcap \twoheaduparrow d \subseteq \bigcap \uparrow d$, b is an upper bound for S. Hence $x \sqsubseteq b \ll a$, what implies $a \in \twoheaduparrow x$. □

Lemma 4.33. *The mapping μ_λ is Scott-continuous.*

Proof. Let $x \sqsubseteq y \in D$. Then since $\twoheaduparrow y \subseteq \twoheaduparrow x$, $\mu_\lambda y \le \mu_\lambda x$. Hence μ is monotone with respect to order on $[0,\infty)^{op}$.

By Lemma 4.32, cocontinuity of λ and monotonicity of μ we have that for every directed subset $S \subseteq D$ with $x = \bigsqcup^{\uparrow} S$

$$\mu_\lambda x = \lambda(\twoheaduparrow x) = \lambda(\mathrm{Int}(\bigcap_{d\in S} \twoheaduparrow d)) = \bigsqcap_{d\in S} \lambda(\twoheaduparrow d) = \bigsqcup{}^{\uparrow} \mu_\lambda(D).$$

□

Lemma 4.34. *For an ω-continuous dcpo D with (†) we have* $\ker\mu_\lambda = \max D$.

Proof. Let $x \in \max D$. Hence $\mu_\lambda x = \lambda(\Uparrow x) = \lambda(\emptyset) = 0$. That is, $x \in \ker\mu_\lambda$. Conversely, if $x \notin \ker\mu_\lambda$, then $\Uparrow x \neq \emptyset$. That is, $\Uparrow x \supset \emptyset$ and by strictness, $\mu_\lambda x = \lambda(\Uparrow x) > 0$. Hence $x \notin \max D$. □

Proposition 4.35. *For an ω-continuous dcpo D with (†) the subset* $\max D$ *is a G_δ set with respect to the Scott topology on D.*

Proof. By Lemmata 4.9 and 4.34. □

Recall that a Scott-continuous mapping $\mu\colon P \to [0,\infty)^{op}$ on a continuous poset is a measurement iff for all $x \in D$, if (x_n) is a sequence in $\Downarrow x$ with $\lim_{n\to\infty} \mu x_n = \mu x$, then (x_n) is directed with supremum x.

Proposition 4.35 can be proved explicitly as follows:

Construction 4.36. *For every countably based continuous dcpo D with (†), the subset* $\max D$ *is G_δ with respect to the Scott topology on D.*

Proof. Let $B := \{b_1, b_2, b_3, ...\}$ be a countable basis for D. For every $i = 1, 2, 3, ...$ we set $w(b_i) := 2^{-i}$. We have $\sum_{b\in B} w(b_i) = 1$. For every $A \in \mathcal{P}D$ define

$$\lambda(A) := \sum_{b\in A\cap B} w(b).$$

We have $\lambda(D\setminus A) = \lambda(D) - \lambda(A) = 1 - \lambda(A)$. Note also that λ restricted to ΩD, i.e. $\lambda\colon \Omega D \to [0,1]$ is a Scott-continuous valuation. All of the properties are straightforward and their justification is omitted. Define $\mu\colon D \to [0,\infty)^{op}$ by $\mu x := \inf\{\lambda(U) \mid x \in U \in \Omega D\}$.

Monotonicity of the map follows from the well-known characterisation of the order:

$$x \sqsubseteq y \text{ iff } (x \in U \Rightarrow y \in U)$$

for any $x, y \in D$ and $U \in \Omega D$. Scott-continuity of μ is a consequence of the fact that if a Scott-open set contains the supremum of a directed set, then it already contains some element from the set.

Assume $x \notin \max D$. Since B is dense in the Scott topology, there exists a $b \in \Uparrow x$. Therefore, for any Scott-open set U around x we have $\lambda(U) \geq \lambda(\Uparrow x) \geq w(b) > 0$. Thus, $\mu x \geq \lambda(\Uparrow x) \geq w(b) > 0$. That is, $x \notin \ker\mu$. Conversely, suppose that $x \in \max D$. The set $D\setminus\cup_{i=1}^{n}\downarrow b_i$ is Scott-open set around $\max D$. For every $n \in \omega$ we have

$$0 \leq \mu x \leq \lambda(D\setminus\bigcup_{i=1}^{n}\downarrow b_i) = \lambda(D) - \lambda(\bigcup_{i=1}^{n}\downarrow b_i) = 1 - \sum_{i=1}^{n}\sum_{b\in\downarrow b_i} w(b).$$

Therefore, in the limit

$$0 \leq \mu x \leq 1 - \sum_{i=1}^{\infty} \sum_{b \in \downarrow b_i} w(b) = 1 - 1 = 0.$$

That is, $x \in \ker\mu$. $\square$

4.2.6 Models of developable spaces

In this section we present a general construction of a model for a wide class of first-countable topological spaces.[2]

Lemma 4.37. *If X is a developable and T_1 topological space, then it is homeomorphic to the subset of maximal elements of an optimal poset P equipped with a measurement $\mu\colon P \to [0,1)^{op}$ with $\ker\mu = \max P$.*

Proof. ($\Rightarrow$). Let $\{\mathcal{U}_n\}_{n\in\omega}$ be a development for X. We can assume that elements of each cover are nonempty. Define $X' := \{\{a\} \mid a \in X\}$ and $P := \bigcup \mathcal{U}_n \cup X'$. Assign a number

$$n(x) := \sup\{m \mid x \in \mathcal{U}_m\}$$

to every element x in $P \setminus X'$ and put $n(x) := \infty$, whenever $x \in X'$.

Note that if $z \notin X'$, then $n(z) \in \omega$. For, suppose the contrary: $z \in \mathcal{U}_m$ for every $m \in \omega$. Since $z \notin X'$, there exist at least two distinct elements $a, b \in z$. By T_0 property of the space, there exists $v \in \tau$ with either $a \in v$ and $b \notin v$ or we have $a \notin v$ and $b \in v$. In the former case there is $k \in \omega$ such that $a \in \mathbf{St}(a, \mathcal{U}_k) \subseteq v$. But by our supposition, $z \subseteq \mathbf{St}(a, \mathcal{U}_k)$ and so $z \subseteq v$, a contradiction witnessed by $b \in z \setminus v$. The latter case is symmetric.

Consider a partial order $\sqsubseteq$ between elements of P defined as follows:

$$x \sqsubseteq y \text{ iff } (y \subseteq x \text{ and } n(y) > n(x)) \text{ or } x = y.$$

Note that if $x \in X'$ (say, $x = \{a\}$ for some $a \in X$) and $x \sqsubseteq y$ for some $y \in P$, then $y \subseteq \{a\}$ and hence $y = \{a\} = x$. Therefore $X' \subseteq \max P$. Conversely, if $x \notin X'$, then $x \in \mathcal{U}_n$ for some $n \in \omega$. Choose any $b \in x$ and note that $x \sqsubset \{b\}$ in P. Hence $x \notin \max P$. We conclude that $X' = \max P$.

Observe that

$$\forall x, y \in P.\ n(x) = n(y) \text{ implies } (x = y \text{ or } x \# y), \tag{4.7}$$

[2]Theorem 4.38 was announced by Mike Reed at the First Irish Conference on the Mathematical Foundations of Computer Science and Information Technology (MFCSIT2000). We present a proof, which has been found independently.

by the definition of the order on P (recall that $x\#y$ means that the subset $\{x, y\}$ of P has no upper bound).

Define a mapping $\mu\colon P \to [0,1)^{op}$ by $\mu x = 2^{-n(x)}$ if $x \in P \setminus X'$ and $\mu x = 0$ otherwise. By definition and (4.7), $\ker\mu = X' = \max P$. It is also clear that the map is monotone and strictly monotone.

We will show that the function μ is Scott-continuous. Let D be a directed subset of P with supremum x.

Assume $x \in P \setminus X'$. Suppose that for any $d \in D$ we have $n(d) < n(x)$. Since D is nonempty, choose $d_1 \in D$ such that for any other $e \in D$, $n(e) \leq n(d_1)$. Now, if for arbitrary $d_2 \in D$ we have $d_2 \sqsubseteq d_1$, then $d_1 = x$, a contradiction as $n(d_1) < n(x)$. Otherwise, there is $d_2\#d_1$ and by directness of D, there exists $d_3 \in D$ with $n(d_3) > n(d_1)$, a contradiction with our choice of d_1. We conclude that there exists an element $d \in D$ with $n(d) = n(x)$ and hence $x = d \in D$ by (4.7). We have proved that

$$\forall D \in P.\ (x = \bigsqcup\nolimits^{\uparrow} D \text{ and } x \notin X') \text{ implies } x \in D. \tag{4.8}$$

Therefore, $\bigsqcup^{\uparrow}\mu(D) = \mu x$.

Assume that $x \in X'$. Suppose that there exists $m \in \omega$ such that $n(d) \leq m$ for any $d \in D$. Without loss of generality we may choose the number m in such a way that $m = n(e)$ for some $e \in D$. If all elements of D are below e, then $x \sqsubseteq e$ and hence $x = e$, by maximality of x. This implies that $n(e) = n(x) = 0$, a contradiction. Otherwise, there exists $e_1 \in D$ with $e_1\#e$. By directness of D, there is $e_2 \sqsupseteq e_1, e$ with $n(e_2) > n(e)$, which is again a contradiction. We have shown that

$$\forall D \in P.\ (x = \bigsqcup\nolimits^{\uparrow} D \text{ and } x \in X') \text{ implies } \{n(d) \mid d \in D\} \text{ is unbounded.} \tag{4.9}$$

Hence, $\bigsqcup^{\uparrow}\mu(D) = 0 = \mu x$.

We conclude that the mapping μ is Scott-continuous.

We claim that every non-maximal element is compact. Let $z \in P \setminus X'$ and $z \sqsubseteq x = \bigsqcup^{\uparrow} D$ for some directed subset D of P. If $x \notin X'$, then $z \sqsubseteq x \in D$ by (4.8). Otherwise, say $x = \{a\}$ for some $a \in X$, and so there exists $k \in \omega$ such that $a \in \mathbf{St}(a, \mathfrak{U}_k) \subseteq z$. Without loss of generality, $k > n(z)$ and $n(e) = k$ for some $e \in D$ (the latter follows from (4.9)). Hence $a \in e \subseteq z$ and so $z \sqsubseteq e$. We have shown that $z \ll z$, whenever $z \in P \setminus X'$.

It is now easy to see that for any $x \notin X'$ we have $\twoheaddownarrow x = {\downarrow} x$ and so $x = \bigsqcup^{\uparrow} \twoheaddownarrow x$. Otherwise, if $x \in X'$ (say $x = \{a\}$), then by construction of P, $\twoheaddownarrow x$ is directed and $\{n(y) \mid y \ll x\}$ is unbounded. Clearly, if $\twoheaddownarrow x \sqsubseteq z$ for any other $z \in P$, then $n(z) = \infty$ and so $z \in X'$. Then $z = x$ by the T_1 axiom of the space X. We conclude that $x = \bigsqcup^{\uparrow} \twoheaddownarrow x$. Therefore, P is an optimal poset. Also, from the construction of P it is immediate that $\tau = \sigma(P)_{|X'}$.

Finally, we will show that the mapping μ measures P. Let $x \in P$ and $x \in \Uparrow z \in P$. If $x \notin X'$, taking $\varepsilon := \mu x/2$ proves the claim. Otherwise, $x = \{a\}$ for some $a \in X$ and the claim follows from the fact that there exist $k \in \omega$ with $a \in \mathbf{St}(a, \mathcal{U}_k) \subseteq z$. □

Theorem 4.38. *The following are equivalent:*

1. *X is developable and T_1,*
2. *X is the kernel of a measurement on an optimal poset,*
3. *X is the kernel of a measurement on a continuous poset.*

Proof. (1)⇒(2) follows from the lemma above. (2)⇒(3) is trivial. (3)⇒(1) is proved by Martin in [Mar00], Proposition 5.3.5, page 137. □

A metrizability theorem by Alexandroff and Urysohn (cf. e.g. [Wil70], Theorem 23.7, page 169) is now a corollary of the above characterization.

Corollary 4.39 (Alexandroff & Urysohn, 1923). *A T_0 space is metrizable iff it has a development $\{\mathcal{U}_n\}_{n\in\omega}$ such that whenever $y, z \in \mathcal{U}_n$ and $y \cap z \neq \emptyset$, then $y \cup z \subseteq w$ for some $w \in \mathcal{U}_{n-1}$.*

Proof. (⇒). Take $\mathcal{U}_n := \{B(x, \frac{1}{4^n}) \mid x \in X\}$. (⇐). T_1 separation follows easily from the assumptions. Then, build P as in Theorem 4.37. Take any pair of consistent elements $x, y \in P$. Then $x, y \in P \setminus X$. Without loss of generality suppose that $x \in \mathcal{U}_k$, $y \in \mathcal{U}_n$ and $k > n$. Since $\mathcal{U}_k$ refines $\mathcal{U}_n$, there exists $z \sqsubseteq x$ with $z \in \mathcal{U}_n$. By assumption, we can choose $w \in \mathcal{U}_{n-1}$ with $w \sqsubseteq z, y$. Therefore, $w \sqsubseteq x, z$ and $\mu w = 2\max\{\mu x, \mu y\}$. By Theorem 5.4.2 of [Mar00], the function μ is a Lebesgue measurement on X. Hence X is metrizable. □

4.2.7 Choquet completion via rounded ideal completion

In his "Lectures on analysis" [Cho69] Gustave Choquet proposed a notion of completeness for topological spaces.

Choquet's idea is easily understood in terms of certain topological *games*. Let X be a topological space and α and β be two "players". We assume that β moves first. Players take turns and each one chooses a nonempty set lying in the opponent's previously chosen open set. In addition β always specifies a point lying in his chosen open set and α's open set must contain this point. We say that a space is *strongly α-favourable* if α has a *winning tactic (strategy)* in the game, that is, if α can choose sets V_n so that $\bigcap V_n \neq \emptyset$ no matter what β chooses (in which case α "wins" the run (V_n) of the game).

One could favour player α even more and in every particular run of the game give him a chance to base his choice of the next move not only on the last β's move but on *all* moves played so far. If, under this additional assumption, α wins every possible run of the game, we say that the space X is *Choquet-complete.* We formalize this situation as follows:

Definition 4.40. *Let $\langle X, \tau\rangle$ be a topological space and*

$$\tau_* = \{(U, x) \mid x \in U, U \in \tau\}.$$

The space X is Choquet-complete *if there exists a sequence of functions*

$$f_n\colon \underbrace{\tau_* \times ... \times \tau_*}_{n\ times} \to \tau, \qquad n = 1, 2, ...$$

such that

1. *for each $((U_1, x_1), ..., (U_n, x_n))$ we have*

$$x_n \in f_n((U_1, x_1), ..., (U_n, x_n)) \subseteq U_n$$

   and

2. *for any sequence (V_n, x_n) in τ_* with $V_{n+1} \subseteq f_n((V_1, x_1), ..., (V_n, x_n))$ for all $n = 1, 2, ...$ we have*

$$\bigcap V_n \neq \emptyset.$$

Basic facts about Choquet completeness are:

Proposition 4.41. *The following hold:*

1. *A Choquet complete space is Baire.*
2. *A metric space is Choquet complete iff it is completely metrizable.*
3. *G_δ subspaces of Choquet complete spaces are Choquet complete.*
4. *A locally compact sober space is Choquet complete.*

Proof. First two claims are demonstrated in [Cho69]. For the proof of the third one, we refer to Exercise 8.16 of [Kec95]. Finally, the last fact is proved in [Mar99]. □

As has been already remarked in [Mar00], Choquet completeness does not assume any separation axioms to hold and, moreover, captures two fundamental aspects of computing: approximation and convergence. It seems therefore well-suited as a topological notion that captures completeness of continuous domains. This reasoning leads us to a conclusion that there must be a close connection between Choquet completeness and *sobriety*. It is indeed: in his doctoral thesis Keye Martin proved that if the Scott topology on a continuous poset is sober (equivalently: the poset is a dcpo), then it is Choquet-complete. Results of this section implicitly show that the Choquet completion of continuous posets equipped with abstract measurements agrees with the rounded ideal completion and hence, by Lawson's observation [Law97a], it is a special case of sobrification. The precise relationship between the two kinds of completions needs further study; at the moment we focus only on the application to modelling topological spaces.

Theorem 4.42. *Any topological space X that can be modelled as a kernel of an abstract measurement on a continuous poset P is Choquet completable.*

Proof. Let $f\colon P \to [0,\infty)^{op}$ be an abstract measurement on P. Since kernels of Scott-continuous maps are G_δ subsets (cf. Lemma 4.9), the space X can be modelled as a G_δ subset of an optimal poset E (cf. Theorem 4.25). However, according to Proposition 4.26, the abstract measurement f on P restricts to an abstract measurement μ on E with $\mathrm{ker}\mu = \max E$. By Theorem 4.18, the map μ can be extended to a measurement $\bar{\mu}$ on the rounded ideal completion of E. We claim that $\mathrm{ker}\bar{\mu} = \max\mathfrak{I}(E)$. Indeed, every ideal x with $\bar{\mu}x = r$, for some $r > 0$, has a maximum, say $a \in x$, with $\mu a = r$. Hence x cannot be maximal and we have shown that $\max\mathfrak{I}(E) \subseteq \mathrm{ker}\bar{\mu}$. Conversely, suppose that $\bar{\mu}x = 0$ for some $x \in \mathfrak{I}(E)$ and $x \sqsubseteq y$ for $y \in \mathfrak{I}(E)$ with $\bar{\mu}y = 0$. Then since the measurement $\bar{\mu}$ is strictly monotone, we have $x = y$. This means that $x \in \max\mathfrak{I}(E)$. Therefore, $\mathrm{ker}\bar{\mu} \subseteq \max\mathfrak{I}(E)$, as required. In particular, $\max\mathfrak{I}(E)$ is a G_δ subset of a continuous dcpo and hence is Choquet complete by Proposition 4.41.(3)-(4). It remains to show that the $(\cdot)^*$-image of X is dense in $\max\mathfrak{I}(E)$. Let $x \in \max\mathfrak{I}(E)$ and we can assume that $x \neq b^*$ for each $b \in E$. Take any open subset U around x in $\max\mathfrak{I}(E)$. Then $U = V \cap \max\mathfrak{I}(E)$ for some Scott-open set V of $\mathfrak{I}(E)$. Since the $(\cdot)^*$-image of E is a basis for $\mathfrak{I}(E)$, there is $b \in E$ with $b^* \ll x$ and $b^* \in V$. But since $\bar{\mu}b^* > 0$, then $\mu b > 0$ and so $b \in K(E)$. Therefore, referring to the construction of E from P, which is performed in Theorem 4.25, there exists $c \in X$ with $b \ll_P c$. This implies $b \ll_E c$ and so $b^* \sqsubseteq c^*$ in $\mathfrak{I}(E)$. But then $c^* \in U$, which proves density of X in $\max\mathfrak{I}(E)$. $\square$

Reinhold Heckmann in his paper [Hec99] demonstrates that the completion of a metric space X is the space of maximal elements of the poset of rounded ideals of the formal ball model $\mathbf{B}X$. We will show that Heckmann's construction is a special case of ours.

Proposition 4.43 (Heckmann). *The completion of a metric space $\langle X, \rho \rangle$ is the subspace of maximal elements of the rounded ideal completion of the formal ball model* $\mathbf{B}X$*.*

Proof. Recall that $\mathbf{B}X$ is equipped with a measurement $\mu \colon \mathbf{B}X \to [0, \infty)^{op}$ given by $\mu(x, r_x) := r_x$ for all $(x, r_x) \in \mathbf{B}X$. First, we prove that the function μ is an abstract measurement. Let A be a subset of $\mathbf{B}X$ and D be a $\ll$-directed subset of $\mathbf{B}X$ with $A \ll_H D$ and $\inf\mu(A) = \inf\mu(D)$. Set $s := \inf\mu(A) = \inf\mu(D)$. Let $(c, r_c), (d, r_c) \in D$ with $(c, r_c) \ll (d, r_d)$. Choose $\varepsilon > 0$ and $(a, r_a) \in A$ with $s < r_a < s + \varepsilon/2$. Since $A \ll_H D$ and D is directed, there exists $(e, r_e) \in D$ with $\rho(a, e) < r_a - r_e$ and $\rho(d, e) < r_d - r_e$. Then

$$\rho(d, a) \leq \rho(d, e) + \rho(e, a) < r_d + r_a - 2r_e.$$

But since $s \leq r_e$, we have $r_a < s + \varepsilon/2 \leq r_e + \varepsilon/2$. Therefore,

$$\rho(d, a) < r_d - r_a + \varepsilon$$

for any $\varepsilon > 0$ and so $(d, r_d) \sqsubseteq (a, r_a)$. Thus, $(c, r_c) \ll (a, r_a)$. We have shown that $D \ll_H A$ in $\mathbf{B}X$ and so μ is an abstract measurement on $\mathbf{B}X$. Therefore, it extends to a measurement $\bar{\mu}$ on the rounded ideal completion of $\mathbf{B}X$. Heckmann [Hec99] proves that $\max\mathbf{B}X$ embeds onto $\max\mathfrak{I}(\mathbf{B}X)$. Thus, the subset of maximal elements of the rounded ideal completion of $\mathbf{B}X$ is the Choquet-completion and hence the metric completion of X. □

4.2.8 Models of metric spaces

In this section we characterize models of metric and ultrametric spaces. We want to stress one particular aspect of this characterization. We do *not* assume that the models satisfy the Lawson condition ("*the Scott and Lawson topologies agree at the top*"). Hence subspaces of maximal elements of our *algebraic* models are *not* zero-dimensional in general. Dropping the Lawson condition enables us to observe an interesting correspondence between the structure of metric spaces and certain properties of their models.

Theorem 4.44. *The following are equivalent for a topological space X:*

1. *X is metrizable;*

2. *X is the kernel of a Lebesgue measurement on a continuous poset;*

3. *X is the kernel of a partial metric that induces the Scott topology on a continuous poset;*

4. *X is the kernel of a partial metric that induces the Scott topology on an optimal poset.*

Proof. For (1)⇒(3) use the formal ball model $\mathbf{B}X$. (3)⇒(2) holds since the self-distance mapping of a partial metric is a Lebesgue measurement. For (2)⇒(4), use the construction of an optimal poset from Proposition 4.27. (4)⇒(1) is trivial. □

Analogous statement can be made about complete spaces.

Theorem 4.45. *The following are equivalent for a topological space X:*

1. *X is completely metrizable;*

2. *X is the kernel of a Lebesgue measurement on a continuous dcpo;*

3. *X is the kernel of a partial metric that induces the Scott topology on a continuous dcpo;*

4. *X is the kernel of a partial metric that induces the Scott topology on an optimal dcpo.*

Proof. For (4)⇒(1), it is trivial that X is metrizable. Since X is a G_δ subset of a Choquet complete space (cf. Theorem 4.41(3)-(4)), it is completely metrizable. The rest of the proof obeys the same pattern as in the theorem above and is therefore omitted. □

Finally, we characterize Polish spaces:

Theorem 4.46. *The following are equivalent for a topological space X:*

1. *X is Polish;*

2. *X is the kernel of a Lebesgue measurement on an ω-continuous dcpo;*

3. *X is the kernel of a partial metric that induces the Scott topology on a ω-continuous dcpo;*

4. *X is modelled by an ω-continuous dcpo P such that* $\max P$ *is regular with respect to the subspace Scott topology.*

5. *X is modelled by an ω-continuous dcpo, which satisfies the Lawson condition* **(Law)**.

6. *X is modelled by a countably-based Lawson-compact dcpo.*

Proof. For (1)⇒(3) use the formal ball model $\mathbf{B}X$. (3)⇒(2) is immediate. (2)⇒(4). Let p_μ be the partial semimetric induced by a Lebesgue measurement μ on P. By Theorem 2.28, the induced semimetric d_{p_μ} is metrizable. Hence $\max P$ is metrizable and thus regular. (4)⇒(1) is proved by Martin in [Marc]. The argument is as follows: Since second-countability is hereditary, the subspace of maximal elements of P is second-countable and thus metrizable by Urysohn's theorem (cf. [Wil70], Theorem 23.1, page 166). The space $\max P$ is also a G_δ subset of a Choquet complete separable metric space, and hence Polish. (1)⇒(5) is proved in [CFKb]. (5)⇒(6) is immediate. To complete the proof, we show (6)⇒(2). Let P be a countably-based Lawson-compact model of X. By a result of Lawson [Law97b], P admits a radially convex metric d for the Lawson topology. Now, Martin proves (cf. Theorem 5.6.1 of [Mar00], page 150) that P admits a kernel measurement μ such that for all $y \sqsubseteq x$ we have $d(x, y) \leq \mu y - \mu x$. Let K be a Scott-compact subset of a Scott-open set U in P. Hence K is Lawson-compact and U is Lawson-open. By the Lebesgue covering lemma (cf. [Wil70], Theorem 22.5, page 163), there exists $\varepsilon > 0$ with $B_d(K, \varepsilon) \subseteq U$. Let $x \in K$ and $y \in \mu(x, \varepsilon)$. Then $d(x, y) \leq \mu y - \mu x \leq \mu y < \varepsilon$. Therefore $y \in B_d(K, \varepsilon) \subseteq U$ and we conclude that $\mu(K, \varepsilon) \subseteq U$, as required. □

Definition 4.47. *A* tree *is a continuous poset P such that*

$$\forall x, y \in P.\ \ x{\uparrow}y \text{ implies } (x \sqsubseteq y \text{ or } y \sqsubseteq x).$$

A tree is complete *if P is a dcpo. We will also assume that every tree has a bottom element $\perp$.*

Theorem 4.48. *The following are equivalent for a topological space X:*

1. *X is ultrametrizable (completely ultrametrizable; completely ultrametrizable and second countable);*
2. *X is the kernel of a Lebesgue measurement on an optimal tree (optimal complete tree; countably based, optimal complete tree);*
3. *X is the kernel of a partial metric that induces the Scott topology on an optimal tree (optimal complete tree; countably based, optimal complete tree).*

Proof. (1)⇒(3). Set

$$\mathfrak{U}_n := \{B_d(x, 1/4^n) \mid x \in X\}$$

with respect to some ultrametric $d\colon X \times X \to [0,\infty)$ compatible with the topology on X. The collection $\{\mathcal{U}_n\}$ is a development for X. We build an optimal model P of X as in Lemma 4.37 (and use the notation of the lemma). Since d is an ultrametric, for every x, y in $P \setminus X'$ such that $x{\uparrow}y$ we have either $x \sqsubseteq y$ or $y \sqsubseteq x$, and so P has a tree structure. Therefore the measurement μ (as defined in the proof of the lemma) is vacuously weakly modular. Hence, the induced partial semimetric p_μ is a partial metric with $\ker p_\mu = X$. (3)$\Rightarrow$(2) is clear. For (2)$\Rightarrow$(1), observe that any partial metric p on a tree P satisfies

$$p(x,y) \leq \max\{p(x,z), p(z,y)\}$$

for any $x, y, z \in P$. Hence, the mapping p restricts to an ultrametric on its kernel.

For the "completeness" part we need to prove only (1)$\Rightarrow$(3) as the rest follows the same pattern as above. The model P of X is algebraic by Proposition 4.24 and admits a Lebesgue measurement μ. Hence μ is an abstract measurement on P by Corollary 4.20. The Choquet completion of X is modelled by a continuous dcpo $\mathcal{I}(P)$. But since X is already Choquet complete by Proposition 4.41.(4), this means that $\mathcal{I}(P)$ is a complete model for X equipped with a measurement $\bar{\mu}$. Observe that since P is a tree, its rounded ideals are chains and hence the tree structure is inherited by $\mathcal{I}(P)$. Moreover, $\bar{\mu}$ is vacuously weakly modular, and hence induces a partial metric $p_{\bar{\mu}}$.

The proof of the claim about second-countability presents no difficulties.
□

4.2.9 Topological spaces and their models - an overview

The diagram

The network diagram on page 119 shows relations between various classes of topological spaces and their models. All objects are numbered and a list of full names is presented below.

All the modelled spaces are assumed to be T_1. All the distance maps on models are assumed to induce the Scott topology and satisfy the kernel condition (for a partial metric p on a continuous poset P the kernel condition is the requirement that $\ker\mu_p = \max P$, where μ_p is the self-distance of p).

The schematic naturally splits into four parts. Two upper ones contain spaces (developable on the left, metric ones on the right). Two lower parts show various models, again sorted from the most general on the left to the most specific on the right.

On the other hand, both the "models" and "spaces" part are divided into three classes indicated by dotted lines. The "levels" from top to down represent respectively:

1. second-countable, complete spaces,
2. complete spaces,
3. "plain" spaces,
4. "plain" models,
5. complete models,
6. countably-based, complete models.

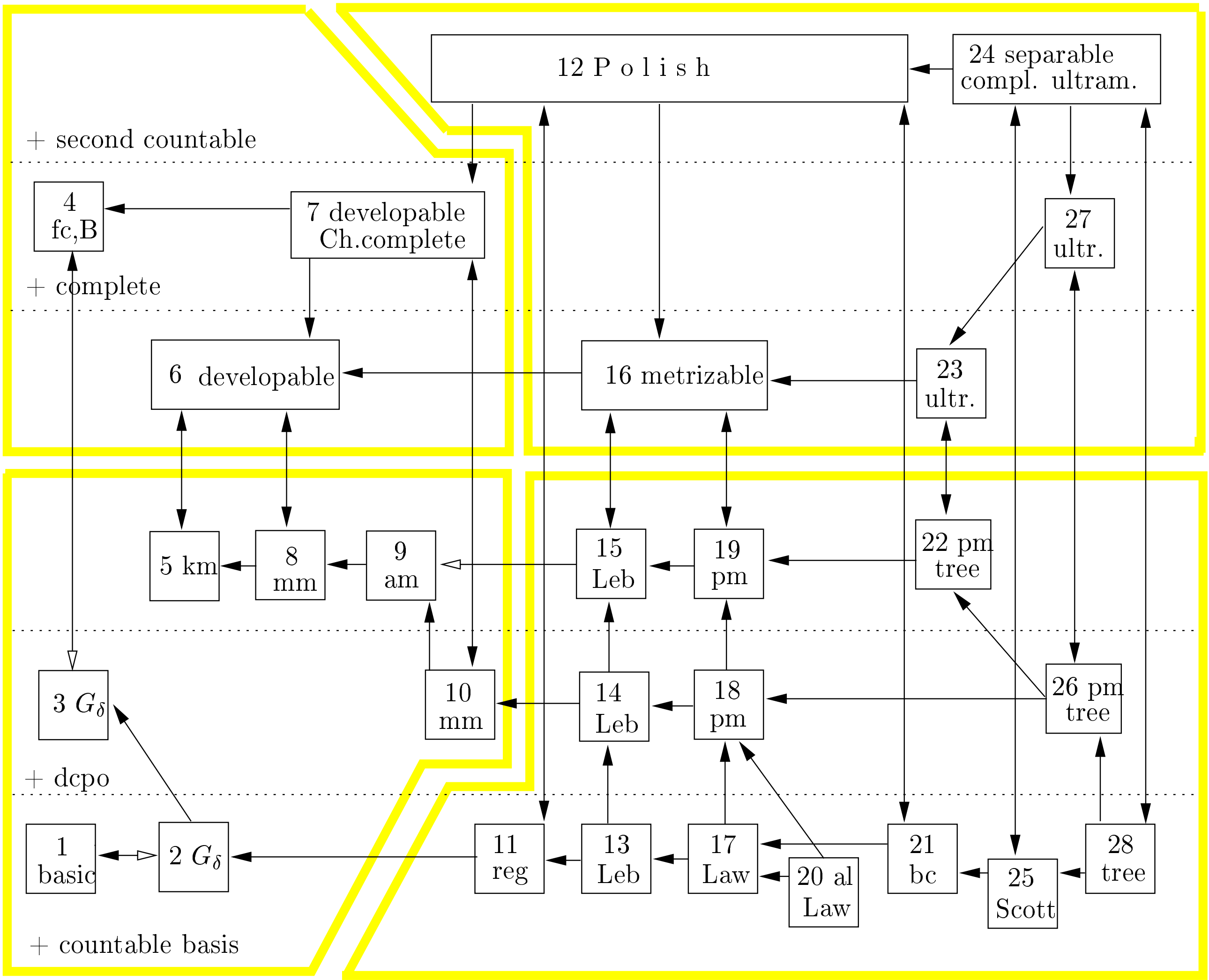
Arrows between objects of the same type represent inverse inclusion of classes. For example, (25)$\leftarrow$(28) indicates that countably based, complete trees form a subclass of the class of all Scott-domains. A white head of the arrow indicates an open question, for instance it is not known if (1)$\rightarrow$(2) holds.

Other arrows indicate how spaces are related to models and *vice versa*, e.g. the correspondence (12)$\leftrightarrow$(22) states that a space is Polish if and only if it can be modelled by a bounded-complete ω-continuous dcpo.

Description of the entries

1. "basic model": a countably-based continuous dcpo,
2. A countably-based continuous dcpo P such that max$P$ is a G_δ subset with respect to the subspace Scott topology,
3. A continuous dcpo P such that max$P$ is a G_δ subset with respect to the subspace Scott topology,
4. A first-countable Baire space,
5. A continuous poset P with a kernel measurement μ.
6. A developable space,
7. A Choquet-complete developable space,
8. A continuous poset P with a measurement,
9. A continuous poset P with an abstract measurement,
10. A continuous dcpo P with an abstract measurement

11. A countably-based continuous dcpo P such that $\max P$ is a regular space with respect to the subspace Scott topology,
12. A Polish space,
13. A countably-based continuous poset P with a Lebesgue measurement,
14. A continuous dcpo P with a Lebesgue measurement,
15. A continuous poset P with a Lebesgue measurement,
16. A metrizable space,
17. An ω-continuous Lawson-compact dcpo,
18. A partially metrized continuous dcpo,
19. A partially metrized continuous poset
20. An ω-algebraic dcpo such that the Scott and the Lawson topologies agree at the subspace of maximal elements,
21. An ω-continuous bounded-complete dcpo,
22. A partially metrized tree,
23. An ultrametrizable space,
24. An separable, completely ultrametrizable space,
25. A Scott-domain,
26. A directed-complete, partially metrized tree,
27. A completely ultrametrizable space,
28. A directed-complete, countably based tree.



+ second countable
+ complete
+ dcpo
+ countable basis
12 P o l i s h
24 separable compl. ultram.
4 fc,B
7 developable Ch.complete
27 ultr.
6 developable
16 metrizable
23 ultr.
5 km
8 mm
9 am
15 Leb
19 pm
22 pm tree
3 G_δ
10 mm
14 Leb
18 pm
26 pm tree
1 basic
2 G_δ
11 reg
13 Leb
17 Law
20 al Law
21 bc
25 Scott
28 tree

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Symbols

Order

Topology

Distance and measurement